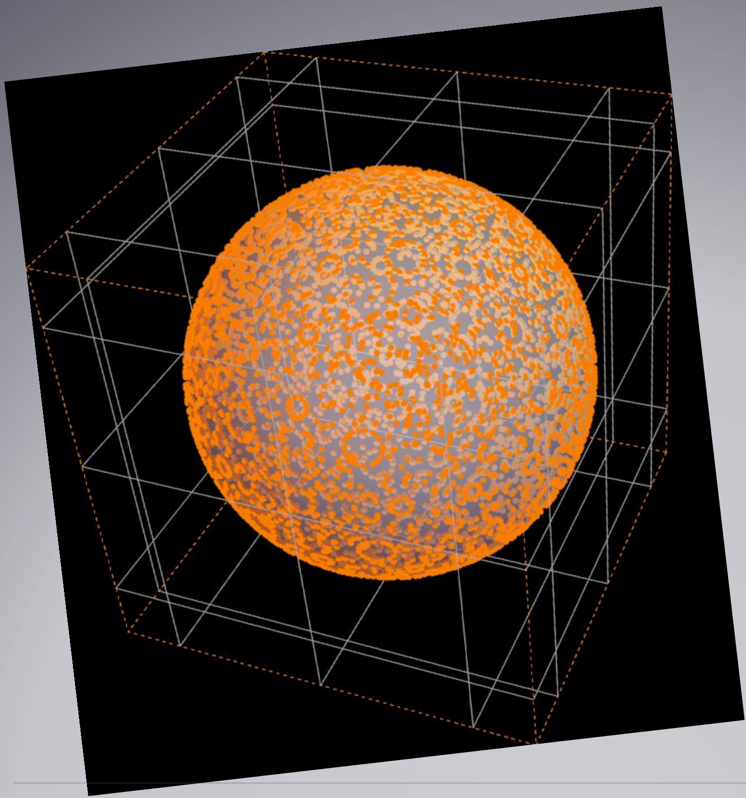
Karl-Heinz Kuhl



PRIME NUMBERS– THINGS LONG-KNOWN AND THINGS NEW-FOUND

A JOURNEY THROUGH THE LANDSCAPE OF THE PRIME NUMBERS

Amazing properties and insights – not from the perspective of a mathematician, but from that of a voyager who, pausing here and there in the landscape of the prime numbers, approaches their secrets in a spirit of playful adventure, eager to experiment and share their fascination with others who may be interested.

Third, revised and updated edition (2020)

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Dipl.-Phys. Karl-Heinz Kuhl

Parkstein, December 2020

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$
(Ramanujan)

Web:
https://yapps-arrgh.de

(Yet another promising prime number source:
amazing recent results from a guerrilla hobbyist)

Link to the latest online version https://yapps-arrgh.de/primes_Online.pdf

Some of the text and Mathematica programs have been removed from the free online version. The printed and e-book versions, however, contain both the text and the programs in their entirety. Recent supplements to the book can be found here: https://yapps-arrgh.de/data/Primenumbers_supplement.pdf

Please feel free to contact the author if you would like a deeper insight into the many Mathematica programs.

Contact: info@yapps-arrgh.de

For Michèle

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The illustration on the title page shows the graphic from Figure 82, Chapter 9.2.

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1 Table of Contents

2 INTRODUCTION

Prime numbers – scarcely any other concept in mathematics can have fascinated and inspired so many people. Prime numbers seem devoid of the very properties usually associated with mathematical ‘objects’: computability, neatness, order…

Prime numbers exhibit no discernible regularity; they just sit randomly and aimlessly between the other natural numbers. One has the impression that God, when creating the natural numbers, just scattered the primes in among them to grow wild like weeds. Mathematicians have been known, of course, to use on occasion more positive and poetic imagery when speaking of prime numbers and their related functions: instead of ‘weeds’, one hears terms like ‘pearls’ or ‘gems’ (an allusion, perhaps, to the fact that very large prime numbers are as hard to find as precious stones), and the zeta function, which is closely related to prime numbers (see Chapter 1), is sometimes spoken of as a ‘landscape crying out for exploration’.

This ‘unfathomability’ – this ‘quantum of chaos’, if you will – is the source of their appeal; for although prime numbers have exercised a fascination over mankind for hundreds of years, many of the questions surrounding them remain unresolved, despite the best efforts of some of the greatest mathematicians who have ever lived or are alive today!

The number of books devoted to prime numbers has grown considerably in recent years. These fall for the most part into either of two categories: popular-scientific books, which contain hardly any mathematical formulae, and academic treatises written in highly technical language, which consist mainly of mathematical derivations, proofs and formulae that even ambitious hobby-mathematicians find difficult to understand.

This book seeks to provide a different approach to mathematics. Wherever possible, language has been used that is simple and easy to understand. The reader will find here very few proofs. The author has made no attempt, though, to dispense with formulae and graphs. To the contrary: the book contains an abundance of illustrations and formulae. The reason for this is very simple: mathematical formulae have a certain aesthetic and mysterious appeal, even if they are not always understood by the reader. This may awake in readers a certain curiosity and inspire them to seek a more profound understanding of some topics. It is the same with the many graphs and illustrations: a picture is worth a thousand words. The author would even venture the hypothesis that it is possible to appreciate the aesthetics of mathematics without subjecting oneself to the full rigours of the discipline.

No effort has been made here to present formal mathematical proofs of the various theorems discussed. The author regards mathematics – especially the mathematics of the primes – rather as a giant playground to be explored at one’s leisure and within which to experiment freely. Of course such experiments are not such as to satisfy the norms that prevail in the mathematical community, and such a procedure may even make some mathematicians uneasy. It is conceived, though, as a means of offering – even to those with no formal education in the subject – a glimpse of the beauty of mathematics, just as one can enjoy a concerto by J.S. Bach without having previously analysed its structure using the tools of the musicologist.

A constant source of astonishment is the way as one explores the galaxy of prime numbers, wormholes suddenly appear linking supposedly remote areas of the mathematical – and physical – universe with one another.

Without any mathematical knowledge whatever, it must be owned, readers are likely to struggle. Those with high school or GCSE maths will certainly find it of assistance in the understanding of certain chapters. On the other hand, it is possible to appreciate the results (which are mostly presented in the form of illustrations and graphs) without necessarily understanding the minutiae of the methods by which they were obtained…

The exploration of prime numbers was long categorized as 'pure' mathematical research of little use to anyone in everyday life. But all that changed with the need in recent years to develop secure methods of encrypting data flows over the Internet. These methods are based on the characteristics of very large prime numbers (or on the characteristics of large numbers composed of a very large prime numbers). More on this in the chapter "Prime numbers and online banking".

Naturally, this work does not cover all topics concerning prime numbers. Indeed certain themes that might be thought relevant are not touched on at all. Instead, the author has cherry-picked topics that seemed to him to be of particular interest and concentrated exclusively upon them. Most of the themes discussed here can be found in the numerous books devoted to the subject as well as in periodicals and on the Internet. This work is therefore in large part a summary of well-known theorems and techniques of analysis that are to some extent useful also for the understanding of the more detailed parts of the book. These parts are in the nature of an 'anthology of formulae' and most of the selected topics are dealt with in detail on websites such as https://en.wikipedia.org and http://mathworld.wolfram.com.

This book would not have been possible without the software application 'Mathematica'[1] with the aid of which most of the many illustrations and formulae have been created. Readers in possession of this software are encouraged to experiment with the many programs presented here, which can be done simply by copying the program code into a Mathematica notebook and then executing it or by loading the notebooks directly from the CD enclosed with the book.

The author has endeavoured to cite as many sources as possible. However, to pre-empt misunderstandings over any unattributed quotations or sources, the following convention has been adopted: passages displayed or printed in black contain material that has already been presented (by other authors) and published (whether on the Internet or in academic books or periodicals). The material in black therefore constitutes for the most part a summary or amalgam of texts the author considers especially interesting, sourced mainly from well-known websites devoted to the subject. The author craves indulgence if every last one of these sources is not mentioned, but in the age of the Internet, with its powerful

[1] Mathematica: https://www.wolfram.com/mathematica

search engines, locating in any given case the source in question should only take a few seconds.

Themes or formulae that (to the best of the author's knowledge) have not yet been dealt with in the specialist literature – including new conjectures and discoveries – are presented in blue.

The author is aware that the term 'new-found' in the title of this work has a limited shelf-life. What is still new today may, a few years from now, be 'old hat'. Wherever possible, therefore, the author has appended a 'time stamp' to important statements and conjectures.

Readers wishing to delve more deeply into the subject will find a list of suitable material in the Bibliography.

2.1 MATHEMATICAL NOTATION USED IN THIS BOOK

In this publication, besides the elementary mathematical symbols and functions, the following mathematical notation, symbols, function names and abbreviations will be used:

Sets

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$: set of the natural, whole, real and complex integers

$\mathbb{P}$: set of the prime numbers

Operators and symbols

$O(\ldots)$: further remainder terms of order $(\ldots)$

$\sum$: summation

$\prod$: product

p_n: nth prime number

$n!$: factorial

$p\#$: product over all prime numbers $p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n$ to $p_n = p$

F_n: nth Fermat number

M_n: nth Mersenne prime number

ρ_n: nth zero of the zeta function along the 'critical' line

γ: Mascheroni constant (aka Euler constant): $0.57721566 \ldots$

B: Brun's constant (sum of the inverse twin primes): 1.90216054

Π_2: twin prime constant: 0.6601618158

∞: infinity

$\lfloor x \rfloor$: the same as $\text{floor}(x)$: takes the integer part of x

$\binom{n}{k}$: binomial coefficients

(n, m): greatest common divisor, also: $\text{gcd}(n, m)$

$\text{gcd}(n, m)$: see also (n, m)

lcm(n, m): least common multiple, in German: $\text{kgV}(n, m)$

Functions $f(n)$

$\mu(n)$: Moebius function

$$\mu(n) = \begin{cases} (-1)^k \; if \text{ n square free, k: number of prime factors} \\ 0 \text{ otherwise} \end{cases}$$

$M(n)$: Mertens function (summation over Moebius function)

$\Lambda(n)$: Von Mangold function

$$\Lambda(n) = \begin{cases} \ln(p) \text{ if } n = p^k, p \text{ prime and } k > 0 \\ 0 \text{ otherwise} \end{cases}$$

$\varphi(n)$: Euler's phi function (totient function)

$\Phi(n)$: summatory function of $\varphi(n)$

$\sigma_k(n)$: sum of the kth powers of all positive divisors of n

$\sigma(n)$: = $\sigma_1(n)$ (generally called sigma function)

$s(n)$: aliquot sum: sum of all divisors (without n), $s(n) = \sigma_1(n) - n$

$r_k(n)$: number of representations of n as sum of k squares

$r(n)$: $= r_2(n)$ (number of 2-dim. lattice points on a circle with radius n)

$r_4(n)$: $= 8\,\sigma(n) - 32\sigma\left(\frac{n}{4}\right)$, where $\left(\frac{n}{4}\right) = 0$, if $4 \nmid n$

number of 4-dim. lattice points of a 4-dim. sphere with radius n

$\tau(n)$: Ramanujan tau function

$c_q(n)$: Ramanujan sums

$\mathcal{F}_n$: Farey sequence of order n

$\omega(n)$: number of different prime factors of a number n

$\Omega(n)$: number of prime factors of a number n

Functions $f(x)$

$\pi(x)$: counting function for prime numbers: gives the number of prime numbers up to x.

$\pi_2(x)$: gives the number of twin primes up to x

$\pi_3(x), \pi_4(x)$: gives the number of prime triplets / quadruplets up to x

$\pi_n(x)$: gives the number of prime n-tuplets up to x

$\pi'_n(x)$: gives the number of prime pairs with difference n up to x

$\pi_0(x)$: same as $\pi(x)$, but different if x is a prime number:

$\pi_0(x) = \lim\limits_{\varepsilon \to 0} \frac{\pi(x-\varepsilon)+\pi(x+\varepsilon)}{2}$ or: $\pi_0(p) = \pi(p) - \frac{1}{2}$

$\Theta(x), \vartheta(x)$: 1st Chebyshev function: $= \sum_{p \le x} \ln(p)$ (sum over log values of all prime numbers $\le n$)

$\psi(x)$: Chebyshev psi function: summatory function of Von Mangoldt function $\psi(x) = \sum_{p^k \le x} \ln(p) = \sum_{n \le x} \Lambda(n)$ (2nd Chebyshev func.)

$\psi_0(x)$: same as $\psi(x)$, but different if x is a prime number:

$\psi_0(x) = \lim\limits_{\varepsilon \to 0} \frac{\psi(x-\varepsilon)+\psi(x+\varepsilon)}{2}$

$\zeta(s)$: Riemann's zeta function

$P(s)$: prime zeta function

$\xi(s)$: variant of Riemann's zeta function (has the same zeros along the critical line as $\zeta(s)$, but real function values)

$\Gamma(s)$: gamma function

$R(x)$: Riemann function

$\ln(x)$, $\mathrm{Li}(x)$: natural logarithm, integral logarithm

$\mathrm{Ei}(x)$: integral exponential function

$\mathrm{E}_n(x)$: exponential integral function of order n

$Z(t), \vartheta(t)$: Riemann-Siegel functions

$L(s)$: Ramanujan tau Dirichlet L function

$Z(t)$: Ramanujan tau Z function

$\Theta(t)$: Ramanujan tau theta function

$\mathrm{rad}(n)$: radical: product of distinct prime factors:

$$\mathrm{rad}(n) = \prod_{\substack{\boldsymbol{p}|\boldsymbol{n} \\ \boldsymbol{p}\in\mathbb{P}}} \boldsymbol{p}$$

$\mathfrak{Z}(p)$: Z function

Other abbreviations

o: OCRON

$g(o)$: Gödel number of an OCRON

OEIS: Online Encyclopedia of Integer Sequences (http://oeis.org)

OCRON: 'Operator Chain Representation Of Number'

GOCRON: 'Gödelized Operator Chain Representation Of Number'

EOCRON: "Enhanced" OCRON, also EOCRON4, EOCRON6… (types)

EGOCRON: "Enhanced" GOCRON, also EGOCRON4, EGOCRON6… (types)

RG numbers: sequence built by recursive application of the algorithm used for computing Gödel numbers

3 BASICS OF PRIME NUMBERS

Let us begin, first of all, with some important fundamental statements about prime numbers such as can be found in any handbook for mathematical beginners:

- A prime number is a natural number greater than 1 that has exactly two integer divisors: '1' and the number itself. Prime numbers are not divisible by any other integers.
- The first prime numbers read: 2,3,5,7,11,13,17,19, ... etc. The sequence of prime numbers starts with 2 and not with 1.
- Prime numbers become rarer the further we ascend in the number region[2]. This raises the question as to whether there exists a last, greatest prime number. However, as the ancient Greek mathematician Euclid proved 2000 years ago:
- There are infinitely many prime numbers. Euclid's proof is so easy to understand that it can be stated in a few lines:

First, let us suppose the opposite of Euclid's statement: that there exists a greatest prime number p_n. Next build the product from all n prime numbers and add 1:

$$N = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n + 1$$

Obviously, N is much greater than p_n and must be therefore be divisible, as we have assumed a greatest prime number $p_n < N$. After a moment's reflection, it will be clear that N cannot be divisible by 2, nor by 3, 5 …It cannot be divisible by any of the primes p_n. Thus N must be a prime number or must be divisible by a prime number $p > p_n$. This is, however, a contradiction to our assumption. Thus the assumption of the existence of a greatest, last prime number p_n is wrong!

The set $\mathbb{P}$ of prime numbers can be easily extended to the Gaussian complex numbers, leading to the set of 'Gaussian primes'. 'Primality' can also be generalized and defined for other sets of elements. These are commonly called 'prime elements'.

A book about prime numbers deserves at least a few lists of prime numbers (generated by Mathematica):

```
If[PrimeQ[#], Framed[#], #] & /@ Range[1, 100]
```

$\{1, \boxed{2}, \boxed{3}, 4, \boxed{5}, 6, \boxed{7}, 8, 9, 10, \boxed{11}, 12, \boxed{13}, 14, 15, 16, \boxed{17}, 18, \boxed{19}, 20, 21, 22, \boxed{23}, 24, 25, 26, 27, 28, \boxed{29}, 30, \boxed{31}, 32, 33, 34, 35, 36, \boxed{37}, 38, 39, 40, \boxed{41}, 42, \boxed{43}, 44, 45, 46, \boxed{47}, 48, 49, 50, 51, 52, \boxed{53}, 54, 55, 56, 57, 58, \boxed{59}, 60, \boxed{61}, 62, 63, 64, 65, 66, \boxed{67}, 68, 69, 70, \boxed{71}, 72, \boxed{73}, 74, 75, 76, 77, 78, \boxed{79}, 80, 81, 82, \boxed{83}, 84, 85, 86, 87, 88, \boxed{89}, 90, 91, 92, 93, 94, 95, 96, \boxed{97}, 98, 99, 100\}$

[2] Please refer to the table: 'number of primes…' in the Appendix

```
If[PrimeQ[#], Framed[#], #] & /@Range[10^4, 10^4 + 100]
```

```
{10 000, 10 001, 10 002, 10 003, 10 004, 10 005, 10 006, 10 007, 10 008, 10 009, 10 010, 10 011, 10 012, 10 013, 10 014,
 10 015, 10 016, 10 017, 10 018, 10 019, 10 020, 10 021, 10 022, 10 023, 10 024, 10 025, 10 026, 10 027, 10 028, 10 029,
 10 030, 10 031, 10 032, 10 033, 10 034, 10 035, 10 036, 10 037, 10 038, 10 039, 10 040, 10 041, 10 042, 10 043,
 10 044, 10 045, 10 046, 10 047, 10 048, 10 049, 10 050, 10 051, 10 052, 10 053, 10 054, 10 055, 10 056, 10 057, 10 058,
 10 059, 10 060, 10 061, 10 062, 10 063, 10 064, 10 065, 10 066, 10 067, 10 068, 10 069, 10 070, 10 071, 10 072,
 10 073, 10 074, 10 075, 10 076, 10 077, 10 078, 10 079, 10 080, 10 081, 10 082, 10 083, 10 084, 10 085, 10 086,
 10 087, 10 088, 10 089, 10 090, 10 091, 10 092, 10 093, 10 094, 10 095, 10 096, 10 097, 10 098, 10 099, 10 100}
```

We see that prime numbers become gradually less frequent: in the range 1 to 100 we have 25 prime numbers, from 10000 to 10100 there are still 11, and in the region between 10^{20} and $10^{20} + 100$ there is only one prime number!

```
If[PrimeQ[#], Framed[#], #] & /@Range[10^20, 10^20 + 100]
```

```
{100 000 000 000 000 000 000, 100 000 000 000 000 000 001, 100 000 000 000 000 000 002,
 100 000 000 000 000 000 003, 100 000 000 000 000 000 004, 100 000 000 000 000 000 005,
 100 000 000 000 000 000 006, 100 000 000 000 000 000 007, 100 000 000 000 000 000 008, 100 000 000 000 000 000 009,
 100 000 000 000 000 000 010, 100 000 000 000 000 000 011, 100 000 000 000 000 000 012, 100 000 000 000 000 000 013,
 100 000 000 000 000 000 014, 100 000 000 000 000 000 015, 100 000 000 000 000 000 016, 100 000 000 000 000 000 017,
 100 000 000 000 000 000 018, 100 000 000 000 000 000 019, 100 000 000 000 000 000 020, 100 000 000 000 000 000 021,
 100 000 000 000 000 000 022, 100 000 000 000 000 000 023, 100 000 000 000 000 000 024, 100 000 000 000 000 000 025,
 100 000 000 000 000 000 026, 100 000 000 000 000 000 027, 100 000 000 000 000 000 028, 100 000 000 000 000 000 029,
 100 000 000 000 000 000 030, 100 000 000 000 000 000 031, 100 000 000 000 000 000 032, 100 000 000 000 000 000 033,
 100 000 000 000 000 000 034, 100 000 000 000 000 000 035, 100 000 000 000 000 000 036, 100 000 000 000 000 000 037,
 100 000 000 000 000 000 038, 100 000 000 000 000 000 039, 100 000 000 000 000 000 040,
 100 000 000 000 000 000 041, 100 000 000 000 000 000 042, 100 000 000 000 000 000 043, 100 000 000 000 000 000 044,
 100 000 000 000 000 000 045, 100 000 000 000 000 000 046, 100 000 000 000 000 000 047, 100 000 000 000 000 000 048,
 100 000 000 000 000 000 049, 100 000 000 000 000 000 050, 100 000 000 000 000 000 051, 100 000 000 000 000 000 052,
 100 000 000 000 000 000 053, 100 000 000 000 000 000 054, 100 000 000 000 000 000 055, 100 000 000 000 000 000 056,
 100 000 000 000 000 000 057, 100 000 000 000 000 000 058, 100 000 000 000 000 000 059, 100 000 000 000 000 000 060,
 100 000 000 000 000 000 061, 100 000 000 000 000 000 062, 100 000 000 000 000 000 063, 100 000 000 000 000 000 064,
 100 000 000 000 000 000 065, 100 000 000 000 000 000 066, 100 000 000 000 000 000 067, 100 000 000 000 000 000 068,
 100 000 000 000 000 000 069, 100 000 000 000 000 000 070, 100 000 000 000 000 000 071, 100 000 000 000 000 000 072,
 100 000 000 000 000 000 073, 100 000 000 000 000 000 074, 100 000 000 000 000 000 075, 100 000 000 000 000 000 076,
 100 000 000 000 000 000 077, 100 000 000 000 000 000 078, 100 000 000 000 000 000 079, 100 000 000 000 000 000 080,
 100 000 000 000 000 000 081, 100 000 000 000 000 000 082, 100 000 000 000 000 000 083, 100 000 000 000 000 000 084,
 100 000 000 000 000 000 085, 100 000 000 000 000 000 086, 100 000 000 000 000 000 087, 100 000 000 000 000 000 088,
 100 000 000 000 000 000 089, 100 000 000 000 000 000 090, 100 000 000 000 000 000 091, 100 000 000 000 000 000 092,
 100 000 000 000 000 000 093, 100 000 000 000 000 000 094, 100 000 000 000 000 000 095, 100 000 000 000 000 000 096,
 100 000 000 000 000 000 097, 100 000 000 000 000 000 098, 100 000 000 000 000 000 099, 100 000 000 000 000 000 100}
```

Mathematica offers many ways to generate prime numbers, e.g. for the region between 10^9 and $10^9 + 100$:

```
Reduce[10^9<x<10^9+100,x,Primes]
```

3.1 QUICK START: WHAT DO WE KNOW FOR CERTAIN?

Below, the reader will find a shortened description of the most important theorems about prime numbers and arithmetical functions related to them that are proven (as of Nov. 2016):

1. There are infinitely many prime numbers.
2. Each integer that is composite (i.e. not a prime number) can be unambiguously represented as a product of at least two prime numbers.
3. The number of primes $\pi(n)$ denotes the number of primes that exist up to a limit n. For $\pi(n)$ there exist many (more or less precise) estimates that make it possible to compute $\pi(n)$ approximately. There are also exact formulae for $\pi\ (n)$ (see 8.6).
4. For computing the nth prime number, formulae also exist for an approximate calculation, however, also exact formulae (see ‘Formulae for calculating the *nth* prime number‘).
5. The ‘gaps’ between adjacent prime numbers can be of any size. The largest gap currently known includes an area of $3.311.852$ consecutive composite numbers (as of Oct. 2015).
6. The sum of the reciprocals of all prime numbers diverges (goes towards infinity).
7. The largest currently known prime number is: $2^{\mathbf{82589933}} - 1$. It has 24862047 digits if written in the decimal system (as of Dec. 2020).
8. There exists no arithmetic sequence of integer numbers that delivers only prime numbers, such as for example Euler’s formula $n^2 + n + 41$, which generates only prime numbers for $0 \leq n < 40$ but not for $n = 40$! However it remains true that many arithmetic sequences create (among others) infinitely many prime numbers.
9. Currently there are 51 known Mersenne prime numbers. The first Mersenne prime exponents are:
   $2, 3, 5, 7, 13, 17, 19, 31$ (sequence A000043 in OEIS) (as of Dec. 2020).
10. If M_p is a prime, then p is also a prime.
11. Currently there are 5 known Fermat primes $F_n = 2^{2^n} + 1$ $(\mathrm{n} = 0\ \ldots 4)$ These are:
    $3, 5, 17, 257, 65537$ (sequence A000215 in OEIS) (as of Nov. 2016).
    F_5 to F_{32} are composite numbers. F_{33} is the first Fermat number of which it is not known whether it is composite or prime (as of Nov. 2016).
12. Each even perfect number N (i.e. the sum of its positive divisors without N gives N) has the form $2^{n-1}(2^n - 1)$in which $2^n - 1$is prime, i.e. to each Mersenne prime number belongs a perfect number!!
13. If $\phi(n) + \sigma(n) = 2\,n, n \geq 2$, then n is a prime number, in which $\phi(n)$ is Euler’s totient function and $\sigma(n)$ the ‘sum-of-divisors function’.
14. If $\binom{n-1}{k} \equiv (-1)^k \pmod n$, then n is a prime number, of which $\binom{n}{k}$ are the binomial coefficients.
15. For each prime number p, the following relations to the σ function obtain:
    $\sigma_0(p) = 2$ (Each prime number has only two divisors: itself and 1)
    $\sigma_0(p^n) = n + 1$
    $\sigma_1(p) = p + 1$

3.2 QUICK START: WHAT ARE OUR (UNPROVEN) CONJECTURES?

Here are (in shortened form) the most important statements and conjectures about prime numbers and about the closely related zeta function that are probably true but still unproved (as of Nov. 2016):

1. Each even natural number $n > 2$ can be represented as the sum of two prime numbers (strong Goldbach conjecture). This conjecture has been numerically verified up to $n < 4 \cdot 10^{18}$ (as of Apr. 2012).
2. Each odd natural number > 5 can be represented as the sum of three prime numbers (weak Goldbach conjecture). This has been proved for $n > 10^{43000}$!
3. Between n^2 and $(n + 1)^2$ there exists at least 1 prime number (Oppermann's conjecture, 1882).
4. The 'non-trivial' zeros of the zeta function are all located in the Gaussian complex plane on a straight line having a real part of 0.5. This is the famous Riemann conjecture, which Riemann formulated in the year 1859, and which remains unproved to this day (as of Nov. 2016). It ranks among the 'Top Seven unsolved mathematical problems'. A reward of one million US dollars has been offered for its solution. The conjecture has been numerically verified up to the first 10^{13}. Thus there is overwhelming numerical evidence for the truth of Riemann's conjecture.
5. There are infinitely many Mersenne prime numbers (numbers of the form $M_p = 2^p - 1$).
6. There are infinitely many composite Mersenne numbers.
7. There are only five Fermat prime numbers.
8. There are no odd perfect numbers (see above).
9. The 'new Mersenne conjecture':
   if any two of the following conditions hold, then the third condition is also true:
   - $n = 2^k \pm 1$ or $n = 4^k \pm 3$
   - $2^n - 1$ is prime (obviously a Mersenne prime)
   - $\frac{(2^n+1)}{3}$ is prime
10. There are infinitely many twin prime numbers. Twin primes are prime numbers having a difference of 2. It is known that the sum of the reciprocals of the twin primes converges (Brun's constant: 1.902160577783278, proved by Brun in 1919).
11. The number N_{M_p} of Mersenne prime numbers that are smaller than or equal to N is given asymptotically by the formula: $N_{M_p}(N) \sim \frac{e^{\gamma}}{\ln(2)} \ln\ln(N)$.
12. The final digits of consecutive prime numbers show striking correlations.

3.3 QUICK START: WHAT IS STILL UNSOLVED?

Here are (in a shortened form) the most important unsolved questions about prime numbers and related topics, of which we have no idea whether they are wrong or right:

1. Are all Mersenne numbers $M_p = 2^p - 1$ square-free, i.e. does their prime factor decomposition contain each factor only once?
2. Are there infinitely many prime number N-tuplets? (These are tuplets of n consecutive prime numbers having minimal differences, as defined in Chapter 4.3).
3. Are there infinitely many ‘Wagstaff’ prime numbers, i.e. prime numbers of the form $\frac{(2^p+1)}{3}$ (with p being an odd prime number)?
4. Are there infinitely many ‘Sophie Germain’ prime numbers, i.e. prime numbers of the form $2p + 1$ (with $2p + 1$ as a ‘safe prime’ and p as the ‘Sophie Germain’ prime)?
5. Are there infinitely many ‘Fibonacci’ primes, i.e. prime numbers occurring in the Fibonacci sequence?
6. Does the ‘Euclid-Mullin sequence’ contain all prime numbers?
7. Does there exist an efficient factorizing method for the prime factor decomposition of large numbers, i.e. a procedure that accomplishes the factorization process in ‘polynomial time’? Because no such a method is currently known, it is still impossible to factorize large numbers (as the computing time required would be astronomically high). Currently the fastest known methods of factorization are the ‘number field sieve (Pomerance et. al.) and the method using elliptic curves (as of Nov. 2016).

3.4 QUICK START: WHAT IS NEW?

1) A new property of the Fibonacci numbers (see 4.10).
2) Properties of the Reed Jameson sequence and its relation to prime numbers (see 4.10.1).
3) RG number sequences (recursive 'Gödelized') sequences (see 4.17).
4) 'Fun and games' with the product representation of the $\zeta(s)$ in the complex domain (see 5.3).
5) $\mathfrak{Z}(s)$: a 'function' having minima that are located at the prime positions (see 5.3).
6) The Reed Jameson function: zeros at the prime number positions (see 8.5.1).
7) Prime numbers and surfaces of 4-dimensional hyperspheres (glomes) (see 9.3).
8) Of OCRONs and GOCRONs (see Chapter 10).
9) Is it possible to find (typographic) prime number rules using the Matrix software? (Chapter 11).
10) An equation for a plane as a link between GOCRONs and the *abc* conjecture (see 12.1).
11) Prime numbers as rhythmic patterns (Chapter 15.2).
12) Differences and quotients of aliquot sequences (Chapter 20.9.2.5).

4 SPECIAL KINDS OF PRIME NUMBERS

4.1 TWIN PRIMES

Twin primes are prime numbers having a difference of 2. The following applies: n and $n+2$ are twin primes if and only if the following equation obtains:

$$\mathbf{4[(n-1)!+1]+n \equiv 0\ [mod\ n(n+2)]} \tag{1}$$

$$\boldsymbol{\phi(n)\sigma(n)} = \boldsymbol{(n-3)(n+1)}, \textbf{where } \boldsymbol{n} = \boldsymbol{p(p+2)} \text{ (product of a twin prime pair)} \tag{2}$$

$$\boldsymbol{(n, n+2)} \text{ are twin primes if}$$
$$\sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+2}{i} \right\rfloor + \left\lfloor \frac{n}{i} \right\rfloor\right) = 2 + n^a + \sum_{i=1}^{n} i^a \left(\left\lfloor \frac{n+1}{i} \right\rfloor + \left\lfloor \frac{n-1}{i} \right\rfloor\right) \tag{3}$$
where $a \geq 0$ and $\lfloor\ \rfloor$ is the floor() function.

Unfortunately these formulae are not practicable for the computation of twin prime numbers.

Let $\pi_2(x)$ be the number of twin primes up to a given limit x.
Since the 19th century the following estimate has been accepted:

$$\pi_2(x) \leq c\Pi_2 \frac{x}{(\ln\ x)^2} \tag{4}$$

Hardy and Littlewood have conjectured that c = 2 and

$$\boldsymbol{\pi_2(x) \sim 2\Pi_2 \int_2^x \frac{dt}{(\ln\ t)^2} = 2\Pi_2\left(\mathrm{Li}(x) - \frac{x}{\ln(x)} - Li(2) + \frac{2}{\ln\ (2)}\right)} \tag{5}$$

using the twin prime constant:

$$\boldsymbol{\Pi_2} = \prod_{p \geq 3} \frac{\boldsymbol{p(p-2)}}{\boldsymbol{(p-1)^2}} = 0.6601618158$$

$$\boldsymbol{2\Pi_2 = 1.3203236316}$$

The sum of the reciprocals of all twin primes converges (Brun's constant, proved by Brun in 1919).

$$B = \sum_{p=twin} \left(\frac{1}{p} + \frac{1}{p+2}\right) = 1.90216054 \tag{6}$$

Table 1. Number of twin primes and values of the Hardy-Littlewood function

n		$\pi_2(10^n)$	Hardy-Littlewood
1		2	4.84
2		8	13.54
3		35	45.80
4		205	214.21
5		1224	1248.71
6		8169	8248.03
7		58980	58753.82
8		440312	440367.79
9		3424506	3425308.16
10		27412679	27411416.53
11		224376048	224368864.67
12		1870585220	1870559866.69
13		15834664872	15834598303.94
14		135780321665	135780264884.86
15		1177209242304	1177208491777.05
16		10304195697298	10304192553765.33
17		**9094883**9353159	**9094883**3254536.36
18		**8086758**88577436	**8086759**01436127.88

For $n = 10^{18}$ this approximation given by Hardy-Littlewood is exact up to an error of

$1.59 \cdot 10^{-8}$: $$\frac{\pi_2(10^{18})}{\pi_{2_{approx}}(10^{18})} = 0.999999984$$

This matching of the approximations with the exact values for large n is remarkable and could be interpreted as a 'numerical proof' of the infinite number of twin primes (Chapter 4.1).

Mathematica program for creating the table:

```
ile = 2; Do[Do[If[(PrimeQ[2 n - 1]) && (PrimeQ[2 n + 1]), ile = ile +
1], {n, 5*10^m, 5*10^(m + 1)}]; Print[{m, ile}], {m, 0, 7}]
```

Here is a comparison of the exact values with the Hardy-Littlewood formula for the first 3500 twin primes (blue: exact, yellow: Hardy-Littlewood) :

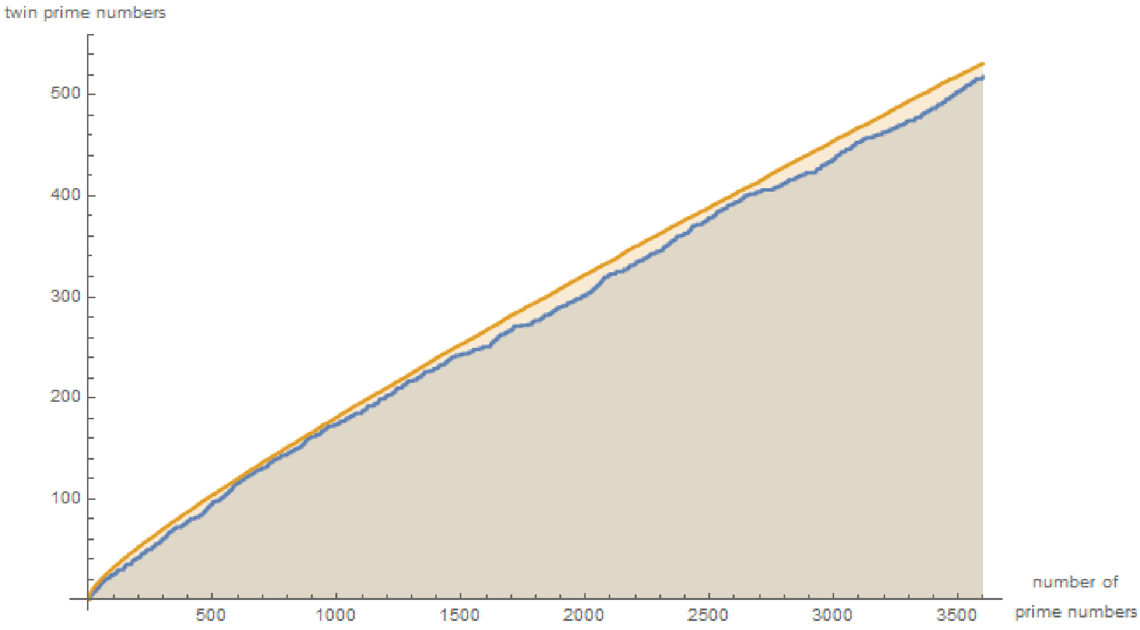


Figure 1. Number of twin primes from 2 to 3500

The Mathematica program to create the plot can be found in the Appendix[3]. The following Mathematica program can be used to compute twin primes (e.g. up to 10000):

```
Select[Range[10000],(PrimeQ[#]&&PrimeQ[#+2])&]
```

The largest known twin prime pair is $\mathbf{3756801695685 \cdot 2^{666669} \pm 1}$
(in decimal form: $\mathbf{8.72966507873032809131455726874 \times 10^{200699}}$)
(as of Oct. 2015).

Polignac's conjecture:

This conjecture says that for every even number n, there exist infinitely many pairs of prime numbers with difference n. For $n = 2$ we get the special case of the twin primes. The Hardy-Littlewood conjecture may be generalized also for this case:

$$\boldsymbol{\pi'_n(x) \sim 2C'_n \int_2^x \frac{dt}{(\ln t)^2} = 2C'_n \left(\mathrm{Li}(x) - \frac{x}{\ln(x)} - Li(2) + \frac{2}{\ln(2)}\right)} \tag{7}$$

where

$$C'_n = \Pi_2 \sum_{q|n} \frac{q-1}{q-2} \tag{8}$$

[3] Mathematica programs: comparison of the number of twin, cousin and sexy primes with the Hardy-Littlewood formula

Special cases:

$n = 4$: **Cousin primes:** here we have $C'_4 = C'_2 = C_2$ primes (with difference 4) and twin primes have the same asymptotic density. There exist the same number of instances of both kinds!!

$n = 6$: **Sexy primes:** here we have $C'_6 = 2C'_2$ primes (with difference 6) having an asymptotic density twice as high as twin primes. There exist twice as many sexy primes as twin primes!!

4.2 PRIME TRIPLETS AND QUADRUPLETS

For prime triplets and prime quadruplets there also exist approximations (Hardy-Littlewood conjecture) for the number of triplets and quadruplets up to a given limit x:

Triplets

$$\boldsymbol{\pi_3(x) \leq \frac{9}{2}\prod_{p\geq 5}\frac{p^2(p-3)}{(p-1)^3}\int_2^x \frac{dt}{(\ln t)^3} = 2.858248596\int_2^x \frac{dt}{(\ln t)^3}} \tag{9}$$

In expanded form:

$$\boldsymbol{\pi_3(x) \sim 2.858248596\left(\frac{1}{2}\mathrm{Li}(x) - \frac{x}{2\ln^2(x)} - \frac{x}{2\ln(x)} + \frac{1}{\ln(2)} + \frac{1}{\ln^2(2)} - \frac{1}{2}\mathrm{Li}(2)\right)} \tag{10}$$

or

$$\boldsymbol{\pi_3(x) \sim 2.858248596\left[\ln^{-2}(x)\left(-\mathrm{E}_3(-\ln(x))\right) - \ln^{-2}(2)\left(-\mathrm{E}_3(-\ln(2))\right)\right]} \tag{11}$$

The largest currently known prime triplet is

$\mathbf{6521953289619 \cdot 2^{55555} + d, d = -5, -1, 1}$ (having 16737 decimals)

Table 2. Number of prime triplets and values of the Hardy-Littlewood function[4]

n	$\pi_3(10^n)$	Hardy-Littlewood	H-L / $\pi_3(10^n)$
1	1	8.49	8.490
2	4	13.86	3.465
3	15	25.57	1.70467
4	55	69.34	1.26073
5	259	279.36	1.07861
6	1393	1446.17	1.03817
7	8543	8591.23	1.00565
8	55600	55490.86	0.99804
9	379508	379802.73	1.00078
10	2713347	2715291.84	1.00072
11	20093124	20089653.88	0.99983
12	152850135	152830566.82	0.99997
13	1189795268	1189763105.37	0.999999
14	9443899421	9443890414.16	0.999999
15	76218094021	76217780005.59	0.999996
16	624026299748	624025187564.06	0.999998

The matching of the approximations with the exact values for large n is remarkable.

Mathematica program for creating the approximate values:

```
ch=2.858248596; (*Pi3!*)
n=3;
Do[Print[N[Re[SetPrecision[ch,50]*((Log[10^i])^(1-n)(-ExpIntegralE[n,-
Log[10^i]])-(Log[2])^(1-n)(-ExpIntegralE[n,-Log[2]]))],{Infinity,3}
]],{i,1,16}]
```

Quadruplets

$$\pi_4(x) \le \frac{27}{2}\prod_{p\ge 5}\frac{p^3(p-4)}{(p-1)^4}\int_2^x \frac{dt}{(\ln t)^4} = 4.151180864\int_2^x \frac{dt}{(\ln t)^4} \qquad (12)$$

or

$$\pi_4(x) \sim 4.151180864\left[\ln^{-3}(x)\left(-E_4(-\ln(x))\right) - \ln^{-3}(2)\left(-E_4(-\ln(2))\right)\right] \qquad (13)$$

[4] Source: Thomas R.Nicely, http://www.trnicely.net/quads/t3a_0000.htm

Table 3. Number of prime quadruplets and values of the Hardy-Littlewood function[5]

n	$\pi_4(10^n)$	Hardy-Littlewood	H-L / $\pi_4(10^n)$
1	1	11.29	11.29
2	2	13.60	6.80
3	5	16.49	3.30
4	12	24.17	2.01
5	38	52.88	1.39
6	166	183.68	1.1065
7	899	862.95	0.9599
8	4768	4734.64	0.99300
9	28388	28396.84	1.00031
10	180529	181074.93	1.00302
11	1209318	1209956.22	1.00053
12	8398278	8394578.03	0.99956
13	60070590	60075438.37	1.00008
14	441296836	441290732.40	0.999986
15	3314576487	3314550290.38	0.999992
16	25379433651	25379441340.00	1.0000000

Here, too, the matching of the approximations with the exact values for large n is remarkable.

```
Mathematica program for creating the approximate values:
ch=4.151180864; (*Pi4!*)
n=4;
Do[Print[N[Re[SetPrecision[ch,50]*((Log[10^i])^(1-n)(-ExpIntegralE[n,-
Log[10^i]])-(Log[2])^(1-n)(-ExpIntegralE[n,-Log[2]]))],{Infinity,3}
]],{i,1,16}]
```

The largest prime quadruplet currently known is (Source: Thomas Forbes[6])

$$\begin{aligned} &\mathbf{2673092556681 \cdot 15^{3048} + d, d = -4, -2, 2, 4} \\ &\qquad = \mathbf{1.42289088832921708944844369162 \cdot 10^{3597}} \end{aligned}$$

(as of Oct. 2015).

4.3 PRIME N-TUPLETS

A prime n-tuplet is generally defined as a sequence of consecutive primes $(p_1, p_2, p_3, \dots\ p_n)$ with a fixed minimal value for the difference between the smallest and the largest prime $s(n) = p_n - p_1$ (see table below). For example, $s(4) = 8$ for quadruplets or $s(5) = 12$ for quintuplets. Generally, there exist more solutions for the

[5] Source: Thomas R.Nicely, http://www.trnicely.net/quads/t4_0000.htm
[6] http://anthony.d.forbes.googlepages.com/ktuplets.htm

corresponding sequence for a given prime n-tuplet with a fixed $s(n)$. For example, prime triplets can have two different forms: $(p, p+2, p+6)$ and $(p, p+4, p+6)$. This degeneration grows quite fast with the length n of the n-tuplets. So, for $n=13$, the degeneration is already 6; for $n=25$, we have a degeneration of 18 distinct ordering possibilities for a prime number 25-tuplet where s(25) = 110.

In order to avoid this ambiguity or degeneration, we use here another definition of the term 'prime n-tuplet'. We construct a sequence of primes assuming that it is located in an arbitrarily high number region, having a maximal density of prime numbers by using the following principle of construction (this method bears a certain similarity to the 'Sieve of Eratosthenes'):

1) **We begin with a new list, assuming that the first element of this list is any arbitrarily large prime number p (obviously an odd number). We mark this first list element with 'p'. All other list elements are still unoccupied ('free').**
2) **We set $n=1$ (thus $p_n=2$, is the first prime number)**
3) **As long as in the range between p and $p+p_n-1$ (between the first and the p_n-th element) there still exist more than one list element which could be divisible by p_n (i.e. all elements following with difference $i \cdot p_n$ do not 'collide' with a 'p-marked' element) we reduce this ambiguity more and more by marking the next free (not yet marked with a divisor number or a 'p') position with a 'p'.**
4) **Now, between p and $p+p_n-1$ (between the first and the p_nth) element there exists only one list element $p+j$, which is divisible by p_n. We sieve (i.e. mark with the value of p_n) all following numbers (list elements) $p+j+i \cdot p_n, i=0,1,\ldots\infty$**
5) **We set the next possible prime number at the next free list position and mark this element with a 'p'.**
6) **We increase our counter $n=n+1$ and continue with instruction 3).**

Thus we get a sequence of (possible) prime numbers (with list positions marked by 'p'), which represent the maximal possible density of prime numbers (independent of the number region in which we have started):

$$\boldsymbol{p, p+2, p+6, p+8, p+12, p+18, p+20, p+26, p+30, p+32, p+36, p+42, p+48, p+50, p+56, p+62, p+68, p+72, p+78, p+86, p+90}$$

The prime number tuplets created by this principle of construction differ from the table of prime n-tuplets T. Forbes used[7]. The numbers have the meaning of indices $\boldsymbol{i}$ for $(\boldsymbol{p+i})$:

6-tuplet: (0-2-6-8-12-18)

Forbes: (0-4-6-10-12-16)

16-tuplet: (0-2-6-8-12-18-20-26-30-32-36-42-48-50-56-62)

[7] http://anthony.d.forbes.googlepages.com/ktuplets.htm

Forbes: (0-4-6-10-16-18-24-28-30-34-40-46-48-54-58-60)

or (0-2-6-12-14-20-26-30-32-36-42-44-50-54-56-60)

From the 16-tuplets on, differences become more and more frequent.

It is interesting that this principle of construction, which creates a maximal prime number density for arbitrarily high number regions, results in the same prime number sequence as the prime number sequence from the number 11. At least, at first glance… If we take a closer look, we notice that the prime number **71** is missing (it should be at position 60 in our list)! **The only possible and plausible interpretation is that, for sufficiently large number regions, there cannot exist any prime 16-tuplets of the form (0-2-6-8-12-18-20-26-30-32-36-42-48-50-56-60) (as represented by the 'natural' prime sequence running from 11 to 71). Above 100, more deviations occur from the 'natural' prime sequence.**

It is also obvious that if we continue this principle of prime construction further and further, the resulting prime number n-tuplets will be more and more spaced out (compared with the natural prime sequence starting from the number 11). This is, of course, reasonable: nobody would expect that all the prime 'constellations' of small numbers would also appear infinitely often in arbitrarily high regions!

.

Conclusion: not only prime constellations within the first 10 natural numbers are unique. For numbers larger than 11 there are also prime constellations that appear only one time (that are unique).

Note: The resulting sequence of possible prime positions having a maximal density reads:

{1,3,7,9,13,19,21,27,31,33,37,43,49,51,57,63,69,73, ... } and it is well-known (see A020498 at https://oeis.org).

Let us take a look at the sequences of the differences, obtained after sieving up to prime no. n. These sequences can be easily obtained by using the following formula:

```
RotateRight[Differences[Select[Range[primorial[n]+1],GCD[#,
primorial[n]]==1&]],nRotation]
```

The sequences have cycles of increasing lengths. The parameter 'nRotation' can be taken from the following table, `primorial[n]` is the product of the first n primes.

The cycle lengths can also be easily calculated by the formula:

```
a(0)=1;for n>0,a(n)=(prime(n)-1)*a(n-1)
```

```
Mathematica:
```

```
RecurrenceTable[{a[0]==1,a[n]==(Prime[n]-1)a[n-1]},a,{n,10}]
```

The patterns resulting from the differences of the positions (generated by sieving) repeat after the following cycles:

n	Sieving up to prime no. n	Length of cycle	Parameter for RotateRight[]	Sequences (start and end)
1	2	1	0	2
2	3	2	1	2,4
3	5	8	6	2,4,2,4,6,2,6,4
4	7	48	47	2,4,2,4,6,2,6,4,2,4,6,6,2,6,4,…,10,2,10
5	11	480	218	2,4,2,4,6,2,6,4,2,4,6,6,2,6,6,…,10,2,10
6	13	5760	2861	2,4,2,4,6,2,6,4,2,4,6,6,2,6,6,…,10,2,10
7	17	92160	2695	(as above)
8	19	1658880	???	(as above)

Figure 1: Lengths of cycles by the sieve method for generating max. prime number density

The sieving process becomes more clear if we take a look at the following table (see appendix for the Mathematica program used). The yellow cells mark the possible positions of primes that can occur in any arbitrarily high number regions. The first line shows the result with existing primes if we start from the number 11. The first deviation occurs at the number 71. More deviations follow if we go to higher regions.

11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
p	2	p	2		2	p	2	p	2		2	p	2		2		2	p	2	p	2		2		2	p	2		2	p	2	p	2		2
p	3	p		3		p	3	p		3		p	3			3		p	3	p		3			3	p		3		p	3	p		3	
p		p		5		p		p	5			p		5				p	5	p				5		p			5	p		p		5	
p		p	7			p		p		7		p					7	p		p				7		p				p	7	p			
p		p			11	p		p				p				11		p		p						p	11			p		p			
p	13	p				p		p				p		13				p		p						p	13			p		p			
p		p				p	17	p				p						p		p				17		p				p		p			
p		p				p		p				p				19		p		p						p				p		p			19

47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
p	2		2		2	p	2		2		2	p	2	p	2		2		2	p	2		2		2	p	2		2		2	p	2		2
p	3			3		p	3			3		p	3	p		3			3	p		3			3	p		3			3	p		3	
p			5			p		5				p	5	p				5		p			5			p		5				p	5		
p		7				p			7			p		p		7				p			7			p				7		p			
p		11				p						p	11	p						p				11		p						p			11
p				13		p						p		p			13			p						p				13		p			
p					17	p						p		p						p		17				p						p			
p						p						p		p				19		p						p						p			

83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113
73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103
p	2		2		2	p	2		2		2		2	p	2		2	p	2		2		2	p	2	p	2		2	p
p	3			3		p	3			3			3	p		3		p	3			3		p	3	p		3		p
p		5				p	5					5		p			5	p				5		p		p	5			p
p	7					p		7						p	7			p				7		p		p			7	p
p						p				11				p				p			11			p		p				p
p						p	13							p				p		13				p		p				p
p			17			p								p				p		17				p		p				p
p	19					p								p				p		19				p		p				p

Figure 2: Sieving method for generating a maximal prime density

The web site of T. Forbes is a true treasure for this topic. The following formulae have been taken in large part from his web site.

We generalize the estimate from Hardy-Littlewood for n (prime n-tuplets). The result is:

$$\boldsymbol{\pi_n(x) \sim C_n \left[\ln^{1-n}(x)\left(-\mathrm{E}_n(-\ln(x))\right) - \ln^{1-n}(2)\left(-\mathrm{E}_n(-\ln(2))\right)\right]} \tag{14}$$

with the constants $\boldsymbol{C_n}$. Here $\mathbf{E}_{\boldsymbol{n}}()$ is the integral exponential function of order n.
The constants $\boldsymbol{C_n}$ can be computed as follows:

$$\boldsymbol{C_n = H_n \cdot K_n}$$

where

$$\boldsymbol{K_n = \prod_{p \geq n+1} \frac{p^{n-1}(p-n)}{(p-1)^n}} \tag{15}$$

Finally, here is a formula for the $\boldsymbol{C_n}$ that converges much faster:

$$\boldsymbol{\ln(C_k) = \sum_{n=2}^{\infty} \ln\left[\zeta(n) \prod_{p\ prime, p \leq k} \left(1 - \frac{1}{p^n}\right)\right] / n \cdot \sum_{d|n} \mu\left(\frac{n}{d}\right)(k^d - k)} \tag{16}$$

Table 4. The Hardy-Littlewood constants

Name	Differences, $p_{max} - p_{min}$	H_n	K_n	$\boldsymbol{C_n}$
Twins	2 (2)	2	0.66016182	1.3203236
Triplets	2-4 (6)	$\frac{9}{2}$	0.63516635	2.8582486
Quadruplets	2-4-2 (8)	$\frac{27}{2}$	0.30749488	4.1511809
5-tuplets	2-4-2-4 (12)	$\frac{15^4}{2^{11}}$	0.40987489	10.131795
6-tuplets (*)	4-2-4-2-4 (16)	$\frac{15^5}{2^{13}}$	0.18661430	17.298612
6-tuplets	2-4-2-4-6 (18)	$\frac{?}{?}$	?	?
7-tuplets	2-4-2-4-6-2 (20)	$\frac{35^6}{3 \cdot 2^{22}}$	0.36943751	53.971948
8-tuplets	2-4-2-4-6-2-6 (26)	$\frac{5^6 \cdot 7^7}{2^{24}}$	0.23241933	178.26195
9-tuplets	2-4-2-4-6-2-6-4 (30)	$\frac{5^9 \cdot 7^8}{2^{31}}$	0.12017121	630.06436
10-tuplets	2-4-2-4-6-2-6-4-2 (32)	$\frac{5^{10} \cdot 7^9}{9 \cdot 2^{30}}$	0.041804051	1704.7409
11-tuplets	2-4-2-4-6-2-6-4-2-4 (36)	$\frac{7^{11} \cdot 11^{10}}{45 \cdot 2^{45}}$	0.094530829	3062.0793
12-tuplets	2-4-2-4-6-2-6-4-2-4-6 (42)	$\frac{7^{12} \cdot 11^{11}}{25 \cdot 2^{49}}$	0.035393260	9931.3156

Table 5. Number of prime quintuplets and values of the Hardy-Littlewood function

$\boldsymbol{n}$	$\boldsymbol{\pi_5(10^n)}$	Hardy-Littlewood	**H-L** / $\boldsymbol{\pi_5(10^n)}$
1	**1**	27.93	-
2	2	14.84	7.42
3	3	30.90	10.30
4	4	33.17	**8.29**
5	10	39.79	**3.98**
6	34	64.49	**1.90**
7	160	173.20	**1.0825**
8	697	711.00	**1.02009**
9	3633	3615.11	**0.99508**
9.59868	**10000**	**10094.08**	**1.00941**
10	**20203**	20401.37	**1.00982**
11		122857.37	**?**
12		776698.49	**?**
13		5.10724390*10^6	**?**
14		3.4706125667*10^7	**?**

15		2.42544985095*10^8	?
16		1.73651359676*10^9	

Mathematica program for generating the approximations:

```
ch=10.131795; (*Pi5!*)
n=5;
Do[Print[N[Re[SetPrecision[ch,50]*((Log[10^i])^(1-n)(-ExpIntegralE[n,-Log[10^i]])-(Log[2])^(1-n)(-ExpIntegralE[n,-Log[2]]))],{Infinity,3}]],{i,1,16}]
```

(Values in blue have been calculated analytically using the Hardy-Littlewood formula and are not exact!…)

Table 6. Number of prime n-tuplets dependent upon n

n	$\pi_n(10^3)$	$\pi_n(10^4)$	$\pi_n(10^5)$	$\pi_n(10^6)$	$\pi_n(10^7)$	$\pi_n(10^8)$	$\pi_n(10^9)$
1	168	1229	9592	78498	664579	5761455	50847534
2	35	205	1224	8169	58980	440312	3424506
3	15	55	259	**1393**	**8543**	**55600**	**379508**
4	5	12	38	166	899	4768	28388
5	**3**	4	**10**	**34**	**160**	697	3633
6	**2**	3	**3**	**6**	**16**	**63**	
7							
8							
9							
10							
11							
12							

n	$\pi_n(10^{10})$	$\pi_n(10^{11})$	$\pi_n(10^{12})$	$\pi_n(10^{13})$	$\pi_n(10^{14})$
1	455052511	4118054813	37607912018	346065536839	3204941750802
2	27412679	224376048	1870585220	15834664872	135780321665
3	**2713347**	**20093124**	**152850135**	**1189795268**	**9443899421**
4	180529	1209318	8398278	60070590	441296836
5	20203	122857	776698	5107243	34706125
6	?	?	?	?	**?**
7	424	1334	6185	**33412**	**193160**
8	?	?	?	**4601**	**21405**
9					
10					
11					
12					

n	$\pi_n(10^{15})$	$\pi_n(10^{16})$
1	29844570422669	279238341033925
2	1177209242304	10304195697298
3	**76218094021**	**624026299748**
4	3314576487	25379433651
5	242544985	1736513596
6		
7		
8		
9		
10		
11		

4.4 CORRELATIONS OF THE LAST DIGITS IN THE PRIME NUMBER SEQUENCE

In the spring of 2016, some exciting news appeared in the mathematical press: mathematicians had found striking patterns in prime numbers. The statistical frequency of the last digits of consecutive primes showed clearly relevant correlations. As a prime number can only end with one of the four digits 1,3,7,9 (apart from the small primes 2 and 5), one would ordinarily expect that the final digits 1,3,7,9 would occur with equal regularity (because of the 'randomness' of the primes) and in fact, this is the case: an evaluation of the last digits of the first million prime numbers reveals that 1,3, 7 and 9 occur with equal regularity (25% in each case).

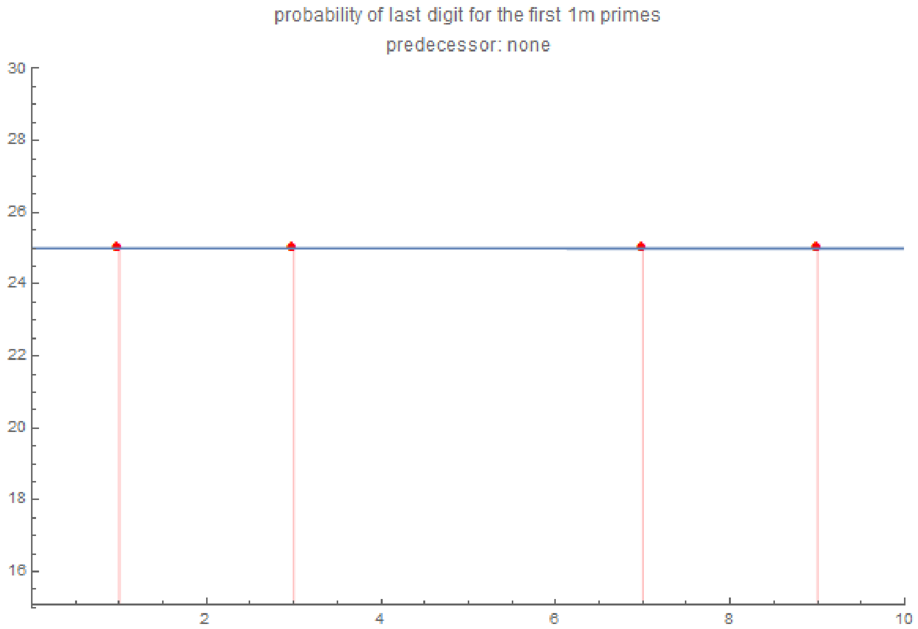


Figure 2. Incidence of the last digits in the prime sequence (without predecessor)

```
Mathematica:
data={{1,24.99},{3,25.01},{7,25.00},{9,24.99}}
line=Fit[data,{1,x},x]
Show[ListPlot[data,PlotStyle->Red,AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic,PlotRange->{{0,10},{15,30}},PlotLabel-
>TextString["probability of last digit for the first 1m
primes\npredecessor: none"],ImageSize->Large],Plot[line,{x,0,10}]]
```

If, however, we examine the statistical properties of possible prime successors for a fixed given, e.g. 1, then we observe that the incidence of the following prime also having a 1 as last digit lies markedly below 25 %. The incidence of the other possible successor digits also show noticeable deviations from the figure of 25 % one would normally expect. In the case of a 1 being the last digit, the incidence of the next prime number also having a 1 as last digit is only 18 %. One could say: prime numbers in the normal ascending sequence do not like to repeat their last digit. In fact, this tendency can be observed for all possible digits. For the first 10m prime numbers, we find the following statistical dependencies of the last digits:

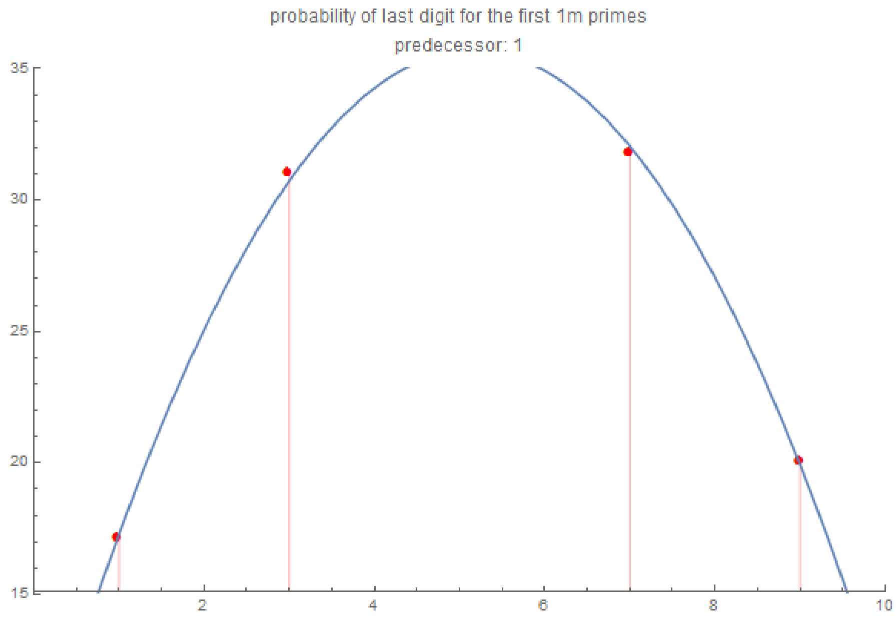


Figure 3. Incidence of the last digits in the prime sequence (predecessor: ‘1’)

```
Mathematica:
data={{1,17.15},{3,31.00},{7,31.79},{9,20.07}}
line=Fit[data,{1,x,x^2},x]
Show[ListPlot[data,PlotStyle->Red,AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic,PlotRange->{{0,10},{15,35}},PlotLabel-
>TextString["probability of last digit for the first 1m
primes\npredecessor: 1"],ImageSize->Large],Plot[line,{x,0,10}]]
```

Here are the results for all four possible last digits:

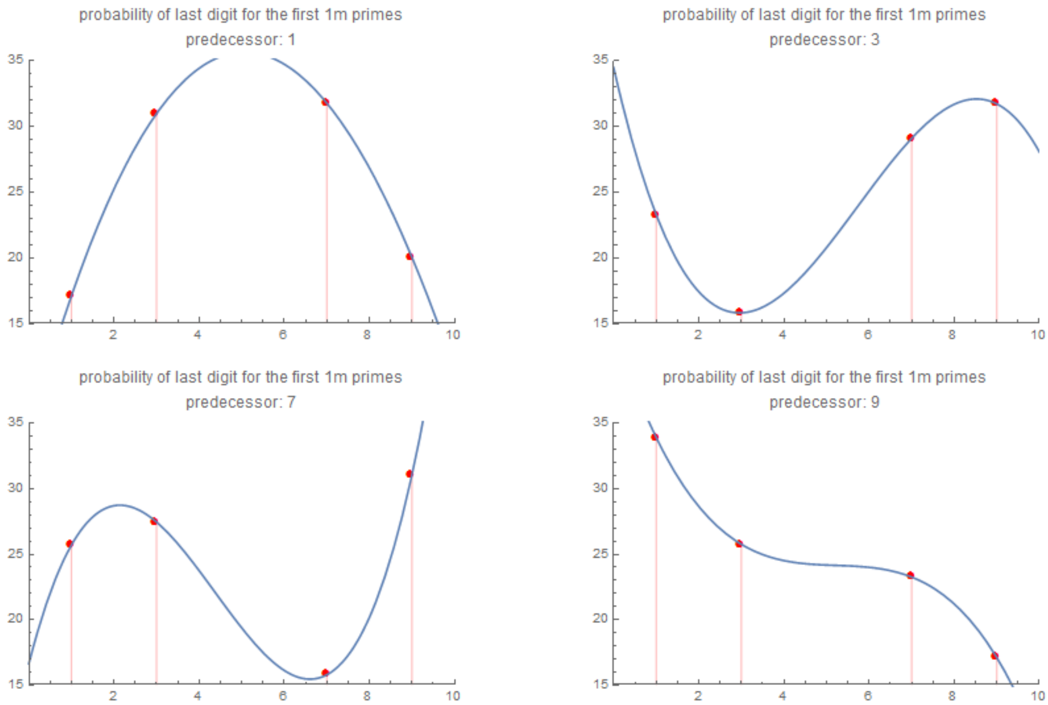


Figure 4. Incidence of the last digits in the prime sequence (all possible predecessors)

`Mathematica:`
(programs see Appendix).

One may wonder what these statistical anomalies look like, if even more preceding primes are included in this exploration. The results, if not only predecessors are included but also pre-predecessors, can be found in the Appendix (Chapter 20.1).

These correlations of the last digits of consecutive primes do not appear exclusively in the decimal system. They appear also in representations of systems having different number bases (e.g. the binary system).

More refined examinations which have been carried out in the meantime have shown that the observed correlations are a direct consequence of the (yet unproven) Hardy-Littlewood formula (see Formula (14) in Chapter 4.3). The observation that these correlations are becoming weaker if we examine prime sequences in very high regions is also a consequence of the Hardy-Littlewood conjecture. Probably the anomalies will steadily disappear if the tests are performed in arbitrarily high number regions. These regions must, however, be very high – probably astronomically high – because the anomalies tend to thin out only very gradually.

The slow pace of this thinning-out-process is actually the only strange thing in this story.

4.5 MERSENNE PRIME NUMBERS

There are a vast number of publications dealing with Mersenne prime numbers. In this book, we will only mention some of the more important and interesting formulae and statements:

Currently 51 Mersenne prime numbers are known (as of Dec. 2020). Many questions about Mersenne primes still remain open (see 3.2 Basics of prime numbers).

Mersenne prime numbers have the form $M_n = 2^p - 1$ with p necessarily being a prime number. However, not every prime number p in this term gives a Mersenne prime M_n. Mersenne primes are very rare, and searching for them is a little bit like searching for gems among the numbers. The largest known prime numbers are all Mersenne primes. That is because for this type of prime there exists a very fast primality test that makes it possible to test even gigantic numbers for primality. The largest currently known prime number is the Mersenne prime number $2^{82589933}$-1. It has 24862048 digits when expressed using the decimal number system (as of Dec. 2020).

The fastest test for Mersenne primes is the Lucas-Lehmer Test[8], which is refined by combination with other methods. A primality test for a number of this order of magnitude needs approx. one month of computing time, if performed on a fast PC with 4 CPU kernels (as of Oct. 2015). The Lucas-Lehmer test and the involved factorizing methods

[8] https://de.wikipedia.org/wiki/Lucas-Lehmer-Test

(P1 test and trial factoring) have been documented and described many times in detail and need not be explained here. [9]

The 51 currently known Mersenne prime exponents are (as of Dec. 2020):

$\mathbf{2, 3, 5, 7, 13, 17, 19, 31, 61, 89,}$
$\mathbf{107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423,}$
$\mathbf{9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243,}$
$\mathbf{110503, 132049, 216091, 756839, 859433, 1257787, 1398269,}$
$\mathbf{2976221, 3021377, 6972593, 13466917, 20996011, 24036583,}$
$\mathbf{25964951, 30402457, 32582657, 37156667, 42643801, 43112609}$
$\mathbf{57885161, 74207281, 77232917, 82589933}$

```
Mathematica program for creating Mersenne prime numbers:

Flatten[Position[EulerPhi[2^#-]+2==EulerPhi[2^#]&/@Range[1,100],True]-1]
```

The range of the first 48 Mersenne prime numbers has been exhaustively tested. The indices of the Four last numbers (49 to 51) are still uncertain, i.e. it may be possible that in this region more Mersenne primes could be discovered.
(sequence A000043 in OEIS) (as of Dec. 2020)

Unresolved questions about Mersenne prime numbers

Are there infinitely many Mersenne prime numbers? Everything indicates that the answer is ‘yes’.
Is the ‘new Mersenne conjecture’ true ‘?

This states that, if any two of the following conditions hold, then the third condition is also true:

1) $n = 2^k \pm 1$ or $n = 4^k \pm 3$
2) $2^n - 1$ is a prime (obviously a Mersenne prime)
3) $\frac{(2^n+1)}{3}$ is a prime

Are there infinitely many composite Mersenne numbers? Probably: yes.
The number N_{M_p} of Mersenne prime numbers that are less than or equal to N is asymptotically:

$$\boldsymbol{N_{M_p}(N) \sim \frac{e^{\gamma}}{\ln(2)} \ln\ln(N)} \tag{17}$$

Graph:[10]

[9] http://www.mersenne.org/various/math.php
[10] http://primes.utm.edu/notes/faq/NextMersenne.html

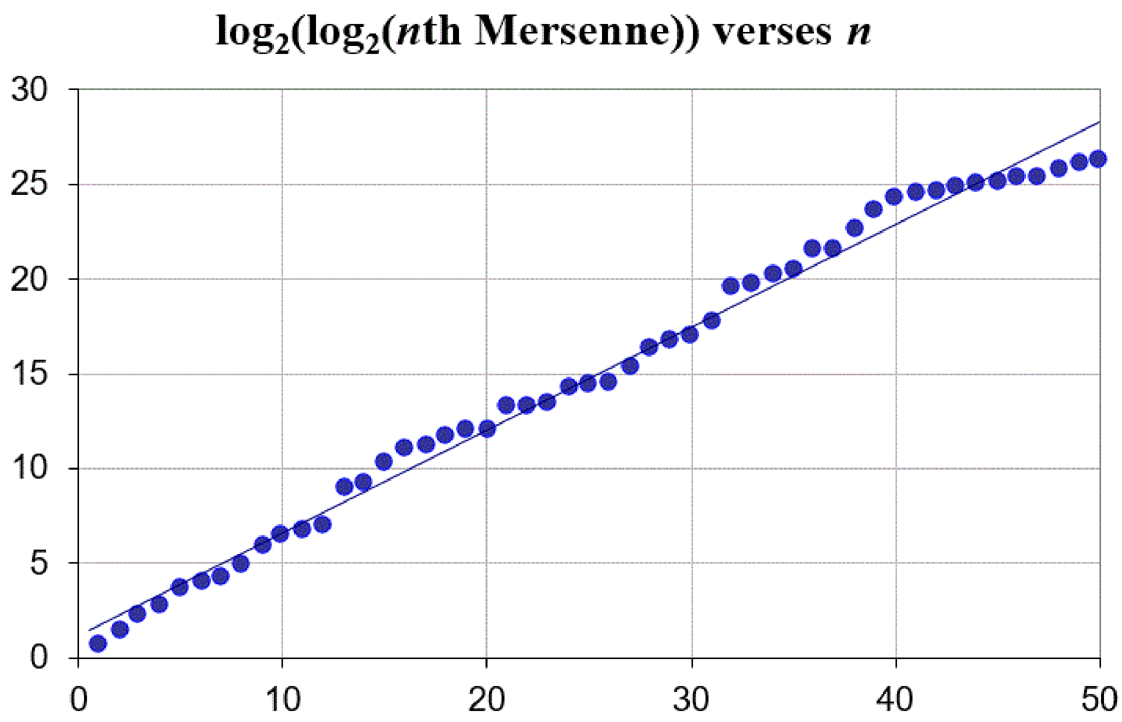


Figure 5. nth Mersenne prime number (double logarithmic plot)

Clearly the asymptotic estimate fits very well.

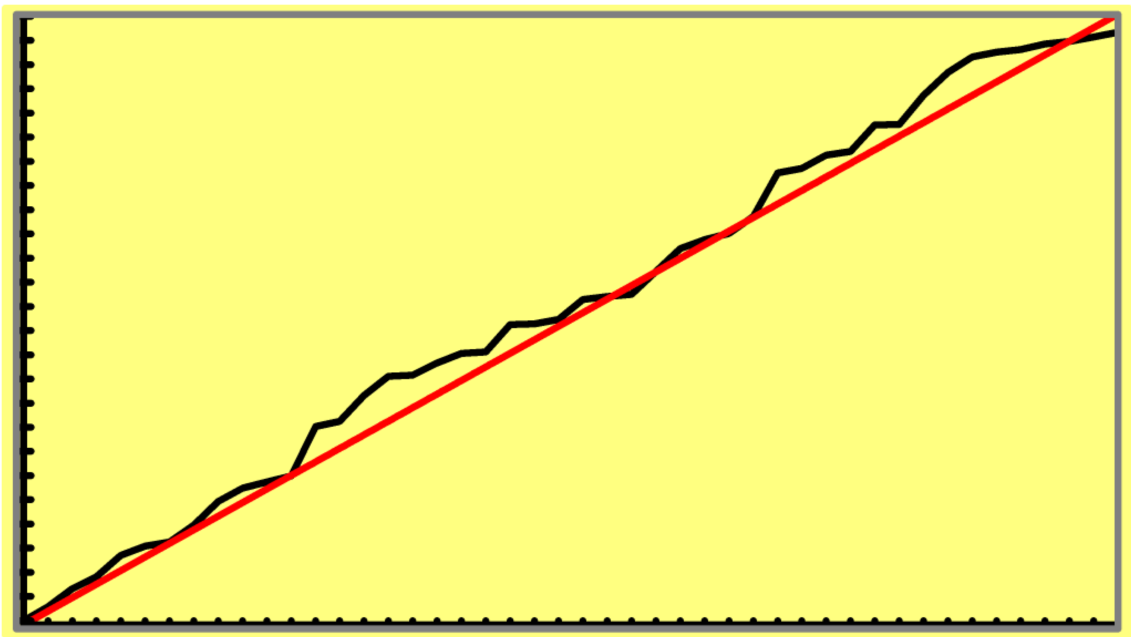


Figure 6. nth Mersenne prime number (double logarithmic plot), created by KVEC

Illustration: estimate (red), ln (ln (M_p)) (black)

Created by KVEC and the following parameter file:

```
vnull
MersennePrimesAsymptotic_KVEC.png
```

```
-antialias 2 -dimension 1024 -xdim 1025 -ydim 576
-format png -xmin 0.000000 -xmax 45.000000
-drcolor 0 0 0 -bkcolor 255 255 128 -nstep 2000 -lwidth 200
-scmode 2 -mode aniso -reduce all -smooth on
 function
imin 0; imax 51; drcolor 0 0 0;
f1(x)=log(KV_MPRIMES[x])/M_LN2;
drcolor 255 0 0;
f2(x)=exp(-M_G)*x+0.8255;
endfunc
```

The few things we know or assume about the analytic mathematics of the Mersenne prime are documented in detail here: http://primes.utm.edu/notes/faq/

The following graphic is a plot of a phase-space representation of logarithmic values of the Mersenne prime numbers:[11]

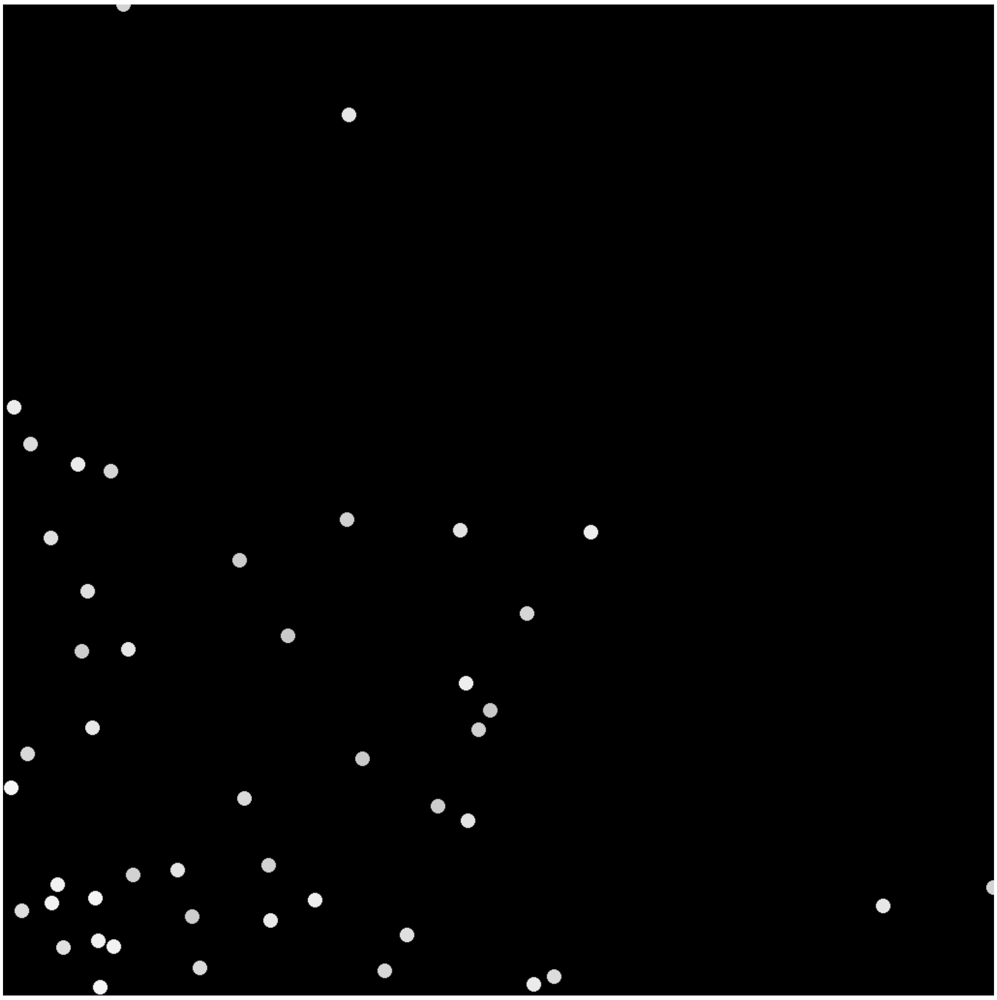


Figure 7. *n*th Mersenne prime number (double logarithmic phase-space representation)

Created by KVEC using the following parameter file:

```
null
```

[11] Created by KVEC (http://www.kvec.de)

```
Mersenne_Exponents_In_PhaseSpace.png
-antialias 2 -dimension 1024 -format png -mode aniso -random 24 703
```

Are there symmetric structures inside? What will this image look like if we take 100 or 1000 Mersenne primes instead of only 51 Mersenne primes?

```
KVEC-program for creating the first 50 Mersenne prime numbers:
vnull
(null).swf
-debug plot –function imax 51; f1(i)=KV_MPRIMES[i]; endfunc
```

Yet another an image created by 'playing around' with Mersenne primes: Lisssajous figure, created with all Mersenne prime number exponents.
The KVEC program used reads:

```
vnull
plot_circles_MersennePrimes_Iteration.jpg
-antialias 2 -xdim 847 -ydim 1025 -format jpeg
-drcolor 50 0 24 -bkcolor 128 196 255 -nstep 500000 -grit 8 -scmode 2
-paper user 600.000000 200.000000 -pattern outin 128 128 128 function
C1=0.9; x1=0.5; y1=0.25;
object markfilledcircle;
msize 0.1; imax 500000;
x1()=(1.0-x1*y1*C1)*cos(log(KV_MPRIMES[II%48])+II);
y1()=(x1-y1)*sin(log(KV_MPRIMES[II%48])-II);
endfunc
```

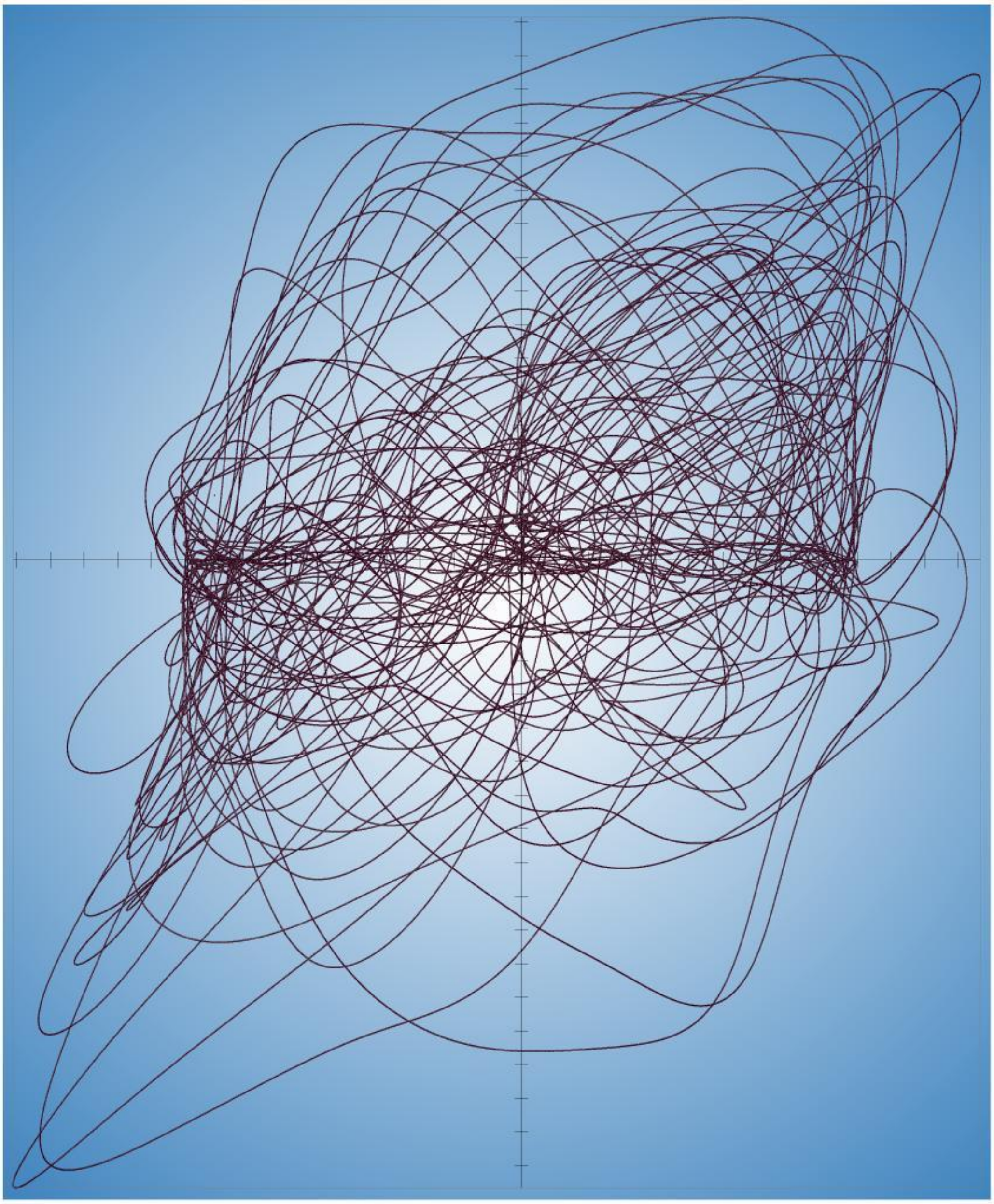


Figure 8. Lissajous-like graphic, created with Mersenne prime number exponents

4.5.1 GIMPS – THE GREAT INTERNET MERSENNE PRIME SEARCH

The GIMPS is an Internet project to which volunteers contribute the computing power of their own PCs. The distributed computer power from thousands of users is enlisted to search for Mersenne prime numbers.

It would be woefully remiss, of course, not to mention this successful research project in a book about prime numbers!

Anyone who wants to participate in this project can download the appropriate software for their operating system from the GIMPS web site[12].

The project has been very successful during the last few years. Since the foundation of the project (1996), GIMPS has discovered the 16 largest Mersenne primes.

The total computing power of the project reaches between 300 and 950 TFLOP/sec (as of Oct. 2016). Just a reminder: a FLOP is a 'Floating Point Operation' (an operation with floating point numbers). A TFLOP/sec (=TeraFLOP/sec) means that 10^{12} floating point numbers are evaluated every second.

At peak times GIMPS achieves speeds of almost one PFLOP/sec (PetaFlop/sec = 10^{15} FLOP/ sec = 1.000.000.000.000.000 FLOP/sec).

Here are the GIMPS statistics of the author, who, naturally, is participating in this project (as of Nov. 2016):

Lifetime Stats

Type	Rank	of	GHz-days	Count
TF	5,185	14,226	27	312
P-1	1,377	10,998	99	47
LL	408	10,384	33,525	427
LL-D	356	11,843	5,048	162
ECM	-	2,242	-	-
ECMF	-	461	-	-
Overall	579	26,893	38,699	948

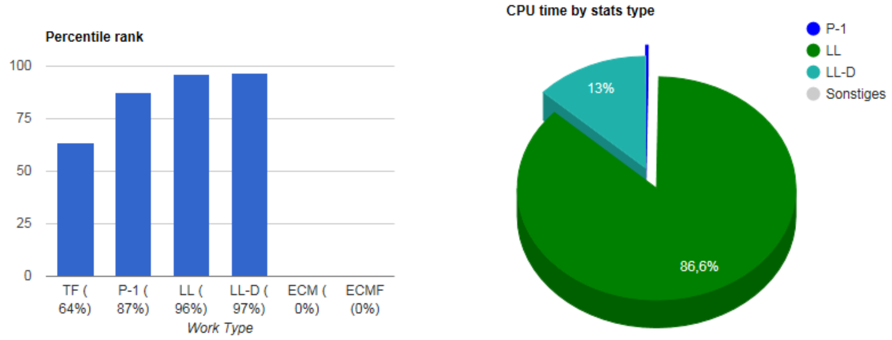


Figure 9. GIMPS statistics of the author

4.6 FERMAT PRIME NUMBERS

There is also an immense amount of literature devoted to Fermat prime numbers. Here, in brief, are the most important issues concerning Fermat primes F_n:

Fermat primes are primes of the form

$$\boldsymbol{F_n = 2^{2^n} + 1} \tag{18}$$

[12] http://www.mersenne.org

It has been known for hundreds of years that numbers of the form $2^m + 1$ can only be primes if m has the form 2^n. Unfortunately, however, not every Fermat number $2^{2^n} + 1$[13] is necessarily a prime (as Fermat believed). In fact, to this day, only five Fermat primes are known, namely $F_0$to F_4 :

$$\mathbf{3, 5, 17, 257, 65537}$$

All other Fermat numbers are probably composite.

F_5 to F_{32} are composite numbers. F_{33} is the first Fermat number of which we do not know whether it is composite or prime (as of Nov. 2016).
Already in 1732, Euler had shown that each factor of a Fermat number $F_n, (n \geq 2)$ must have the form $k \cdot 2^{n+2}$+1.

Since the year 1877, Pépin's prime number test for Fermat numbers has been known.

The following conditions are equal (note: k is usually taken as 3):

- F_n is a prime number and $\left(\frac{k}{F_n}\right) = -1$, $\left(\frac{k}{F_n}\right)$ is the Jacobi symbol[14]
- $k^{(F_n-1)/2} \equiv -1 \pmod{F_n}$

Some basic properties of Fermat prime numbers:[15]

$$\boldsymbol{F_n = (F_{n-1} - 1)^2 + 1}, \textbf{for } \boldsymbol{n \geq 1}$$

and (for $n \geq 2$):

$$\boldsymbol{F_n = F_{n-1} + 2^{2^{n-1}} F_0 \cdots F_{n-2}}$$

$$\boldsymbol{F_n = {F_{n-1}}^2 - 2(F_{n-2} - 1)^2}$$

$$\boldsymbol{F_n = F_0 \cdots F_{n-1} + 2}$$

In the decimal system, the last digit of every Fermat number (with the exception of the first two) is 7.

Fermat primes are not 'Brazilian' numbers, which are numbers of the form:

$$\boldsymbol{P_k = 1 + n + n^2 + n^3 + \cdots + n^k, n > 1, k > 1} \tag{19}$$

Note: it is not known whether there are infinitely many Brazilian prime numbers…)

```
Mathematica program for generating Fermat prime numbers:
Select[Table[2^(2^n) + 1, {n, 0, 4}], PrimeQ]
```

[13] Sequence https://oeis.org/A000215
[14] http://mathworld.wolfram.com/JacobiSymbol.html
[15] https://en.wikipedia.org/wiki/Fermat_number

The number of digits $D(n)$ of a Fermat number $\boldsymbol{F_n}$ (in the decimal system) comes to:

$$\boldsymbol{D(n) = 1 + \lfloor 2^n \ln 2 \rfloor}$$

Remarkable is also the relation (discovered by Gauss back in the 18th century) between the ability to construct a regular polygon with *n* points using a compass and straight edge and the Fermat prime numbers:

A regular polygon having $\boldsymbol{n}$ corners can be constructed with a compass and straight edge if $\boldsymbol{n}$ is the product of a power of 2 and Fermat prime numbers (in distinct pairs).

It is curious that it should be possible to construct a regular pentagon or a polygon with 17 corners by this method but not one with 7 or 11 corners…

4.7 LUCKY PRIMES

Lucky numbers must not be confused with 'happy' numbers (which are defined quite differently[16]).
First of all, 'lucky' numbers are defined as follows[17]. 'Lucky' numbers are constructed according to a procedure that resembles the 'Sieve of Eratosthenes': beginning with the list of natural numbers 1,2,3,4,5,6, ... we remove elements from the list in accordance with the following principle:

- The 1 is 'lucky' by definition:
  `(01,02,03,04,05,06,07,08,09,10,11,12,13,14,15,16,17,18,19,20,…)`
- The next number is the 2, so we remove every second number; the 3 survives:
  `(01,03,05,07,09,11,13,15,17,19,21,23,25,27,29,31,33,35,37,39,…)`
- The next number is the 3, so we remove every third number; the 7 survives:
  `(01,03,07,09,13,15,19,21,25,27,31,33,37,39,43,45,49,51,55,57,…)`
- The next number is the 7, so we remove every 7th number; the 9 survives:
  `(01,03,07,09,13,15,21,25,27,31,33,37,43,45,49,51,55,57,…)`
- The next number is the 9, so we remove every 9th number; the 13 survives:
  `(01,03,07,09,13,15,21,25,31,33,37,43,45,49,51,55,…)`
  `… etc.`

What remains is the sequence of 'lucky numbers'.
The sequence of lucky numbers has much in common with the sequence of prime numbers: they both have the same density, which is proportional to $\frac{1}{\ln(n)}$. Twin primes and twin 'luckies' seem also to exhibit the same density, as the following table suggests:

[16] http://mathworld.wolfram.com/HappyNumber.html
[17] https://oeis.org/A000959

Table 7. Lucky numbers up to 1E15

Region	Number of lucky numbers	Number of primes	Number of twin luckies	Number of twin primes
10^0	1	0	1	0
10^1	4	4	2	2
10^2	23	25	7	8
10^3	153	168	33	35
10^4	1118	1229	178	205
10^5	8772	9592	1162	1224
10^6	71918	78498	7669	8169
10^7	609237	664579	55548	58980
10^8	5286238	5761455	419174	440312
10^9	46697909	50847534	3274570	3424506
10^{10}	418348044	455052511	26298112	27412679
10^{11}	3790060378	4118054813	?	224376048
10^{12}	34652117969	37607912018	?	1870585220
10^{13}	319239995375	346065536839	?	15834664872
10^{14}	2960006060823	3204941750802	?	135780321665
10^{15}	27596305747873	29844570422669	?	1177209242304

Mathematica:

```
luckies=2*Range@500-1;
f[n_]:=Block[{k=luckies[[n]]},luckies=Delete[luckies,Table[{k},{k,k,Le
ngth@luckies,k}]]];Do[f@n,{n,2,30}];luckies
(*or:*)
sieveMax = 10^6; luckies = Range[1, sieveMax, 2];
sieve[n_] := Module[{k = luckies[[n]]}, luckies = Delete[luckies,
Table[{i}, {i, k, Length[luckies], k}]]]; n = 1; While[luckies[[n]] <
Length[luckies], n++; sieve[n]]; luckies
```

Result:

```
{1,3,7,9,13,15,21,25,31,33,37,43,49,51,63,67,69,73,75,79,87,
93,99,105,111,115,127,129,133,135,141,151,159,163,169,171,18
9,193,195,201,205,211,219,223,231,235,237,241,259,261,267,27
3,283,285,289,297,303,307,319,321,327,331,339,349,357,361,36
7,385,391,393,399,409,415,421,427,429,433,451,463,475,477,48
3,487,489,495,511,517,519,529,535,537,541,553,559,577,579,58
3,591,601,613,615,619,621,631,639,643,645,651,655,673,679,68
5,693,699,717,723,727,729,735,739,741,745,769,777,781,787,80
1,805,819,823,831,841,855,867,873,883,885,895,897,903,907,92
5,927,931,933,937,957,961,975,979,981,987,991,993,997}
```

Please note: this type of 'lucky' number must also not be confused with Euler's 'lucky' numbers (prime numbers n of the form $m^2 - m + n$ such that $m^2 - m + n$ gives a prime number, for $m = 0,1, \dots, n-1$).

The set of 'lucky' primes is simply the set of 'lucky' numbers that are prime.[18] It is not known if there are infinitely many 'lucky' prime numbers (as of Oct. 2015).

4.8 PERFECT NUMBERS

4.8.1 GENERAL ISSUES AND DEFINITION

Perfect numbers are closely related to Mersenne prime numbers (see Chapter 4.5).

Definition

A (positive whole) number is perfect if it is identical to the sum of its divisors (where the number itself is excluded as a divisor). This sum of divisors is often called the aliquot sum $S(n)$, in contrast to the complete sum of divisors $\sigma_1(n)$, for which the number itself is also included in the sum. From this, it follows that:

a number is perfect if $\boldsymbol{S(n) = n}$ or $\boldsymbol{\sigma_1(n) = 2n}$ (20)

Perfect numbers have been well known since antiquity (Nicomachus[19], Philo Judaeus[20]) (the four numbers $\boldsymbol{6, 28, 496, 8128}$) and were already mentioned in the 'Elements' of the ancient Greek mathematician Euclid. Probably the name comes from the idea that God created the world in 6 days, as well as the fact that the moon's orbit has a duration of 28 days.

There are as many known perfect numbers as known Mersenne prime numbers (as of Dec. 2020). The first 10 are[21]:

[18] https://oeis.org/A031157
[19] Nicomachus (60 – 120 n. Chr.), antique philosopher, musical theorist and mathematician
[20] Philo Judaeus: (25-50 n. Chr.), Greek-Jewish philosopher, lived in Alexandria
[21] https://en.wikipedia.org/wiki/Perfect_number

Table 8. The first 10 perfect numbers

n		Perfect numbers
1		6
2		28
3		496
4		8128
5		33550336
6		8589869056
7		137438691328
8		2305843008139952128
9		2658455991569831744654692615953842176
10		191561942608236107294793378084303638130997321548169216

All known perfect numbers are related to the Mersenne prime numbers (the proof originated from Euler, 18th century).

$$\textbf{If } 2^p - 1 \textbf{ is prime, then } 2^{p-1}(2^p - 1) \textbf{ is a perfect number.} \tag{21}$$

All currently known perfect numbers are even. There are 51 perfect numbers known (as of Dec. 2020). It is unknown if any odd perfect numbers exist. Probably there are infinitely many perfect numbers (as there are also probably infinitely many Mersenne primes).

Perfect numbers also occur in numerology and mysticism.

4.8.2 PROPERTIES

Each even perfect number can be represented as follows:

$$n = 1 + \frac{9}{2}k(k + 1), (\text{where } k = 8j + 2, j > 0 \text{ and } n > 6) \tag{22}$$

The converse does not apply! One does not obtain a perfect number for each j…

For j= 1, 2, ... we obtain: 28, 496, 1540, 3160, 5356, 8128, 11476, 15400, 19900, …

Only the following j will produce perfect numbers:

Mathematica-program for computing the indices that provide perfect numbers:

```
MPrimeExp={2,3,5,7,13,17,19,31,61,89,107,127,521,607,1279,2203,2281,32
17,4253,4423,9689,9941,11213,19937,21701,23209,44497,86243,110503,1320
49,216091,756839,859433,1257787,1398269,2976221,3021377,6972593,134669
17,20996011,24036583,25964951,30402457,32582657}
```

```
PerfectN=Table[2^(MPrimeExp[[k]]-1)(2^MPrimeExp[[k]]-1),{k,1,20}]
```

```
Table[Solve[1+9/2(8j+2)(8j+3)==PerfectN[[i]],j],{i,2,10}]
```

yields:

```
{{j->-(5/8)},{j->0}},{{j->-(13/8)},{j->1}},
{{j->-(45/8)},{j->5}},
{{j->-(2733/8)},{j->341}},
{{j->-(43693/8)},{j->5461}},
{{j->-(174765/8)},{j->21845}},
{{j->-(715827885/8)},{j->89478485}},
{{j->-(768614336404564653/8)},{j->96076792050570581}},
{{j->-(2063233398808967124831873373/8)},{j->257904174851120890603984211}}
```

The sequence $\{0, 1, 5, 341, 5461, \ldots\}$ is the sequence for all n, so $24n + 7$ produces a Mersenne prime number.

More properties

- the sum of the reciprocals of all divisors of a perfect number n is 2:

$$\sum_{k|n} \frac{1}{k} = 2$$

- each perfect number n>6 can be represented as a sum of third powers:

$$n = \sum_{i=1}^{2^{\frac{p-1}{2}}} (2i-1)^3, \text{where } n = 2^{p-1}(2^p - 1)$$

- each perfect number can also be represented (by taking a suitable k) as:

$$n = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

examples: $6 = 1 + 2 + 3 = \frac{3\cdot 4}{2}, 28 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{7\cdot 8}{2}$

There are two types of generalization of the term ‘perfect number’:

1) if the sum of the ‘true’ divisors (aliquot sum) is n times the number itself, then this number is called ‘n-perfect’. Example: 120 is a ‘2-perfect’ number.
2) if p and p^k-m-1 are prime numbers, the equation

$$\sigma_1(x) = \frac{px + m}{p - 1}$$

has the solution $x = p^{k-1}(p^k - m - 1)$.

Odd perfect numbers

No odd perfect numbers are known. It is also not known whether any exist.
Great progress has already been made in the search for such numbers n. Here is the current status of the research results (Oct. 2015). If such numbers n exist, they must satisfy the following conditions:

- $n > 10^{1500}$
- 105 is not a divisor of n
- n has the form $n \equiv 1 (\text{mod } 12)$or $n \equiv 117 (\text{mod } 468)$ or $n \equiv 81 (mod\ 324)$
- the largest prime factor of n is larger than 10^8
- n is composed of at least 101 prime factors

Thus it is very unlikely that odd perfect numbers exist.

4.9 SOPHIE GERMAIN PRIME NUMBERS

A prime number p is called ***Sophie Germain prime*** if $2p + 1$ is a prime number too. The numbers $2p + 1$ are called **‘*safe primes*’**. They are also solutions to the equation (in which $\varphi(n)$ is the Euler phi function, also called totient function):

$$\boldsymbol{\varphi(n) = 2p} \tag{23}$$

The following theorem applies: if p is a ***Sophie Germain prime***, then there are no integer numbers x, y and z from $\mathbb{Z}$ (without 0) such that p is not a divisor of $x \cdot y \cdot z$ and the equation $x^p + y^p = z^p$ holds.

Note: regarding the solutions of the Fermat equation $\boldsymbol{x^n + y^n = z^n}$ two cases are distinguished: in the first case n is not a divisor of x, y or z, i.e. for prime numbers of the type ‘***Sophie Germain’*** the first case of Fermat’s theorem is true.[22]

The first ***Sophie Germain*** primes are:

```
2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179,
191, 233, 239, 251, 281, 293, 359, 419, 431, 443, 491, 509,
593, 641, 653, 659, 683, 719, 743, 761, 809, 911, 953,
1013, 1019, 1031, 1049, 1103, 1223, 1229, 1289, 1409, 1439,
1451, 1481, 1499, 1511, 1559
```

[22] Fermat’s last theorem: there are no integer solutions of $x^n + y^n = z^n$ if $n > 2$.

4.9.1 COMPUTATION AND PROPERTIES

Properties of *Sophie Germain* primes

1) If $p > 3$ is a *Sophie Germain* prime and $p \equiv 3 \pmod 4$, then $2p + 1$ is a divisor of the pth Mersenne number.
2) For all *Sophie Germain* primes, the following obtains: $p \equiv 3 \pmod 4$.
3) If represented in the decimal system, *Sophie Germain* primes can never have a last digit of 7.
4) p and $2p + 1$ are *Sophie Germain* primes, if and only if p is a prime and $2^{2p} \equiv 1(\text{mod } 2p + 1)$.

The following asymptotic estimate of the number of SG primes up to a limit N obtains:

$$\text{NumberOf}_{SG} = 2C_2 \int_2^N \frac{1}{\ln(x)\ln(2x+1)}\,dx \approx \frac{2C_2N}{\ln^2(N)} \tag{24}$$

(with $C_2 = 0{,}6601618158$ being the twin prime constant).

Computation by Mathematica: (e.g. in the interval 1 to 1000):

```
Select[Prime[Range[1000]], PrimeQ[2#+1]&]
```

Conjectures

1) There are infinitely many *Sophie Germain* primes.
2) Between n and $2n$ there is always at least one *Sophie Germain* prime.

Record

Currently the largest SG prime has the value:

$$\mathbf{18543637900515 \cdot 2^{666667} - 1}$$

– a number having 200.701 decimal digits (as of Nov. 2016).

Notes: in mathematical literature sequences of SG primes are called Cunningham chains of the first kind.[23]

Number $a(n)$ of SG primes up to 10^n:

[23] https://de.wikipedia.org/wiki/Cunningham-Kette

Table 9. Number of Sophie Germain primes up to 1E12

n		a(n)
1		3
2		10
3		37
4		190
5		1171
6		7746
7		56032
8		423140
9		3308859
10		26569515
11		218116524
12		1822848478

Computation by Mathematica (example):

```
Accumulate[Table[Boole[PrimeQ[n]&&PrimeQ[2n+1]], {n, 1, 200}]]
```

4.10 FIBONACCI NUMBERS AND OTHER RECURSIVE SEQUENCES

There is an immense amount of literature concerning the Fibonacci numbers. An overview is not given here. You can find interesting information on Michael Becker's homepage.[24]

Only the following formulae are mentioned here (curiosities):

$\frac{1}{F_{11}} = \frac{1}{89} = 0.01123595$ (the decimal expansion starts exactly with the Fibonacci numbers)

(to be more precise, we should actually write):

$$\frac{1}{F_{11}} = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}} \tag{25}$$

The quotient $\frac{F_{n+1}}{F_n}$ of two consecutive Fibonacci numbers is the nth approximation of the continued fraction:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \tag{26}$$

[24] http://www.ijon.de/mathe/fibonacci/node2.html#0002320

The limit is the number of the Golden Ratio: $\Phi = 1.618 = \frac{1+\sqrt{5}}{2}$.

$$F_{12} = 144 = 12^2$$

F_{12} is the only square among the infinitely many Fibonacci numbers. The question arises as to whether there is some more profound reason for this, because it cannot be the product of chance… The reason for this actually exists. It appears as a 'side-product' in A. Wiles proof of Fermat's last theorem. But that is another story ☺…

The following formula is also a curiosity:

$$\left\lceil e^{\frac{n-1}{2}} \right\rceil, n = 0,1,2 \ldots$$

It yields exactly the first 10 Fibonacci numbers, following each other for $n = 1, \ldots, 10$.

```
Mathematica: Table[Floor[Exp[(n-1)/2]]+1,{n,1,25}]
{1,1,2,3,5,8,13,21,34,55,91,149,245,404,666,1097,1809,2981,4915,8104,…
}
```

For practical purposes: the nth Fibonacci number can be calculated very easily:

$$F_n = \text{Round}\left(\frac{\Phi^n}{\sqrt{5}}\right) \tag{27}$$

Finally, J. P. Jones showed in the year 1975 that there exists a polynomial of degree 5 with two variables, whose positive integer values are exactly the set of the Fibonacci numbers (using non-negative arguments). This remarkable fact seems to be more of theoretical interest, at first glance. After all, the statement can be verified by using the following Mathematica program:

```
numbersOfInstances=6;
polynom[x_,y_]:=2x*y^4+x^2*y^3-2x^3*y^2-y^5-x^4*y+2*y;
list=FindInstance[Reduce[polynom[x,y]>0&&x>=0&&y>=0,{x,y},Integers],{x
,y},Integers,numbersOfInstances,RandomSeed->112]
Table[polynom[list[[n]][[1]][[2]],list[[n]][[2]][[2]]],{n,1,Length[lis
t]}]
```

This strange property is not something the author has seen anywhere described, though perhaps it has already been noticed.

If we have the following polynomial:

$$\text{poly}_{Fib}(x,y) = 2xy^4 + x^2y^3 - 2x^3y^2 - y^5 - x^4y + 2y \text{ where } x, y \in \mathbb{N}_0$$

Then

$$\mathbf{poly}_{Fib}(\boldsymbol{F_i}, \boldsymbol{F_{i+1}}) = \boldsymbol{F_{i+1}} \; (\boldsymbol{i} \geq \mathbf{0}) \tag{28}$$

i.e. the arguments for which the polynomial yields the (positive) Fibonacci numbers are precisely the Fibonacci numbers themselves! From this, we get the following equation:

$$\boldsymbol{F_{i+1} = 2F_iF_{i+1}{}^4 + F_i{}^2F_{i+1}{}^3 - 2F_i{}^3F_{i+1}^2 - F_{i+1}{}^5 - F_i{}^4F_{i+1} + 2F_{i+1}} \quad (29)$$

Or, in other words, the 'successor' $\boldsymbol{F_{i+1}}$ in the Fibonacci sequence can be calculated from the predecessor $\boldsymbol{F_i}$ by calculating the positive, integer valued solution of this equation of degree 4, and that all happens without knowing the index i:

$$\boldsymbol{-y^4 + 2F_iy^3 + F_i{}^2y^2 - 2F_i{}^3y + 1 - F_i{}^4 = 0} \quad (30)$$

Mathematica example: $f = F_i = 8$;

```
Solve[-y^4+2f y^3+f^2y^2-2f^3y+1-f^4==0,y,Integers]
{{y->-5},{y->13}}
```

The following conjecture is probably easily proved (if it is true…):
(30) always has real solutions for positive F_i. Integer solutions exist only if F_i is a Fibonacci number.

Note: this explicit formula can, of course, be used to calculate values of n from F_n (e.g. using Mathematica) and, by taking $n + 1$ for the explicit formula, the successor F_{n+1} of F_n can be determined (without knowing n). This procedure is, however, very inconvenient and not as elegant as using the equation of 4th degree (30).
For practical use: the following formula is the fastest one (for $n > 1$):

$$F_{n+1} = \text{Round}(F_n\Phi),\ \Phi = 1.618 = \tfrac{1+\sqrt{5}}{2} \text{ and } \text{Round}(x) = \lfloor x + 0.5 \rfloor \quad (31)$$

For the inverse process, it is also very easy to determine n:

$$n = \text{Round}\left(\frac{\ln F_n + \frac{\ln 5}{2}}{\ln \Phi}\right) \quad (32)$$

A simple test to find out whether n is a Fibonacci number or not:

n is a Fibonacci number, if $5n^2 + 4$ or $5n^2 - 4$ is a square (more precisely: only if…).

Finally it should be mentioned that the last digit of the numbers in the Fibonacci sequence repeats itself with a period of 60 (for the last n digits there also exist periods, whose lengths grow by a factor of 5 for each additional digit).

The Fibonacci sequence was first mentioned in 450 B.C. in the *Chandah-shāstra*, a document written in Sanskrit. It was only, however, through the publication in 1202 of the *Liber Abaci* (The Book of Calculation) by Leonardo Fibonacci,[25] in which he used it to describe the proliferation of rabbits, that it became widely known in the West.

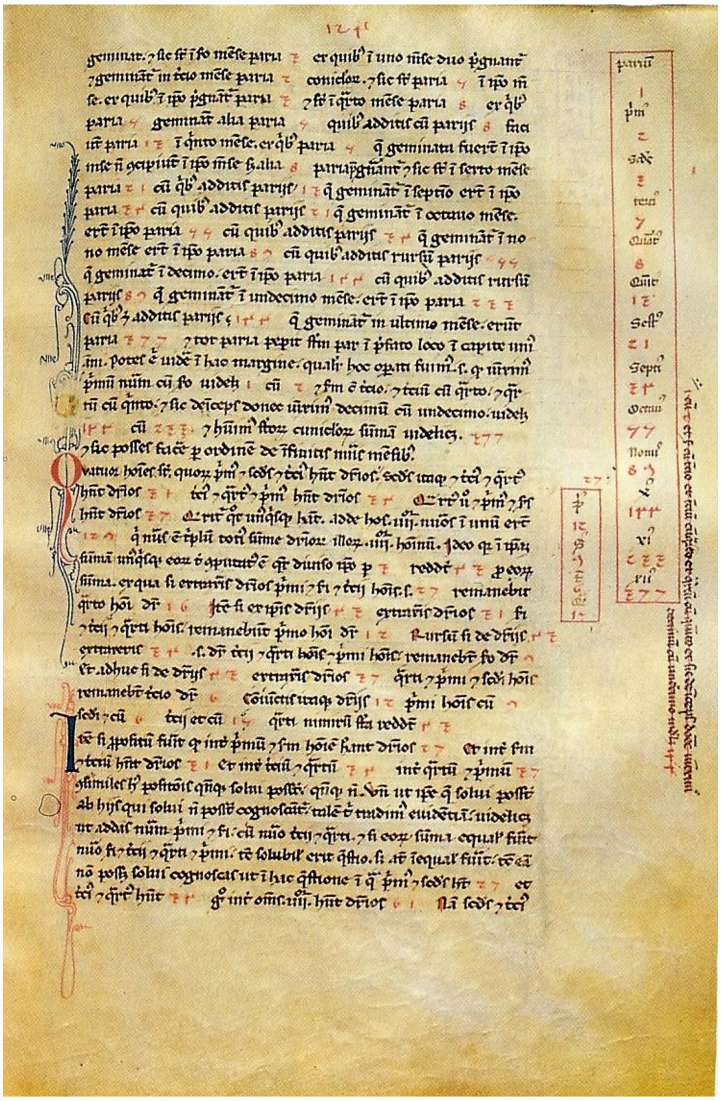


Figure 10. Page from *Liber Abaci* by Leonardo Fibonacci

It is well known in esoteric circles and among conspiracy theorists, as well as appearing in numerous science fiction and fantasy films (e.g. 'Sacrilege'[26], 2004)

The Fibonacci numbers belong to the class of recursively defined sequences (more precisely: linearly recursive), so a brief digression is perhaps in order.

4.10.1 LINEAR RECURSION: A MIGHTY INSTRUMENT

The method of linear recursion, as a principle of construction for arithmetic sequences, yields many interesting consequences that have been the subject of extensive

[25] https://de.wikipedia.org/wiki/Leonardo_Fibonacci
[26] https://de.wikipedia.org/wiki/Sakrileg_(Roman)

mathematical investigations. Here are a few examples: the Fibonacci and its related Lucas, Perrin (aka the 'Skiponacci' sequence, see Chapter 20.3) and Pell sequences.

There is a vast amount of mathematical literature dealing with these sequences. Here, the author would just like to present a few results that appear particularly interesting.

Sequences defined by linear recursion are defined by the linear relation of their sequence members to their preceding sequence members:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \text{with the initial values } a_0, a_1, \ldots, a_{k-1} \tag{33}$$

Table 10. A few linear recursive defined sequences

Recursion	Initial values	Kernel	Expl. formula	Name
$a_n = a_{n-1} + a_{n-2}$	$a_0 = 0,\ a_1 = 1$	{1,1}	$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$	Fibonacci
$a_n = a_{n-1} + a_{n-2}$	$a_0 = 2,\ a_1 = 1$	{1,1}	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	Lucas
$a_n = 2a_{n-1} + a_{n-2}$	$a_0 = 0,\ a_1 = 1$	{2,1}	$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	Pell
$a_n = 2a_{n-1} + a_{n-2}$	$a_0 = 2,\ a_1 = 2$	{2,1}	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	Pell-Lucas
$a_n = a_{n-2} + a_{n-3}$	$a_0 = 1, a_1 = 1, a_2 = 1$	{0,1,1}	(complicated, see 20.6)	Padovan
$a_n = a_{n-2} + a_{n-3}$	$a_0 = 3, a_1 = 0, a_2 = 2$	{0,1,1}	(see Chapter 20.6)	Perrin
$a_n = a_{n-1} + a_{n-2} + a_{n-3}$	$a_0 = 0, a_1 = 1, a_2 = 2$	{1,1,1}	(complicated)	'Tribonacci'
$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$	$a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 4$	{1,1,1,1}	(complicated)	'Quadranacci'
$a_n = a_{n-5} + a_{n-2}$	$a_0 = 5, a_1 = 0, a_2 = 2, a_3 = 0, a_4 = 2$	{0,1,0,0,1}	?	'5'+Sloane A133394
$a_n = a_{n-5} + a_{n-2}$	$a_0 = 0, a_1 = 2, a_2 = 0, a_3 = 2, a_4 = 5$	{0,1,0,0,1}	?	Reed Jameson
$a_n = a_{n-5} - a_{n-3}$	$a_0 = 5, a_1 = 0, a_2 = 0, a_3 = -3, a_4 = 0$	{0,0,-1,0,1}	?	Sloane A136598
$a_n = a_{n-7} + a_{n-4}$	$a_0 = 7, a_1 = a_2 = a_3 = 0, a_4 = 4, a_5 = a_6 = 0$	{0,0,0,1,0,0,1}	?	Sloane A135435 Reed Jameson

Mathematica offers the user the functions: `LinearRecurrence[kernel,init,n]`, `RecurrenceTable[]` and `FindLinearRecurrence[list]`, which are useful for investigations with recursive sequences.

Using `RSolve` and `RSolveValue` recursive equations can be solved, e.g..:

```
RSolve[{f[n]==f[n-1]+f[n-2],f[0]==0,f[1]==1},f[n],n]
{{f[n]->Fibonacci[n]}}   or:

func=RSolveValue[{f[n]==f[n-1]+f[n-2],f[0]==0,f[1]==1},f,n]
```

Note: the sequence ‘Sloane 136598’ is the negative continuation of the sequence ‘Reed Jameson’.

The Fibonacci sequence and the Lucas sequence are closely related with the number $\Phi = 1.618\ldots$ of the Golden Ratio ($\Phi = \varphi$).

The characteristic equation of the Fibonacci sequence and the explicit formulae for the Fibonacci and Lucas sequences read:

$$x^2 - x - 1 = 0 \text{ where the solutions } \varphi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

$$\begin{aligned} F_n &= \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ L_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \end{aligned} \tag{34}$$

There are many other connections between the Lucas numbers and the Fibonacci numbers, which will not be discussed here (e.g. $L_{2n} + 2(-1)^{n-1} = 5{F_n}^2$, or $L_n = F_{n-1} + F_{n+1}$).

Concerning the Reed Jameson sequences, there are similar interesting connections with prime numbers (as with the Perrin sequence (see Appendix)).

More properties of the sequences from Table 10

4.10.1.1 REPRESENTATIONS USING MATRICES

Fibonacci F_n: $Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

Lucas L_n: $Q = \begin{bmatrix} L_2 & L_1 \\ L_1 & L_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad Q^n = \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix}$

Padovan and Perrin P_n: $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad Q^n = \begin{bmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}$

Reed Jameson (‘5# + Sloane A133394, also A136598) RSP_n, RSM_n:

$$Q=\begin{pmatrix}0&1&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix};\ RSP_n=Q^n.\begin{pmatrix}2\\0\\2\\0\\5\end{pmatrix}=\begin{pmatrix}0&1&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}^n.\begin{pmatrix}2\\0\\2\\0\\5\end{pmatrix}$$

$$RSM_n=Q^n.\begin{pmatrix}0\\-3\\0\\0\\5\end{pmatrix}=\begin{pmatrix}0&1&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}^n.\begin{pmatrix}0\\-3\\0\\0\\5\end{pmatrix}$$

Note: RSP_n provides the sequence members in the positive direction, RSM_n in the negative direction.

4.10.1.2 MATHEMATICA PROGRAMS FOR CREATING RECURSIVE SEQUENCES

Fibonacci F_n:

```
LinearRecurrence[{1,1},{0,1},30]

Table[Fibonacci[n],{n,0,30}]

{0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,…}
```

Lucas L_n:

```
LinearRecurrence[{1,1},{2,1},30]

Table[LucasL[n],{n,0,30}]

{2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,3571,5778,…}
```

Pell P_n:

```
LinearRecurrence[{2,1},{0,1},30]

CoefficientList[Series[x/(1-2*x-x^2),{x,0,30}],x]

Expand[Table[((1+Sqrt[2])^n-(1-Sqrt[2])^n)/(2Sqrt[2]),{n,0,30}]]

a=1;b=0;c=0;lst={b};Do[c=a+b+c;AppendTo[lst,c];a=b;b=c,{n,30}];lst

{0,1,2,5,12,29,70,169,408,985,2378,5741,13860,33461,80782,195025,…}
```

Pell-Lucas Q_n

```
LinearRecurrence[{2,1},{2,2},30]

aa={};Do[k=Expand[((1+Sqrt[2])^n+(1-Sqrt[2])^n)];
AppendTo[aa,k],{n,0,30}]; aa

a=c=0; t={b=2}; Do[c=a+b+c; AppendTo[t,c]; a=b;b=c,{n,40}]; t

{2,2,6,14,34,82,198,478,1154,2786,6726,16238,39202,94642, 228486,…}
```

Padovan P_n

```
LinearRecurrence[{0,1,1},{1,1,1},30]

LinearRecurrence[{0,1,1},{1,0,0},30]

a[0]=1; a[1]=a[2]=0; a[n_]:=a[n]=a[n-2]+a[n-3]; Table[a[n],{n,0,30}]

CoefficientList[Series[(1-x^2)/(1-x^2-x^3),{x,0,30}],x]

More Mathematica programs: please contact the author.
```

{1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114,151,200,265,351,…}
or
{ 1,0,0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114,151,200,…}

Tribonacci T:

```
LinearRecurrence[{1,1,1},{0,1,2},30]
```

{0,1,2,3,6,11,20,37,68,125,230,423,778,1431,2632, 4841,8904,16377,…}

Quadranacci Q

```
LinearRecurrence[{1,1,1,1},{0,1,2,4},30]
```

{0,1,2,4,7,14,27,52,100,193,372,717,1382,2664,5135,9898,19079,36776,…}

Perrin P_n

```
LinearRecurrence[{0,1,1},{3,0,2},30]

CoefficientList[Series[(3-x^2)/(1-x^2-x^3),{x,0,30}],x]

explFunc=RSolve[{f[n]==f[n-2]+f[n-3],f[0]==3,f[1]==0,f[2]==2},f[n],n]
Round[Table[Evaluate[f[n]/.First[explFunc]],{n,0,30}]] (*fast*)

More Mathematica programs: please contact the author.
```

{3,0,2,3,2,5,5,7,10,12,17,22,29,39,51,68,90,119,158,209,277,…}

'negative' Perrin P_n

```
LinearRecurrence[{-1,0,1},{3,-1,1},30]

explFunc=RSolveValue[{f[n]==-f[n-1]+f[n-3],
f[0]==3,f[1]==-1,f[2]==1},f,n]
Round[Table[Round[explFunc[n]],{n,0,30}]] (*fast*)
```

{3,-1,1,2,-3,4,-2,-1,5,-7,6,-1,-6,12,-13,7,5,-18,25,-20,2,23,-43,45,-22,-21,66,-88,67,-1,-87}

'5'+ Sloane 133394 (Reed Jameson) R_n

```
LinearRecurrence[{0,1,0,0,1},{5,0,2,0,2},30]          or:
RecurrenceTable[{a[n]==a[n-2]
+a[n-5],a[1]==5,a[2]==0,a[3]==2,a[4]==0,a[5]==2},a,{n,1,30}]

reedJamesonMatrix={{0,1,0,0,1},{1,0,0,0,0},{0,1,0,0,0},{0,0,1,0,0},{0,
0,0,1,0}}; vect={{2},{0},{2},{0},{5}};
Flatten[Table[(MatrixPower[reedJamesonPlusMatrix,n].vect)[[1]],
{n,4-4,100-5}]]

More Mathematica programs: please contact the author.
```

identical
to{5,0,2,0,2,5,2,7,2,9,7,11,14,13,23,20,34,34,47,57,67,91,101,138,158,…
}

Sloane A136598: R_n^*

```
LinearRecurrence[{0,0,-1,0,1},{5,0,0,-3,0},30]

explFunc=RSolve[{f[n]==-f[n-3]+f[n-5],f[0]==5,f[1]==0,f[2]==0,f[3]==-
3,f[4]==0},f[n],n]//Simplify
Round[Table[Evaluate[f[n]/.First[explFunc]],{n,0,100}]]
```

{5,0,0,-3,0,5,3,0,-8,-3,5,11,3,-13,-14,2,24,17,-15,-38,-15,39,55,…}

Sloane A135435

```
LinearRecurrence[{0,0,0,1,0,0,1},{7,0,0,0,4,0,0},30]

explFunc=RSolve[{f[n]==f[n-4]+f[n-7],f[0]==7,f[1]==f[2]==f[3]==0,
f[4]==4,f[5]== f[6]==0},f[n],n]//Simplify
Round[Table[Evaluate[f[n]/.First[explFunc]],{n,0,100}]]
```

{7,0,0,0,4,0,0,7,4,0,0,11,4,0,7,15,4,0,18,19,4,7,33,23,4,25,52,27,11,5
8,75,31,36,110,102,42,94,185,133,78…}

(unknown,'negative A135435')

```
LinearRecurrence[{0,0,-1,0,0,0,1},{7,0,0,-3,0,0,3},30]
```

{7,0,0,-3,0,0,3,7,0,-3,-10,0,3,13,7,-3,-16,-17,3,19,30,4,-22,-46,-
21,25,65,51,-21,-87,-97,0,…}

4.10.1.3 COMPARISON OF THE DIFFERENT METHODS USED FOR CALCULATION

As you can see, there are a number of calculation methods that differ very much in memory demands and computing speed.

1) **`LinearRecurrence[…]`**: best method for situation 1). (alternatively: `recurrenceTable[{a[n]==a[n-2]+…];` this function is more flexible and has more options)
2) Method using matrices: $M^n * \text{initVec}$. Unbeatably fast for Situation 2). Perhaps also suitable for Situation 1).

3) Computation with an explicit function (`RSolveValue[]`): this depends on the complexity of the explicit solution of the recurrence equation. Not suitable for Situation 1).
4) `CoefficientList[…]`
5) Computation by means of the zeros of the characteristic polynom: `Solve[]`…
6) Calculation by the definition (e.g. `Do[c=a+b+c;AppendTo[lst,c];a=b;b=c,…`): slow, but in some cases quite practical!
7) Recursive method: (e.g. : `a[0]=1; a[1]=a[2]=0; a[n_]:=a[n]=a[n-2]+a[n-3];`): impractical!

We distinguish between the following situations:
1) calculation of the recursive sequence from the beginning up to a limit N

2) calculation of single sequence members without knowing a predecessor (will be used for very large indices). In this case mainly the matrix method or the calculation by explicit formulae come into play.

Here is an example of case 2: the sequence '5+Reed Jameson':
`f[n]==f[n-2]+f[n-5],f[0]==5,f[1]==0,f[2]==2,f[3]==0,f[4]==2`

N=10000000. The computation time for the nth sequence member is:

method 1: 476 seconds
method 2: 0.0156 seconds
method 3: 2964 seconds
(The explicit solution is complicated. It requires the calculation of the zeros of polynomials of degree 5. The solution of the recursive equation, although simplified with the Mathematica function '`Simplify`', still covers 15 Mathematica Notebook pages …)

Peter Danzeglocke has come up with a method that uses a function `MatrixPowerMod[]` instead of the function `MatrixPower[]` and that works only with values modulo n. This method can be applied to all mentioned recursive sequences and provides the sequence values modulo n in unbeatably short computational times.

4.10.1.4 CONNECTION TO PRIME NUMBERS

Some of these linear recursively defined sequences show interesting relationships to prime numbers. For a long time it was believed that the terms of the Perrin sequence would always be divisible by prime numbers for prime indices:

$P_n \equiv 0 \ (mod \text{ n})$, if $n \in \mathbb{P}$

This would be a method for checking prime numbers with a single modulus operation. The computation of P_n requires only additions, or additional powers and multiplications, if an explicit formula is known for the nth term of the recursively defined sequence.

Perrin's method of checking primes is perfect at first glance. But only at first glance: it only works up to indices <271441. This index gives the value 0 but it should actually give a result > 0 for the modulus test! This index denoted the first Perrin pseudoprime number. Since then, hundreds more have been discovered (see Chapter 20.3). It is, however, impressive that this method of determining prime numbers works perfectly up to 271441! Once again it becomes clear that numerical evidence cannot be used as a proof.

In principle, the recursive computation of the sequence terms is much faster for small indices (for many sequences the computational time is approximately 100 times faster than the explicit calculation in the range up to 10^6). For very large indices, however, this behaviour probably changes in favour of the explicit calculation

Reed Jameson discovered a similar method for checking prime numbers, which also evaluates the modulus values of sequence members having prime indices. In his method, however, two sequences are used. These are the sequences: **'5+ Sloane 133394'** and the complementary sequence F_n^*: **'Sloane A136598'.** (Note: the complementary sequence results when the 'normal' sequence is extended in the opposite direction towards negative indices). Then the sum of both sequences is evaluated:

$\boldsymbol{S_n} = (\boldsymbol{F_n} \mathbf{\ mod\ } \boldsymbol{n}) + (\boldsymbol{F_n^*\ mod\ n})$. It follows that $\boldsymbol{S_n} = \mathbf{0}$ if (and only if) n is a prime number (we start with index 0). This conjecture has been verified numerically up to n = 10^{10}. However, Peter Danzeglocke discovered numerous Reed Jameson pseudo-primes in the area $n > 10^{15}$.[27]
Still unknown is a method similar to that of Reed Jameson. Here we have the two (mutually complementary) sequences:

$\boldsymbol{F_n}$: **"Sloane 135435"** and the complementary sequence $\boldsymbol{F_n^*}$.

Again we build up the sum sequence $\boldsymbol{S_n} = (\boldsymbol{F_n\ mod\ n}) + (\boldsymbol{F_n^*\ mod\ n})$. The result is that $\boldsymbol{S_n} = \mathbf{0}$, if n is a prime number. This conjecture was checked by Peter Danzeglocke numerically up to $n = 10^9$. So far, no pseudo prime numbers for this sequence are known (as of December 2020). However there are about 10 per cent "trivial" pseudoprimes, which can, however, be separated easily, because for all pseudoprimes of this sequence, the following obtains:

$\boldsymbol{F_n} \equiv \mathbf{0\ (mod\ 2)}\ ||\boldsymbol{F_n} \equiv \mathbf{0\ (mod\ 5).}$

Here are a few plots of the interesting sum sequences of the modulus values of the sequences discussed above:

[27] Danzeglocke tested the Reed Jameson conjecture for all Fermat pseudoprimes to base 2 in the range up to $n < 2^{64}$. See appendix "Reed Jameson pseudo primes".

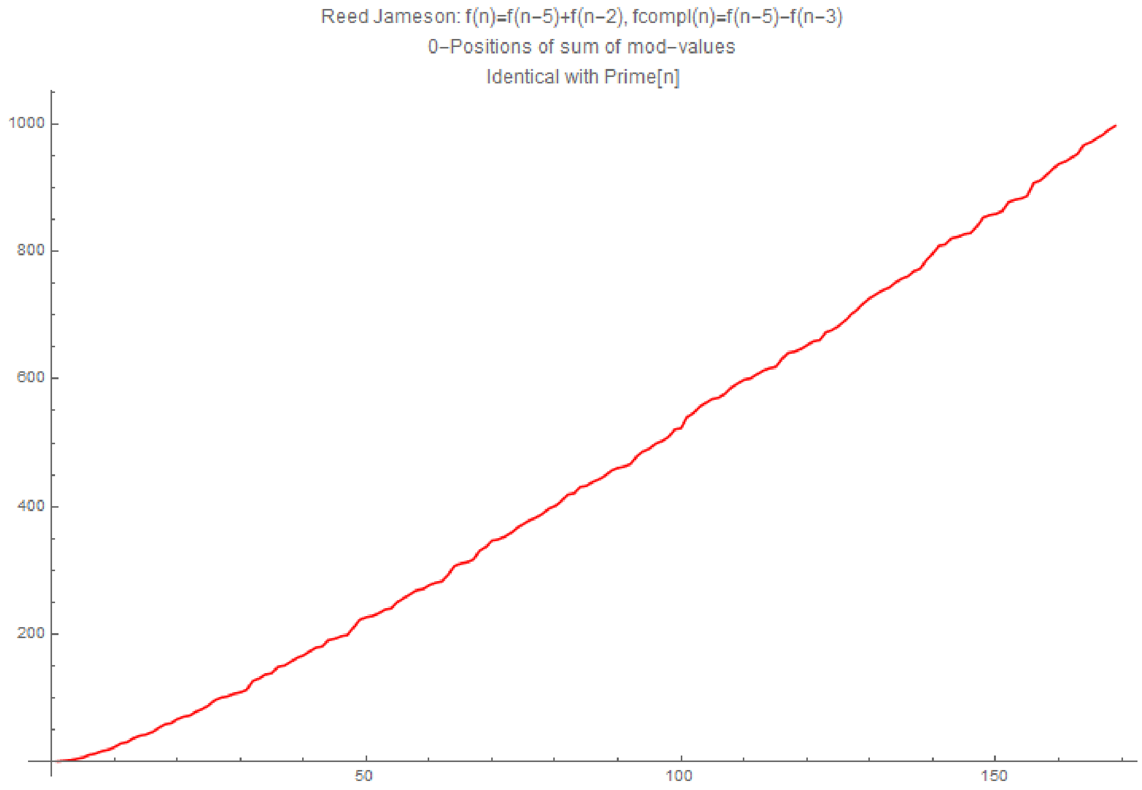


Figure 11. Reed Jameson sequence: plot of the 0 positions of the sum of the modulus values

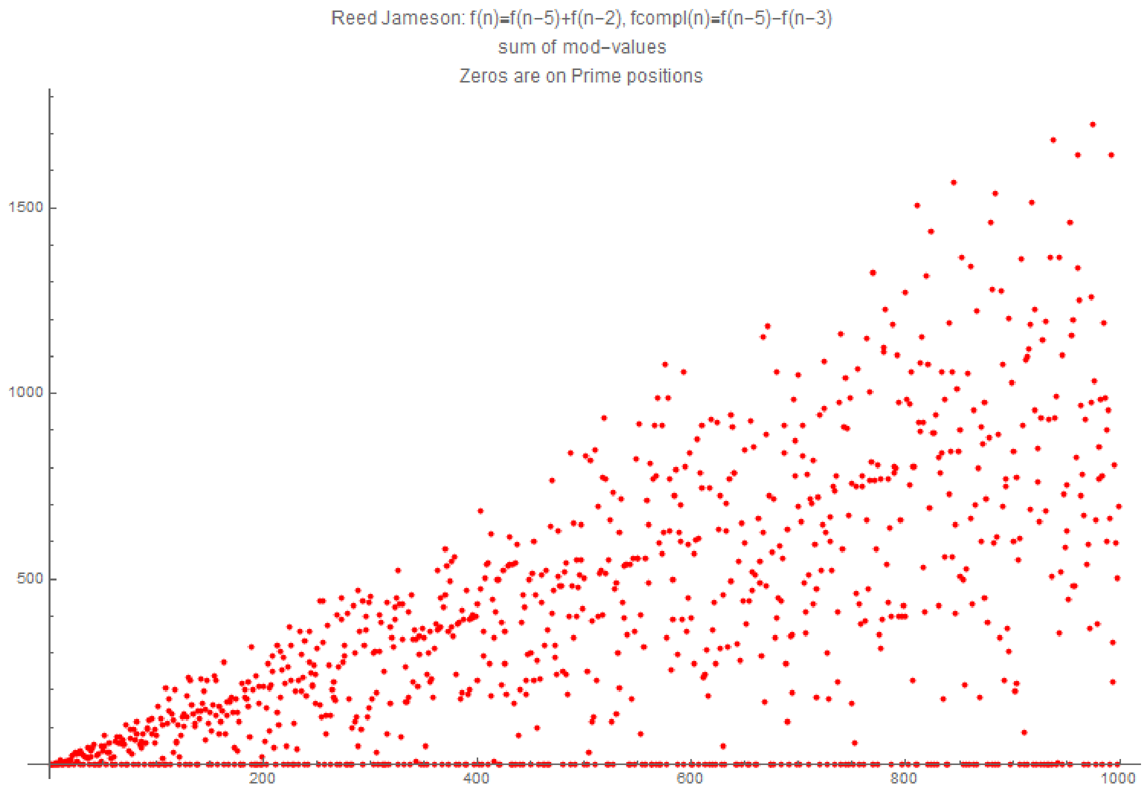


Figure 12. Reed Jameson sequence: plot of the sum of the modulus values

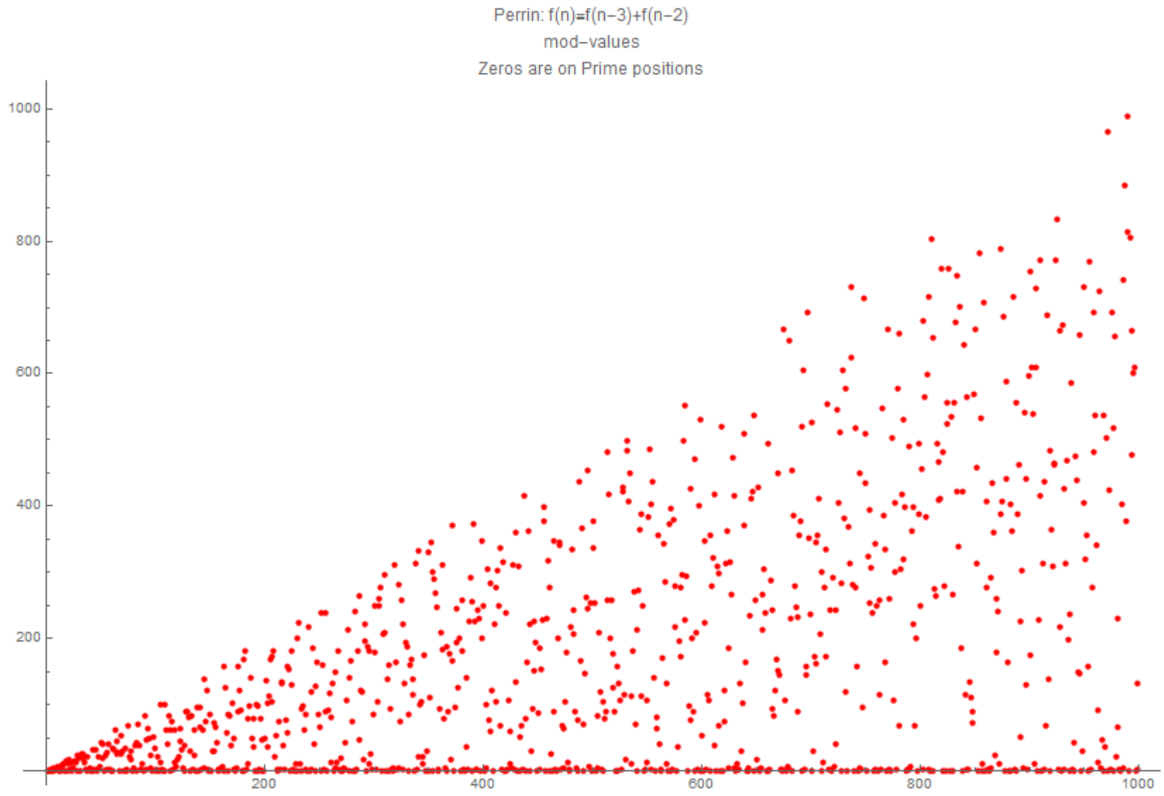


Figure 13. Perrin sequence: plot of the mod values. Zeros are (almost always) at prime positions

Note: Mathematica programs for creating the graphs can be found in the Appendix ().

4.10.2 FIBONACCI PRIME AND PSEUDOPRIME NUMBERS

A Fibonacci prime is a prime number that is also a member of the Fibonacci sequence.

Let us take a closer look at the Fibonacci sequence and mark the values belonging to prime indices:

0,1,**1**,**2**,3,**5**,8,**13**,21,34,55,**89**,144,**233**,377,610,987,**1597**,2584,**4181**,6765,10946,17711,**28657**,46368,75025,121393,196418,317811,**514229**,832040,**1346269**

Now we observe that for many prime indices, the corresponding Fibonacci numbersF_n are prime numbers (hits in red, counterexamples in purple). Indeed the following obtains: If F_n is a prime ($n \neq 2$ and $n \neq 4$), then n is also a prime. The converse of this statement, however, is not true. Fibonacci numbers that belong to prime indices, but are not primes themselves, are a subset of the Fibonacci pseudoprime numbers. Fibonacci pseudoprime numbers are defined as composite numbers for which the congruence $V_n \equiv 1 \pmod n$ obtains.
Furthermore, the following statements obtain:

If n is a prime number, then:

$$F_n \equiv 0 \ (\mathrm{mod}\ n) \text{ if } n \equiv 0 \ (\mathrm{mod}\ 5)$$
$$F_{n-1} \equiv 0 \ (\mathrm{mod}\ n) \text{ if } n \equiv \pm 1 \ (\mathrm{mod}\ 5)$$
$$F_{n+1} \equiv 0 \ (\mathrm{mod}\ n) \text{ if } n \equiv \pm 2 \ (\mathrm{mod}\ 5)$$

The first condition only applies to F_5: $F_5 \ (\mathrm{mod}\ 5) \equiv 5(\mathrm{mod}\ 5) \equiv 0$

Here an example for the second condition: $n = 11;\ n \equiv 1(\mathrm{mod}\ 5) \Rightarrow F_{10} \equiv 0(\mathrm{mod}\ 11)$

An example for the third condition: $n = 13;\ n \equiv -2(\mathrm{mod}\ 5) \Rightarrow F_{14} \equiv 0(\mathrm{mod}\ 13)$

Record
The largest currently known Fibonacci prime number is $\boldsymbol{F_{104911}}$. It has 21925 decimal digits. It is still unknown (as of Dec. 2015) whether there are infinitely many Fibonacci prime numbers.

```
Mathematica:
Select[Fibonacci[Range[400]], PrimeQ]
```

4.10.3 META-FIBONACCI SEQUENCES

In meta-Fibonacci sequences, the next sequence member is not calculated directly from the values of the two preceding members but indirectly via offsets or indices. The first meta-Fibonacci sequence occurring in the literature is Hofstadter's famous Q- sequence[28]:

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), n > 2 \quad \text{with } Q(1) = Q(2) = 1 \tag{35}$$

The sequence seems to be rather chaotic at first sight, but it shows generational-like structures. The sequence is still widely unexplored. The first terms are:

1 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 8, 8, 8, 10, 9, 10, 11, 11, 12, 12, 12, 12, 16, 14, 14, 16, 16, 16, 16, 20, 17, 17, 20

Instead of adding the two preceding values, as in the case of the Fibonacci sequence, the two preceding values tell us how far we have to go back in the sequence to get the numbers that we want to add.

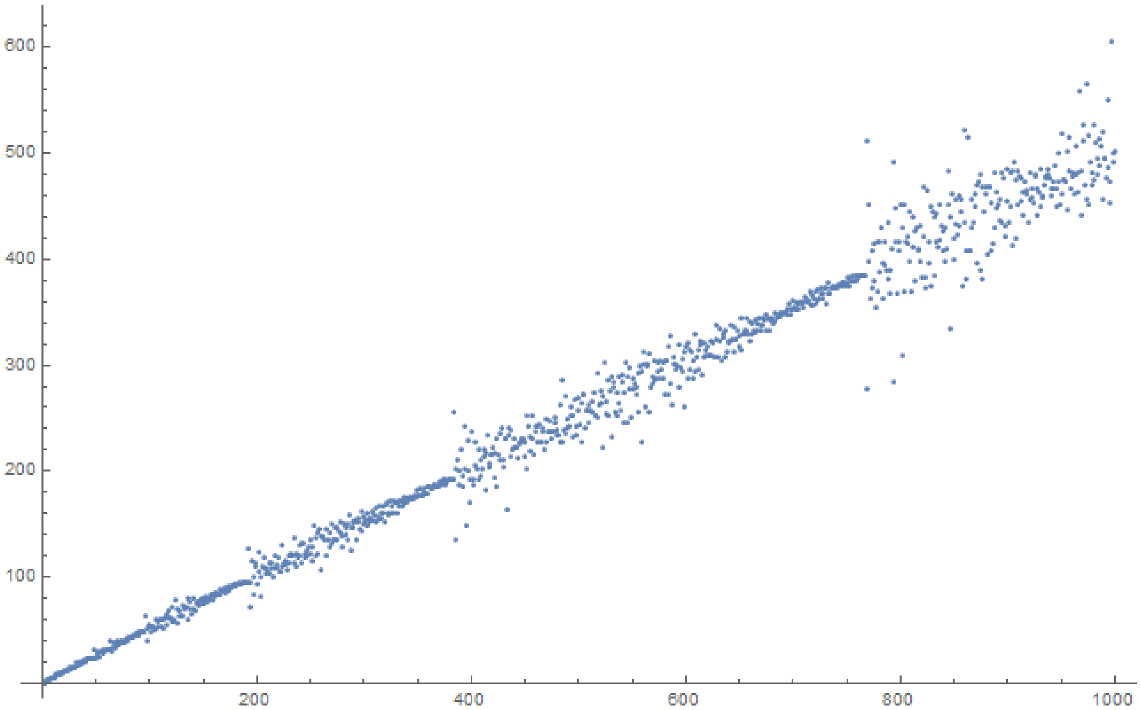


Figure 14. Hofstadter's Q sequence: a meta-Fibonacci sequence

```
Mathematica:
a[1] = a[2] = 1; a[n_] := a[n] = a[n - a[n - 1]] + a[n - a[n - 2]];
ListPlot[Table[{n,a[n]},{n,1,1000}],PlotRange->Full]
```

[28] Hofstadter: Gödel, Escher, Bach p. 149

4.11 CARMICHAEL AND KNÖDEL NUMBERS

A (composite) number n is called a Carmichael number, if:
$a^{n-1} \equiv 1 \pmod{n}$, for all a coprime to n, $a < n$. For the divisors of n the congruence does not hold. The smallest Carmichael number is 561. The prime factor decomposition of 561 is $561 = 3 \cdot 11 \cdot 17$

There are infinitely many Carmichael numbers. Here are all Carmichael numbers up to 10,000:

Carmichael number	Prime factors
561	3·11·17
1105	5·13·17
1729	7·13·19
2465	5·17·29
2821	7·13·31
6601	7·23·41
8911	7·19·67
10585	5·29·73
15841	7·31·73
29341	13·37·61
41041	7·11·13·41
46657	13·37·97
52633	7·73·103
62745	3·5·47·89
63973	7·13·19·37
75361	11·13·17·31

Figure 15. Carmichael numbers up to 10,000

The largest known Carmichael number (as of Dec. 2015) cannot be printed here because it has more than 10 billion prime factors and about 300 million decimal digits (that is, there is only a construction principle).[29] It is easy to prove that every Carmichael number must contain at least three different prime factors and be square-free.

There are construction methods that allow the construction of very large Carmichael numbers. Conversely, it is very difficult and complex to test very large numbers for their Carmichael properties, since they have to be factored for this purpose.

Let $C(n)$ be the number of Carmichael numbers up to a given n. Then the following estimates exist:

$$\boldsymbol{n^{\frac{1}{3}} < C(n) < ne^{\left(-\frac{\ln n \ln\ln\ln n}{\ln\ln n}\right)}}$$

[29] http://math.ucsd.edu/~kedlaya/ants10/poster-hayman.pdf

A generalization of the Carmichael numbers leads to the ‘Knödel’ numbers:

$K_n$denotes the set of composite numbers $\boldsymbol{a^{m-n} \equiv 1 \pmod{m}}$ for all a that are coprime to m and $a < m$. The special case for $n = 1$ results in the Carmichael numbers. Each composite number m is a Knödel number K_n with the property $n = m - \varphi(n)$. The first Knödel sets K_n read:[30]

n	K_n
1	561, 1105, 1729, 2465, 2821, 6601, …
2	4, 6, 8, 10, 12, 14, 22, 24, 26, …
3	9, 15, 21, 33, 39, 51, 57, 63, 69, …
4	6, 8, 12, 16, 20, 24, 28, 40, 44, …

```
Mathematica:
Cases[Range[1, 100000, 2], n_ /; Mod[n, CarmichaelLambda[n]] == 1 && !
PrimeQ[n]]
```

4.12 EMIRP NUMBERS

An emirp number is a prime number that yields a different prime when read backwards. The largest known emirp is (as of Oct. 2015):

$$10^{10006} + 941992101 \cdot 10^{4999} + 1$$

```
Mathematica:
fQ[n_] := Block[{idn = FromDigits@ Reverse@ IntegerDigits@ n}, PrimeQ@
idn && n != idn]; Select[Prime@ Range@ 200, fQ]
```

Curiosities

The following list contains 11 consecutive prime numbers that are all emirps:

$$1477271183, 1477271249, 1477271251, 1477271269, 1477271291, 1477271311, 1477271317, 1477271351, 1477271357, 1477271381, 1477271387$$

4.13 WAGSTAFF PRIME NUMBERS

Wagstaff prime numbers are prime numbers of the form

$$p = \frac{2^q + 1}{3}, \text{where } q \text{ is an odd prime} \tag{36}$$

[30] https://de.wikipedia.org/wiki/Knödel-Zahl

At present, 43 Wagstaff primes p are known (as of Oct. 2015). In red: PRP primes)[31]:

Table 11: Wagstaff prime numbers: exponent q

n	q	n	q
1	3	22	2617
2	5	23	3539
3	7	24	5807
4	11	25	10501
5	13	26	10691
6	17	27	11279
7	19	28	12391
8	23	29	14479
9	31	30	42737
10	43	31	83339
11	61	32	**95369**
12	79	33	**117239**
13	101	34	**127031**
14	127	35	**138937**
15	167	36	**141079**
16	191	37	**267017**
17	199	38	**269987**
18	313	39	**374321**
19	347	40	**986191**
20	701	41	**4031399**
21	1709	42	**13347311**
		43	**13372531**

Wagstaff primes can be calculated using the following Mathematica program:

```
Select[Array[(2^#+1)/3&,190],PrimeQ]
Output:
```

{3,11,43,683,2731,43691,174763,2796203,715827883,2932031007403,
768614336404564651,201487636602438195784363,
845100400152152934331135470251,
56713727820156410577229101238628035243,
62357403192785191176690552862561408838653121833643}

[31] Pseudoprime tests provide PRPs (pseudoprime numbers) and work with probabilistic methods, but they provide reliable statements about primality

Finally, here is a Mathematica program for the prime exponents of the Wagstaff prime numbers:

```
a= {}; Do[c = 1 + Sum[2^(2n - 1), {n, 1, x}]; If[PrimeQ[c],
AppendTo[a, c]], {x, 0, 100}]; a
```

4.14 WIEFERICH PRIME NUMBERS

A prime number satisfying the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ is referred to in the literature as a Wieferich prime. Wieferich, at the beginning of the last century, was the first to explore these numbers. Let us remember the congruence relation of Fermat's little theorem: $2^{p-1} \equiv 1 \pmod{p}$.

This is true for any odd prime. On the contrary, the above Wieferich congruence relationship holds only for very few prime numbers (more precisely, only for two numbers: 1093 and 3511). These are the only Wieferich primes currently known (as of Dec. 2016). It is also known that no further Wieferich primes exist up to $4.968543 \cdot 10^{17}$ (as of Dec. 2015).

The patterns in the binary representation of the value of Wieferich prime numbers (more precisely, in the value reduced by 1) are remarkable:

$$\mathbf{1092 = 10001000100_2, 3510 = 110110110_2}$$

There is a close connection to the so-called 'powerful' numbers P_i, for which the following conditions hold: $p|\, P_i$ and $p^2|\, P_i$.

The first 'powerful 'numbers are: 1,4,8,9,16,25,27,32,36,49 ... They are all of the form a^2b^3 where $a, b \geq 1$. The sum of the reciprocal values of all powerful numbers is finite and has the value:

$$\sum_i \frac{1}{P_i} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.9435964 \tag{37}$$

Primes that do not satisfy the congruence condition $2^p \not\equiv 1 \pmod{p^2}$ are generally referred to as non-Wieferich primes. Both are mutually complementary sets of numbers, i.e. if one of them is finite, then the other must be infinite. There are interesting cross-connections to the *abc* conjecture (Chapter 11.1). The literature about Wieferich primes is very large. There are also many further surprising and interesting connections to Mersenne and Fermat primes, as well as to other areas of number theory.[32]

Wieferich primes can be generalized to have any positive integer basis a if they satisfy the following congruence:

$$a^{p-1} \equiv 1 \pmod{p^2} \tag{38}$$

[32] https://en.wikipedia.org/wiki/Wieferich_prime

The assumption is that there are infinitely many base a Wieferich prime numbers for every natural base a. Here an example: the following base 5 Wieferich primes are known for $a = 5$ (as of Oct. 2016):

$$2, 20771, 40487, 53471161, 1645333507, 6692367337, 188748146801$$

The quotient $q_p(a) = \frac{a^{p-1}-1}{p}$ is generally denoted as the Fermat quotient of p to the base a. The modulo residue of the Fermat quotient $q_p(a)$ has logarithmic properties. If p is not a divisor of ab, then:

$$q_p(ab) \equiv q_p(a) + q_p(b) \pmod{p} \tag{39}$$

More properties:

$$q_p(p-1) \equiv 1 \text{ and } q_p(p+1) \equiv -1 \pmod{p} \tag{40}$$

$$q_p(2) = \frac{1}{p}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{p-1}\right) \pmod{p} \tag{41}$$

Mathematica programs for computing Wieferich primes:

```
Select[Prime[Range[50000]], Divisible[2^(#-1)-1, #^2]&]
Select[Prime[Range[50000]], PowerMod[2, #-1, #^2]==1&]
```

There are other interesting cross-links to other areas of number theory:

To Fermat's theorem:

Let be $k^p + l^p + m^p = 0$ (where $k, l, m \in \mathbb{Z}$ and $p \in \mathbb{P}$). Further we assume: p is not a divisor of the product klm. Then p is a Wieferich prime.

To Mersenne primes and Fermat primes:

Let M_q be a Mersenne number with primal index q and p be a prime number which is a divisor of M_q. If p^2 is also a divisor of M_q, then M_q (and only then) p is a Wieferich prime number.

Double Wieferich primes:

A pair of generalized Wieferich primes, for which

$$q^{p-1} \equiv 1 \pmod{p^2} \text{ and } p^{q-1} \equiv 1 \pmod{q^2} \tag{42}$$

is called a 'double Wieferich prime'. Here is an example:

$(83, 4871)$ are double Wieferich primes.
Note: Catalan's conjecture has been proven using properties of double Wieferich prime numbers (see 20.1). Many interesting monographs on Wieferich primes can be found on well-known websites. Here is a table of some base *a* Wieferich primes that are known (as of Oct. 2016):

Table 12. Generalized Wieferich primes

Base	Wieferich primes with base
1	2,3,5,7,11,13,17,19,23,29,… (all prime numbers)
2	1093,3511
3	11,1006003
4	1093,3511
5	2,20771,40487,53471161,1645333507,6692367337,188748146801
6	66161,534851,3152573
7	5,491531
8	3,1093,3511
9	2,11,1006003
10	3,487,56598313
11	71
12	2693,123653
13	2,863,1747591
14	29,353,7596952219
15	29131,119327070011
16	1093,3511
17	2,3,46021,48947,478225523351
18	5,7,37,331,33923,1284043
19	3,7,13,43,137,63061489
20	281,46457,9377747,122959073
21	2
22	13,673,1595813,492366587,9809862296159
23	13,2481757,13703077,15546404183,2549536629329
24	5,25633
25	2,20771,40487,53471161,1645333507,6692367337,188748146801
26	3,5,71,486999673,6695256707
27	11,1006003,
28	3,19,23
29	2
30	7,160541,947270757
31	7,79,6451,2806861
37	2,3,77867,76407520781
41	2,29,1025273,138200401
43	5,103,13368932516573
47	???
53	2,3,47,59,97
59	2777,18088417183289
61	2
67	7,47,268573
71	3,47,331
73	2,3
79	7,263,3037,1012573,60312841,8206949094581
83	4871,13691,315746063
89	2,3,13
97	2,7,2914393,76704103313

4.15 WILSON PRIME NUMBERS

A prime number satisfying the congruence $(p-1)! \equiv -1 \pmod{p^2}$ is referred to in the literature as a 'Wilson prime'.
Currently, only three Wilson prime numbers are known (Nov. 2016). These are:

$$\mathbf{5, 13 \text{ und } 563}$$

If further Wilson prime numbers exist, these must be larger than $2 \cdot 10^{13}$ (as of Oct. 2016). It is generally believed that there are an infinite number of Wilson prime numbers.
The quotient $W(p)$ is called the Wilson quotient:

$$W(p) = \frac{(p-1)!+1}{p} \tag{43}$$

Wilson's theorem states that all prime numbers p match the congruence

$$(p-1)! \equiv -1 \pmod{p}$$

Here is a plot of the Wilson quotients of the first 100 prime numbers:

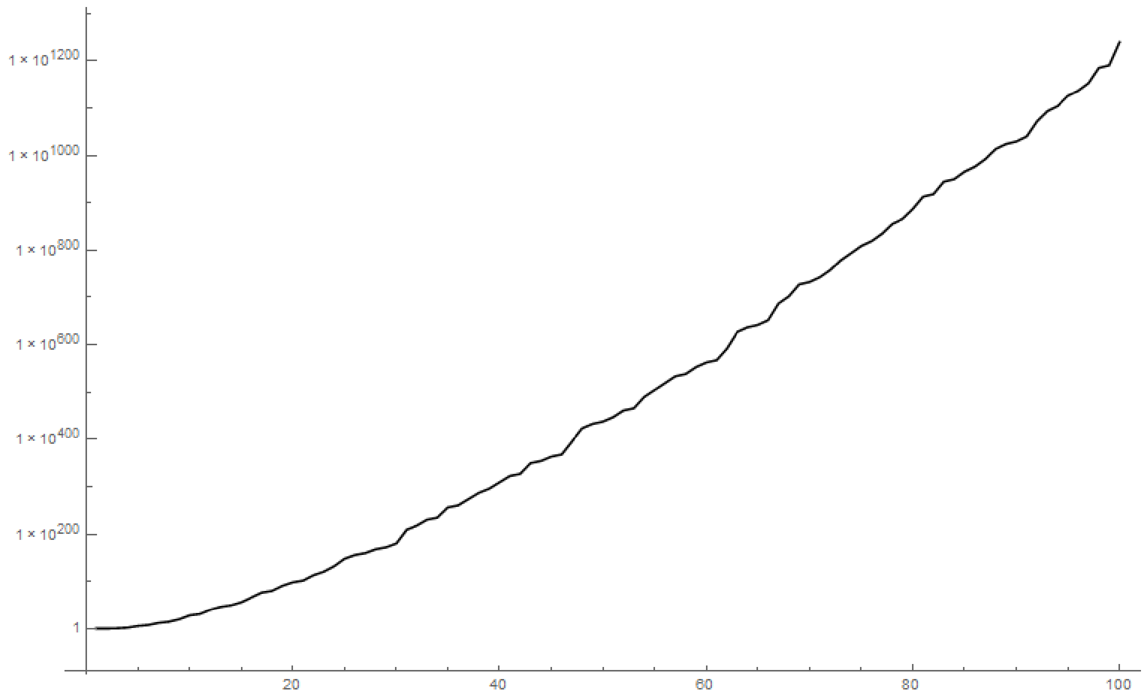


Figure 16. Wilson quotients of the first 100 prime numbers (log representation)

```
Mathematica:
WilsonQuotients=Table[((Prime[i]-1)!+1)/(Prime[i]),{i,1,100}];
ListLogPlot[WilsonQuotients,Joined->True,PlotStyle->Black]
```

For Wilson prime numbers there also exist generalizations, which are described in the literature.

4.16 WOLSTENHOLME PRIME NUMBERS

A prime number satisfying the following congruence is called a Wolstenholme prime number:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4} \tag{44}$$

Please note that according to the Wolstenholme theorem for each prime $p > 3$ the following obtains:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3} \tag{45}$$

The only currently known Wolstenholme prime numbers are (as of Oct. 2016):

16843 and **2124679.**

If there are further Wolstenholme prime numbers, these are greater than 10^9. It is assumed that there are infinitely many Wolstenholme prime numbers.

4.17 RG NUMBERS (= RECURSIVE GÖDELIZED)

4.17.1 GOCRON TYPE 6 ('PRIME OCRONS')

For this we need the definition of the GOCRON type 6 (Prime OCRON, with Gödel codes '*'=0 and 'P'=1, see Chapter 10.2.4).
Let an **RG sequence** in the direction of positive indices be recursively defined as follows:
$\boldsymbol{a(0) = m}$ **(with any integer number $\boldsymbol{m >= 1}$)**
$\boldsymbol{a(n+1) = \mathrm{EGOCRON6}(a(n))}$

The sequence can also be continued in the direction of negative indices:
$\boldsymbol{a(0) = m}$ **(with any integer number $\boldsymbol{m >= 1}$)**
$\boldsymbol{a(n-1) = \mathrm{INVEGOCRON6}(a(n))}$

Here are a few examples:
0,1,2,4,8,14,**9**,19,67,401,409,1103,305999,210535619933…
3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3…
5,**7**,5,7,5,7,5,7,5,7,5,7,5,7,5,7,5…
6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6…
10,**12**,10,12,10,12,10,12,10,12,…
13,13,13,13,13,13,13,13,13,13,…

15,11,17,43,157,2833,3463,59723,4251697,97152271…
20,20,20,20,20,20,20,20,20,20,…
21,37,107,367,37217,363343,30612065639…
25,29,23,59,83,353,379,20719,448693727…
31,31,31,31,31,31,31,31,31,31,…
27,41,71,1153,769,349,8861,5065217,22920311…
33,79,2221,271003,680328533…
35,191, 15299,649093,50511459839…
39, 331,3559,1804973,50220857249
45,101,181,751,304553,627544381
91,547,4463,48266149

Here are a few examples in the direction of positive indices:
15,28,18,30,40,36,96,168,424,12544,6845104128,351820914765360116269056
21,26,22,16,24,56,72,84,122,928,108544,15903336184152064
25,48,64,208,656,1968,116992,30889404792832
27,60,34,58,100,352,3872,16016,73764,503296,360710432,2177877733799636238336
35,112,118,456,2368,6704,10250,25128,1001488,5575424,131365666816,
129617351244588913891122077503488
39, 120, 512, 4032, 6586368, 1065152675904

RG sequences of the 'Prime GOCRONs' have the following properties:
1) They are either periodic or they diverge to infinity in the case of positive and negative indices.
2) An exception is the 9 sequence; it goes towards infinity for negative indices and is not defined for positive indices.

3) Every non-periodic sequence has a 'centre' that lies in the neighbourhood of the minimum. It is the only odd composite number in this sequence. Thus, the sequences can be named after this single central odd number. These are the numerical values marked in red.
4) In the direction of positive indices, there is a strong tendency to increase the degree of 'compositeness'. Towards negative indices there is a tendency to increase 'primality'. This is expressed by the fact that the ‘centre element’ (which is composite and uneven) is the index at which a ‘turn over’ occurs, from the status ‘prime’ to the status ‘composite’ (or vice versa, depending upon the direction from which one is coming).
5) The following RG sequences are periodic (classified according to the central odd composite element):
   3,5,6,7,10,12,13,20,31,61,97,250,457,41112

RG sequences of 'prime OCRONs' (if they are not periodic) come out from infinity as prime numbers, 'hang around' a little bit among the 'finite' numbers, change (turn over) to composite numbers at exactly one index, stay ‘even’ from there on and then disappear again into the infinite. Here are a few plots of RG sequences (with the “transformation point” marked in red, to the left of it: prime numbers, to the right of it: composite, even numbers):

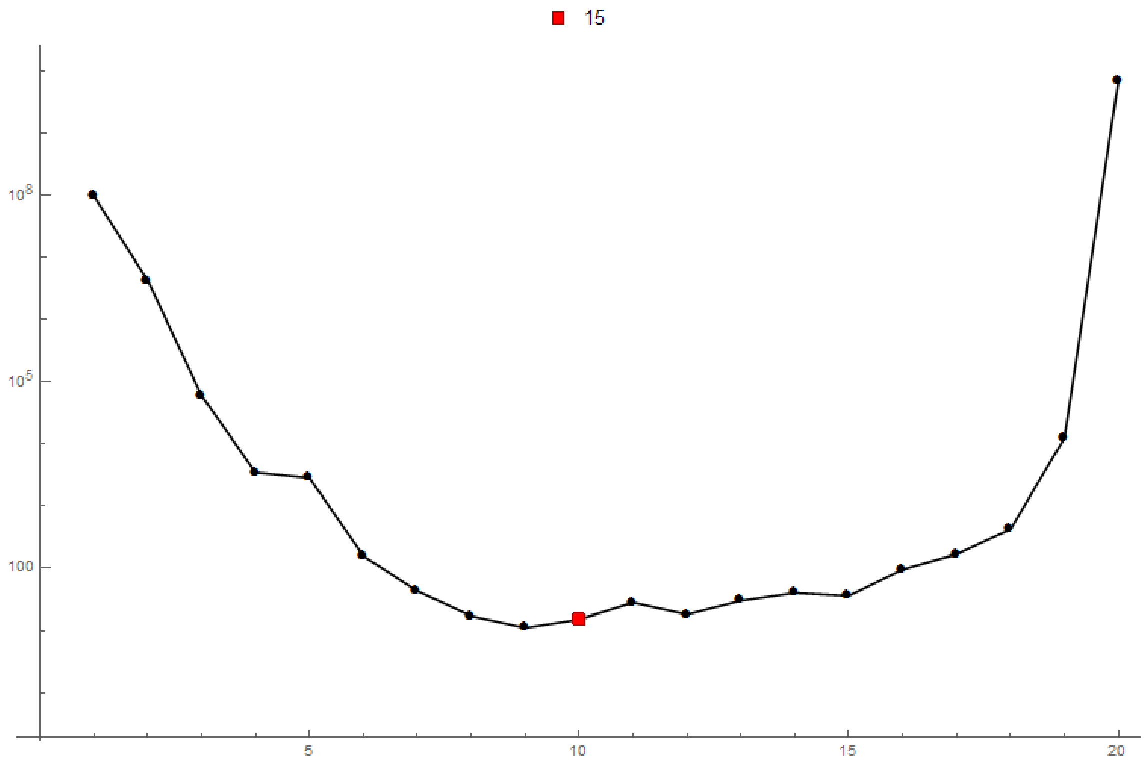


Figure 17. RG sequence ‘15’ (log plot)

Sequence:
97152271,4251697,59723,3463,2833,157,43,17,11,**15**,28,18,30,40,36,96,168,424,12544,6845104128

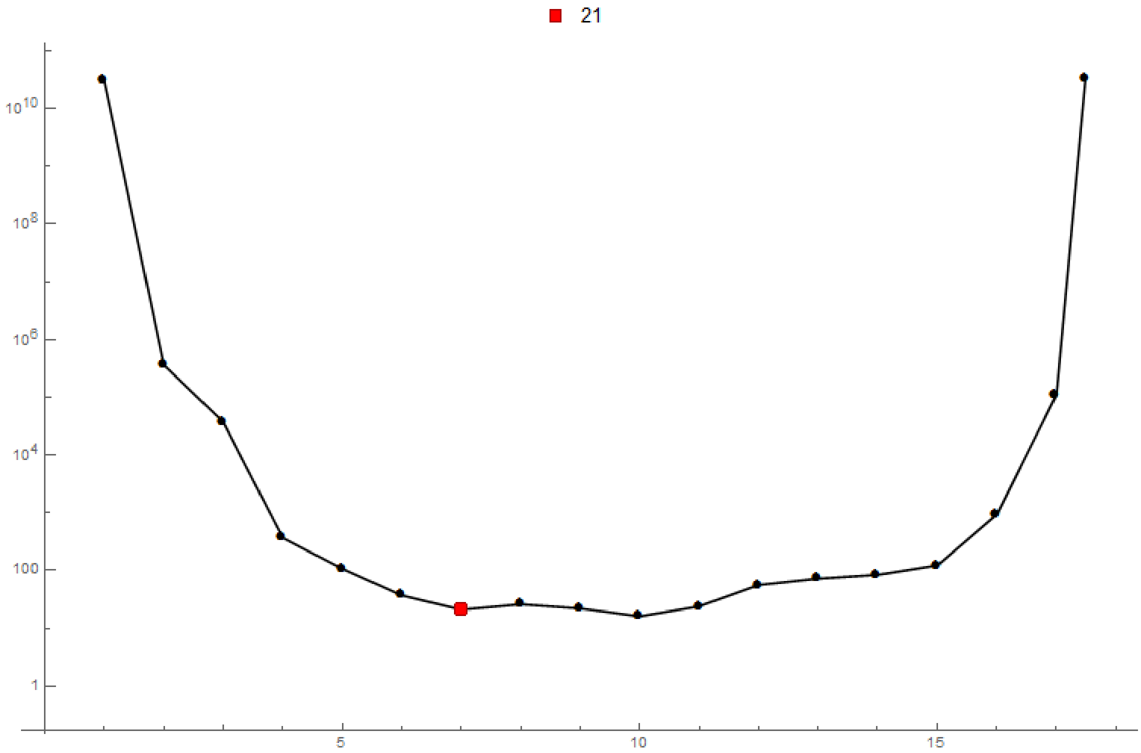


Figure 18. RG sequence '21' (log plot)
30612065639,363343,37217,367,107,37,**21**,26,22,16,24,56,72,84,122,928,10
8544,15903336184152064

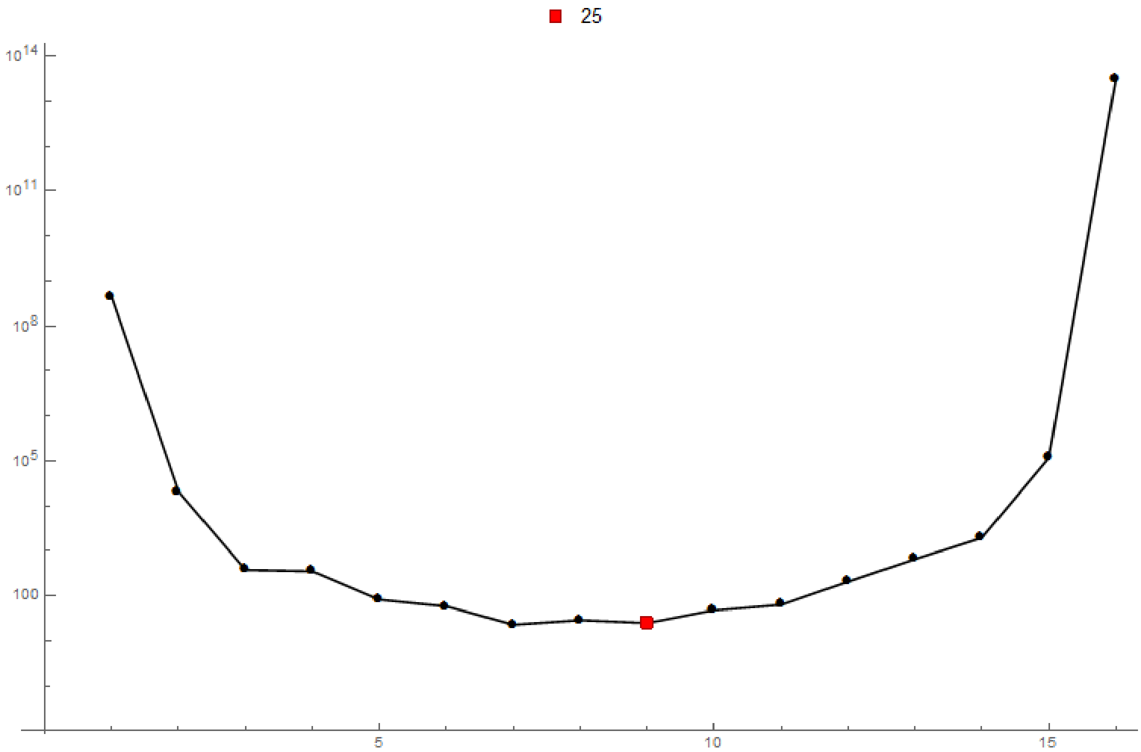


Figure 19. RG sequence '25' (log plot)
448693727,20719,379,353,83,59,23,29,**25**,48,64,208,656,1968,116992,30889
404792832

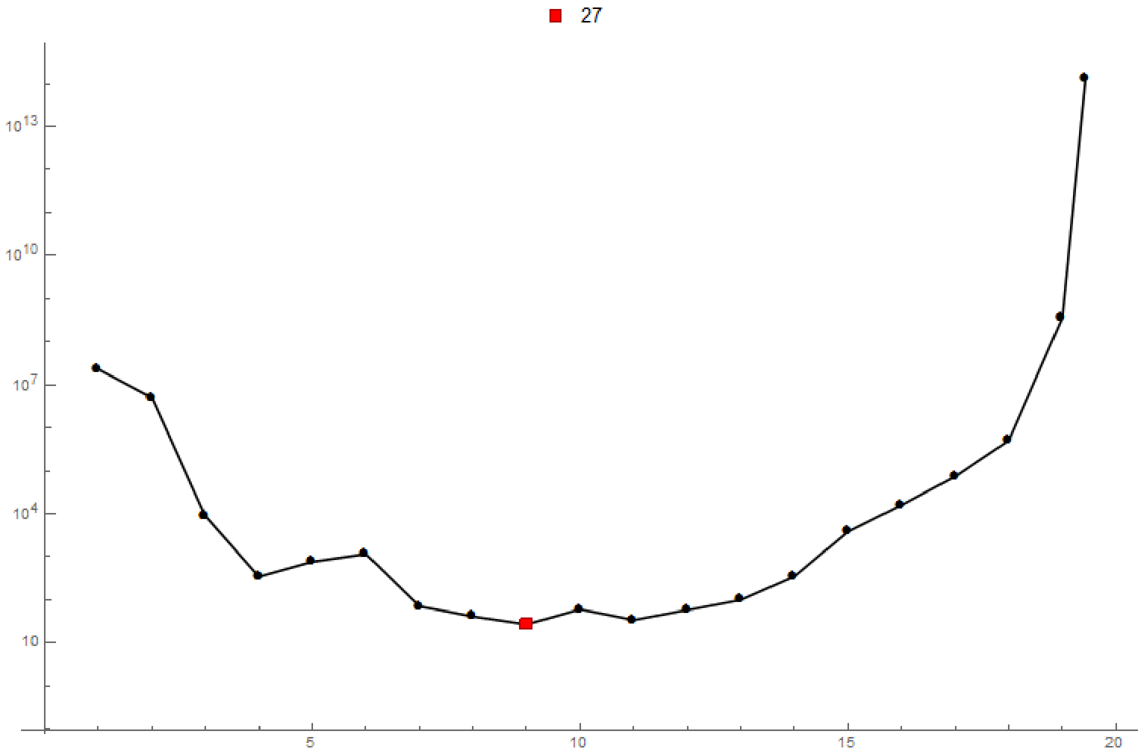


Figure 20. RG sequence ‘27’ (log plot)
22920311,5065217,8861,349,769,1153,71,41,**27**,60,34,58,100,352,3872,1601 6,73764,503296,360710432,21778777337996362383336

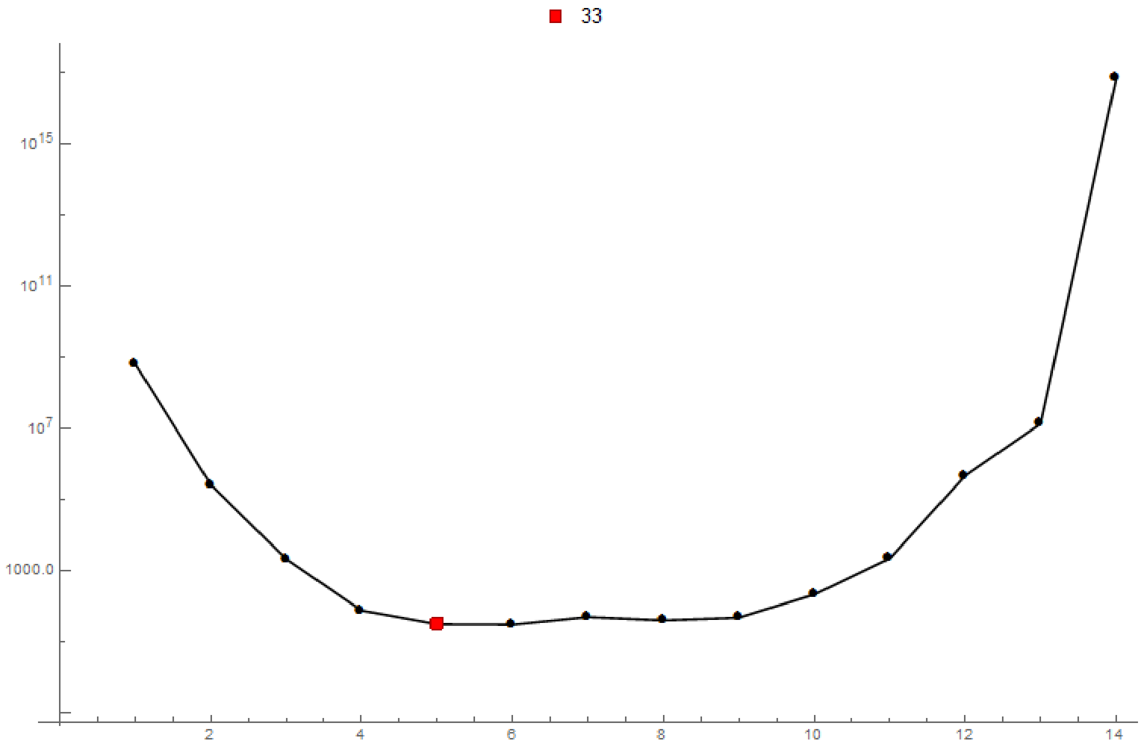


Figure 21. RG sequence ‘33’ (log plot)
680328533,271003,2221,79,**33**,32,52,42,50,224,2304,491776,14160388,70967 016210563072

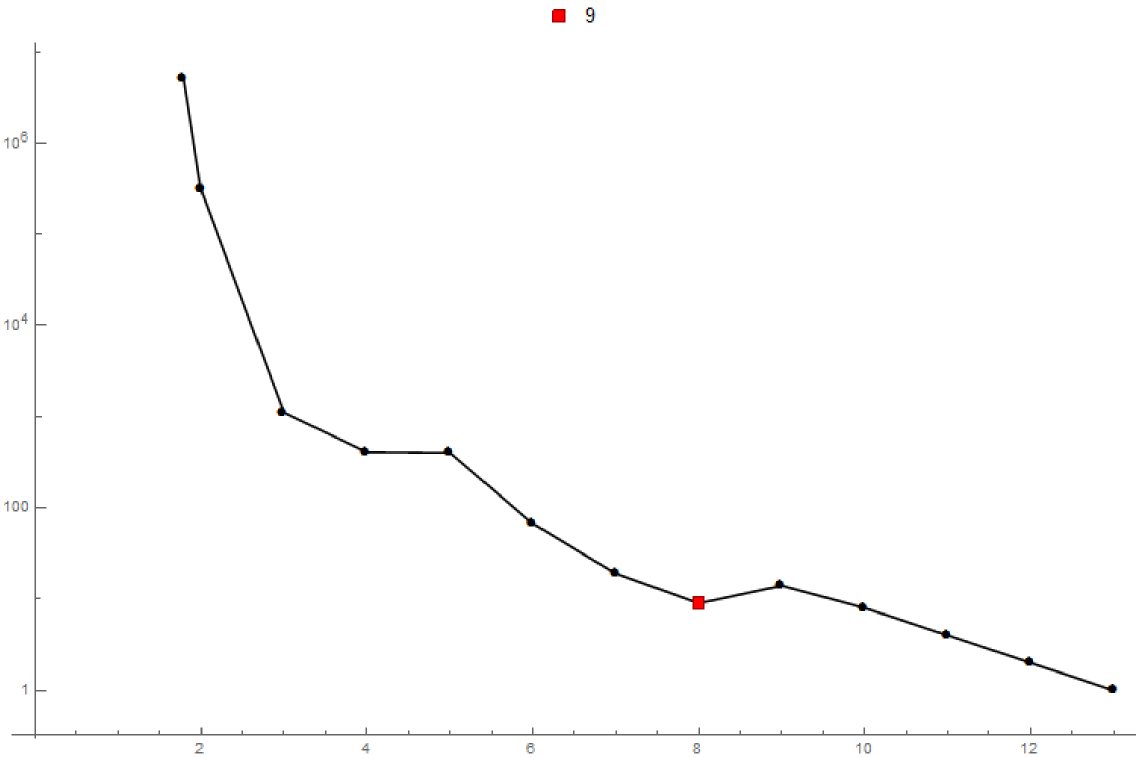


Figure 22. RG sequence '9' (log plot)
210535619933,305999,1103,409,401,67,19,**9**,14,8,4,2,1,0

The application of this recursive rule divides up the set of natural numbers into classes. All numbers of a class end up in the manner described above sooner or later. The similarity with the situation in the aliquot sequences is striking. One could speak here of 'related' numbers that build up a family, as in the case of the aliquot sequences (see Chapter 20.9.2.2).

4.17.2 GOCRON TYPE 4 (WITH THE SYMBOLS ,'2','*','P' AND '^')

For this we need the definition of the GOCRON type 4 (with the Gödel codes '*'=0, 'P'=1, '2'=2 and '^'=3, see Chapter 10.2.2).
Let a **RG sequence** in the direction of positive indices be recursively defined as follows:
$\boldsymbol{a(0) = m}$ **(with any integer number $\boldsymbol{m}$ >= 1)**
$\boldsymbol{a(n+1) = \mathrm{EGOCRON4}(a(n))10}$

The sequence can also be continued in the direction of negative indices:
$\boldsymbol{a(0) = m}$ **(with any integer number $\boldsymbol{m}$ >= 1)**
$\boldsymbol{a(n-1) = \mathrm{INVEGOCRON4}\big(a(n)\big)}$

RG sequences of the 'type 4 EGOCRONs' have the following properties:
1) **In the direction of negative indices (n->EGOCRON4), the RG sequence grows faster than exponentially (see Figure 23) for all initial values > 2.**
2) **In the direction of positive indices (EOCRON4-> n), each RG sequence ends up with the constant value 6. Before this happens, however, the sequence can**

attain astronomically high values before finally ending up on value 6. There is an assumption that is based on the empirical data but it is as yet unproved.

3) **There is always a 'turn over' value, from which point on all sequence members remain even.**

Here are a few plots of RG sequences in the direction of negative indices using different initial values:

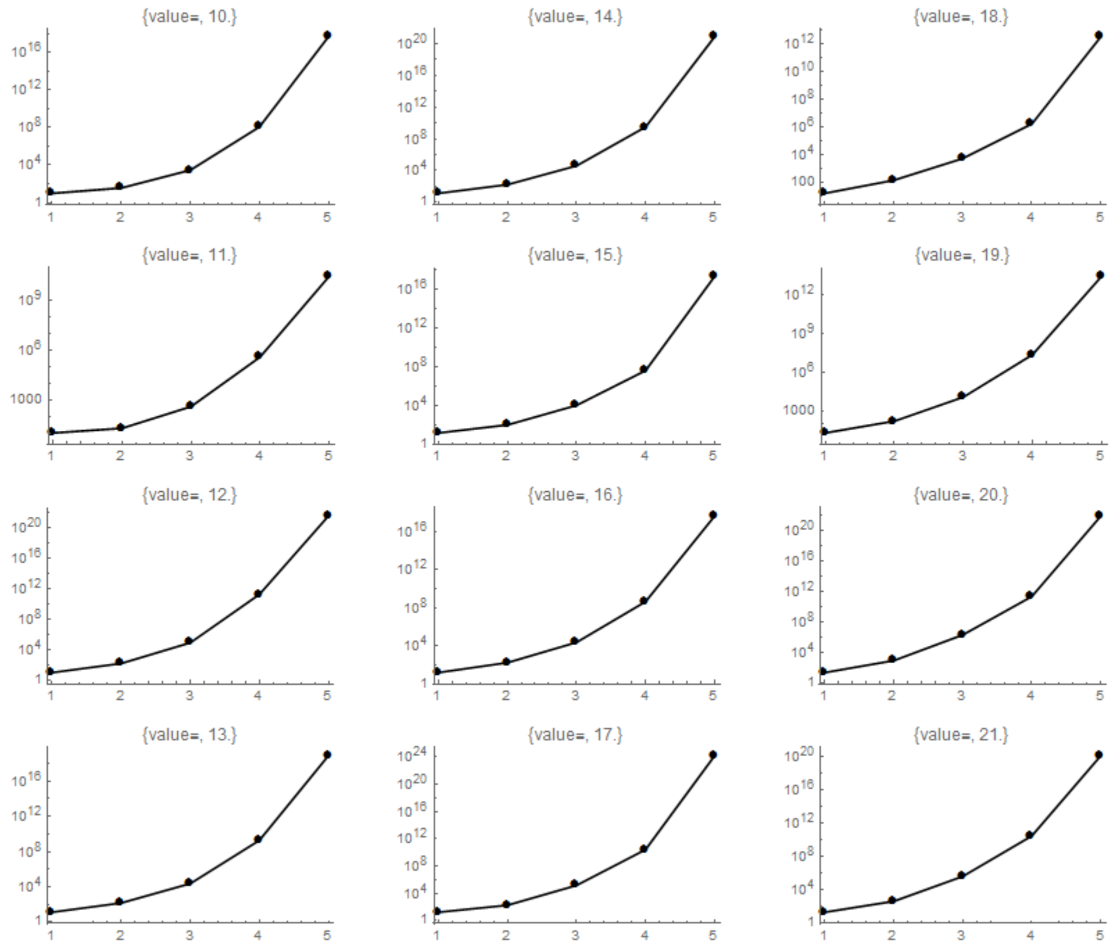


Figure 23. RG sequences in negative direction (type EGOCRON4)

Here are a few graphs of RG sequences in the direction of positive indices with different initial values (value of 'turn over' index, from which all values remain straight, is indicated):

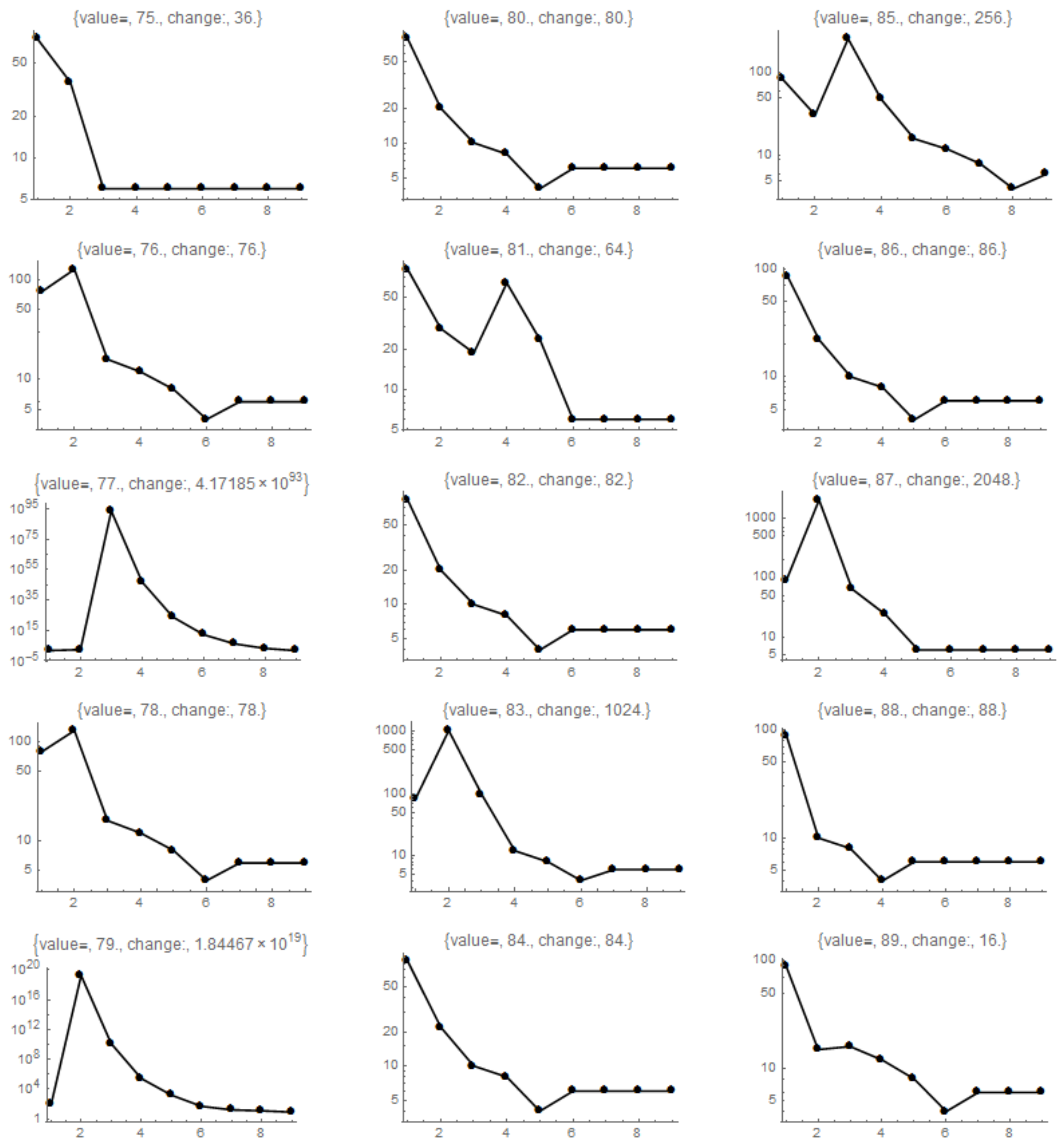


Figure 24. RG sequences in positive direction (type EGOCRON4): always ending up with 6

Further illustrations on this topic can be found in the Appendix 20.7.

The Mathematica programs used to create the graphics can also be found in the Appendix.

The same applies as in the previous chapter: the application of this recursive rule splits up the set of natural numbers into classes. All the numbers of one class are "friends" and end in the same way.

5 DIGRESSION: RIEMANN'S ZETA FUNCTION $\zeta(s)$

5.1 GENERAL

The Riemann zeta function is one of the mysteries of mathematics. Its zeros are especially puzzling. There are the so-called 'trivial' zeros in the real domain. These zeros all lie at even negative integer values $-2, -4, -6, \dots, (-2n)$. But there are infinitely many zeros in the complex domain, all of which lie on the so-called 'critical' line Re (s) = 1/2. There is no simple formula for the position of these zeros. They are seemingly chaotic and randomly distributed and therefore possess similarly mysterious properties as the prime numbers. In fact, the location of the non-trivial zeros is very closely related to the distribution of the prime numbers (see e.g. Chapter 8.6, Formula (131)).
Entire books have been devoted to a discussion of the properties of the zeta function, so we will not discuss them further here. In simple terms, one can say that from the knowledge of the non-trivial zeros, the position (and distribution) of the prime numbers can be calculated and vice versa. This relationship, however, is not a simple one-to-one relationship between zeros and prime numbers but rather resembles a transformation (such as the Fourier transform, which establishes the connection between the time domain and the frequency domain). We could speak here of different domains such as the 'prime number domain' and the 'zeta domain'.

We need to clarify here: the statement that all non-trivial zeros lie on the critical straight line Re (s) = 1/2 (this is the famous Riemann conjecture) has neither been proved nor disproved. However, it is probably true: the numerical evidence in its favour is overwhelming. Although it is so easy to formulate, the 'Riemann conjecture' has so far resisted all attempts (including those of the greatest mathematicians) at proof! For example, it might theoretically happen that there are zeros in unimaginably high number regions that lie apart from the critical line… The history of mathematics has taught us that we cannot always trust in the numerical evidence. Just consider the fact that the asymptotic formula for the prime counting function Li (x) ≈ π (x) always yields too large values[33]. This is certainly true up to 10^{26}, but it has been proved by Skewes (1933) that $Li(x) < \pi(x)$ can happen! Skewes showed that the sign of $Li(x) - \pi(x)$ changes infinitely often, and he proved that the point of the first sign change is less than $10^{10^{10^{34}}}$! This upper limit has now been considerably reduced to $e^{727,95133}$.

Nevertheless, no mathematician believes that Riemann's conjecture is wrong! However, the fact that it has not yet been proved, despite the enormous efforts that have been made, leads many to suppose that Riemann's conjecture belongs to the category of unprovable mathematical propositions. It has been known, at least since the time of Gödel, that there are mathematical propositions that are true but not provable, and Gödel, in fact, proved this. This provides plenty of material for philosophical speculation. Why did God, when he created mathematics and the numbers, also create rules such that something could be 'true' without a compelling reason? For if there were a compelling reason, there would also be a proof…

33 https://en.wikipedia.org/wiki/Prime-counting_function

For enthusiasts: a few special values of the zeta function that can be represented by explicit formulae:

$$\boldsymbol{\zeta(2) = \frac{\pi^2}{6}}$$

$$\boldsymbol{\zeta(3) = \frac{5}{4}\mathrm{Li}_3\left(\frac{1}{\tau^2}\right) + \frac{1}{6}\pi^2 \ln\tau - \frac{5}{6}\ln^3\tau}$$

$$\boldsymbol{\zeta(3) = \cfrac{6}{d(0) - \cfrac{1^6}{d(1) - \cfrac{2^6}{d(2) - \cfrac{3^6}{d(3) - \ddots}}}}}$$

(where $d(n) = 34n^3 + 51n^2 + 27n + 5$)

(with the value of the Golden ratio $\boldsymbol{\tau}$: $\frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}$ as well as $\mathrm{Li}_3(x)$: the polylogarithm function of the 3rd order)[34]

The so called prime zeta function $P(s) = \sum_{p\ prime} \frac{1}{p^s}$

can easily be calculated from the 'normal' zeta function:

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln\ \zeta(ns) \tag{46}$$

The first 20 non-trivial zeros of the zeta function along the critical line (with an accuracy of 10 decimal digits):

```
{14.13472514, 21.02203964,    25.01085758, 30.42487613,
32.93506159, 37.58617816, 40.91871901, 43.32707328,
48.00515088, 49.77383248, 52.97032148, 56.44624770,
59.34704400, 60.83177852, 65.11254405, 67.07981053,
69.54640171, 72.06715767, 75.70469070,77.14484007}
```

```
Mathematica code:
Table[N[Im[ZetaZero[n]],10],{n,20}]
```

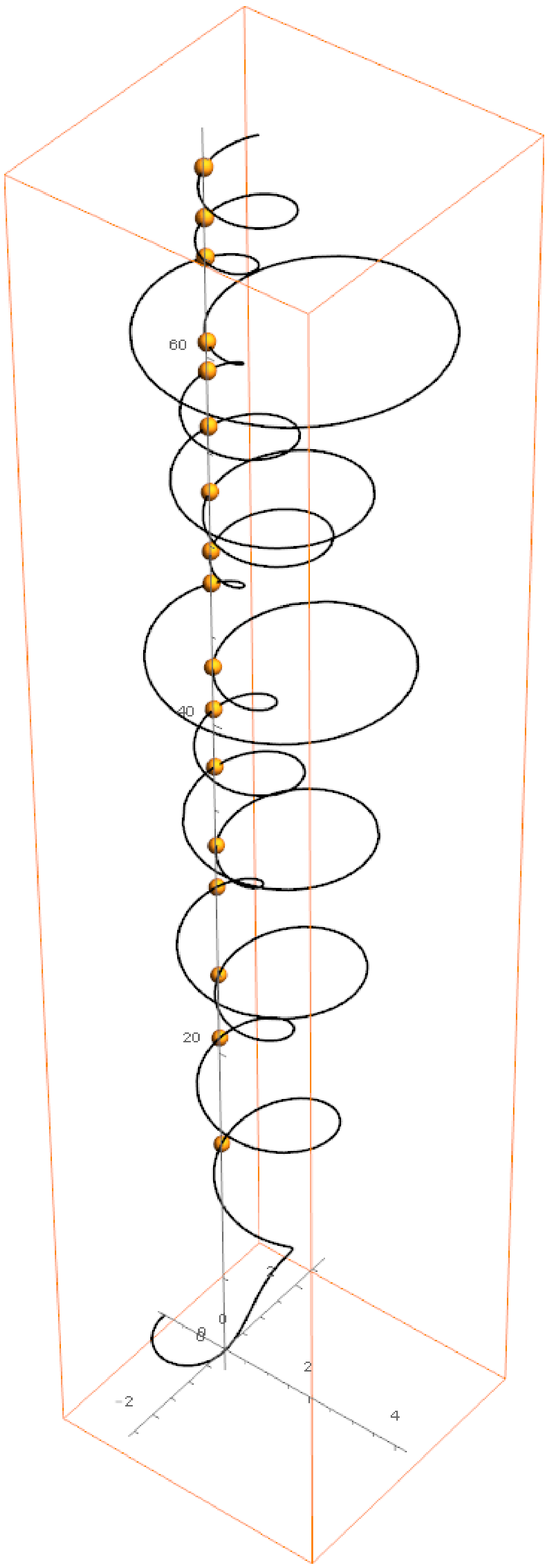
Along the critical line, it is practical to split up the zeta function as follows:

$$\boldsymbol{\zeta\left(\frac{1}{2} + i\,t\right) = Z(t)e^{-i\vartheta(t)}} \tag{47}$$

(where $Z(t)$ and $\vartheta(t)$ are the Riemann-Siegel functions).

[34] Journal of Computational and Applied Mathematics 121 (2000) pp. 247-296

Many books have been written about the zeta function – see, for example, Edwards (1974) and Sautoy (2004) in the Bibliography.
Note: the Riemann zeta function has generalizations (for example, the 'Hurwitz' or 'Lerch' zeta functions, which are mostly named after their discoverers). Of these generalized zeta functions, at least twelve versions are mentioned in the mathematical literature.



The zeta function with zeros as a parametric 3D plot.
The zeta function along the critical line is complex-valued. The critical line goes upwards and the complex function value moves in the x-y plane. The zero points are marked as small spheres. The zeta function spirals clockwise upwards and intersects the vertical z-axis at the zero points.
The Mathematica program for creating the graphics is given in the Appendix under 'Riemann's zeta function'.

Figure 25. Parametric 3D plot (real and imaginary parts) of the zeta function including zeros

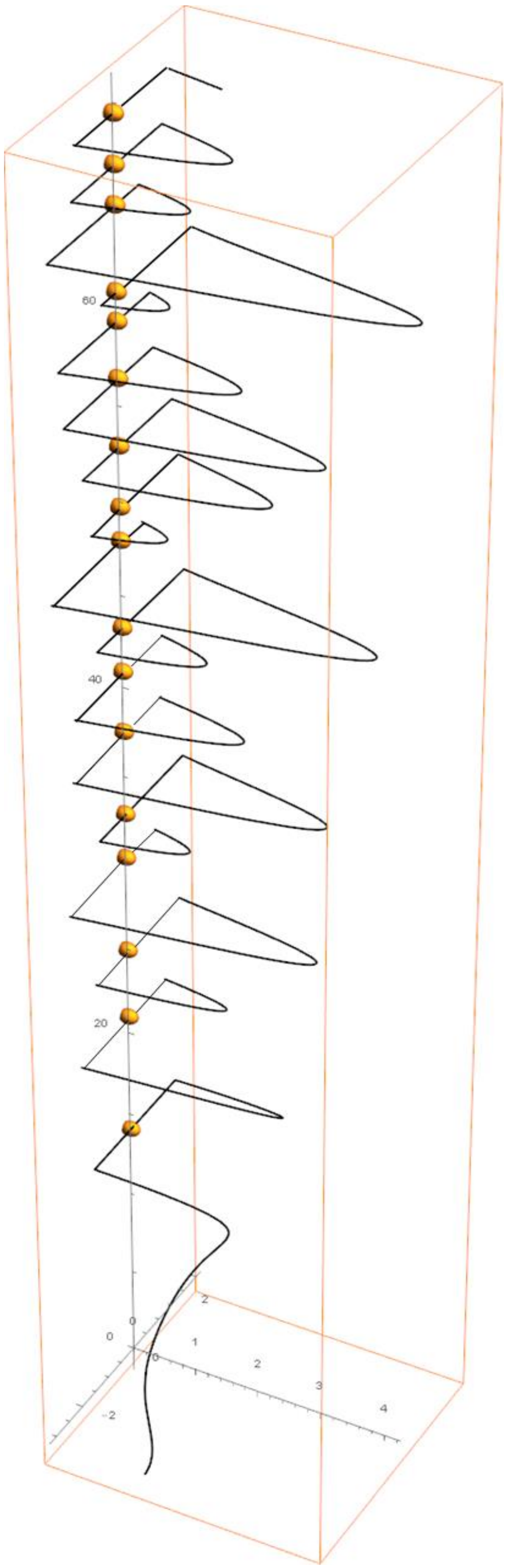


Figure 26. Parametric 3D plot (absolute and argument parts) of the zeta function including zeros

The next graph shows, as Figure 26, the zeta function along the critical line (red: absolute value, black: phase (argument), as a 2D plot. The argument of the zeta function is closely related to the Riemann-Siegel function $\vartheta(t)$ (see (47)).

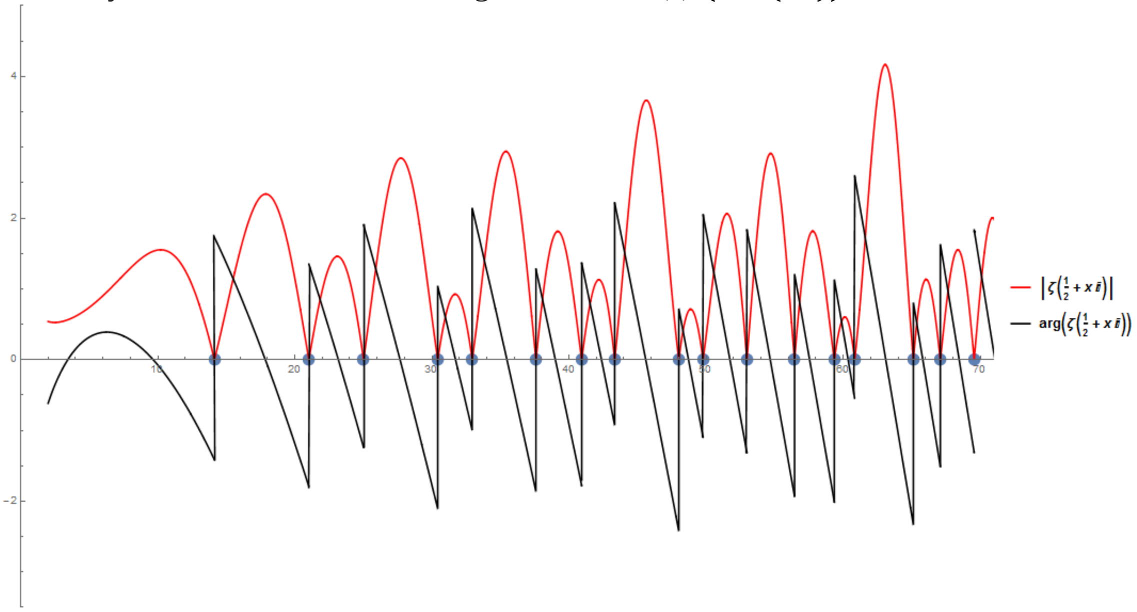


Figure 27. Absolute value and phase of the zeta function along the crit. line (0-70)

The phase of the zeta function jumps around the zeros by the value $+\pi$. The phase of a function is defined only in the interval $[-\pi, +\pi]$. Therefore, its values are limited to this range. The Riemann-Siegel function $\vartheta(t)$, however, describes a continuous phase. Because of the ambiguity of $e^{-i\vartheta(t)}$ the same values are obtained. It can be said that the continuous pieces of the phase of the zeta function can be brought to coincide with the continuous Riemann-Siegel function $-\vartheta(t)$ along the critical straight line by shifting along the y-axis. This is demonstrated in the following graph between the first and second zero (between 14 and 21).

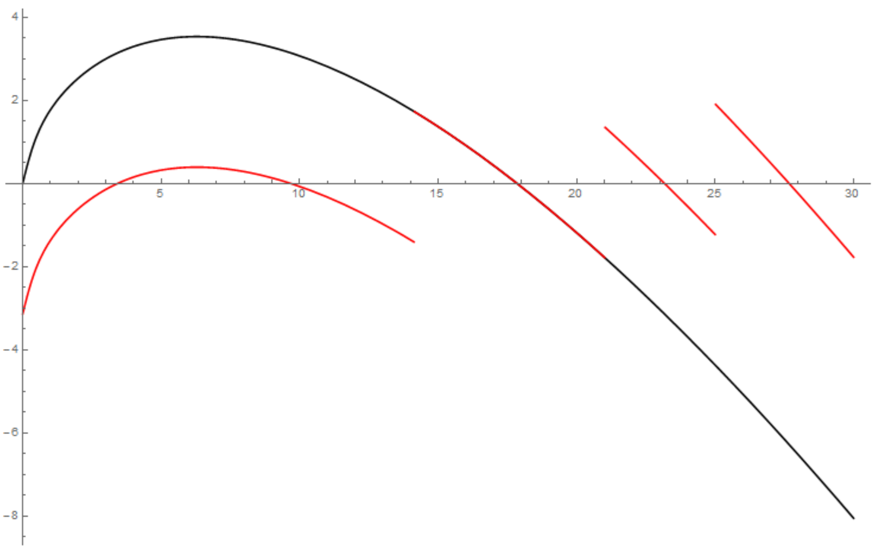


Figure 28. Comparison of the phase of the zeta function with the (negative) Riemann-Siegel function.

Surprisingly the Riemann-Siegel function $\vartheta(t)$ can be calculated without knowledge of the zeta function, only with the aid of the gamma function. It is quite 'inconspicuous', but it has jumps because the arguments can only have values between π and $-\pi$:

$$\vartheta(t) = \mathrm{Im}\left(\ln\left(\Gamma\left(\frac{1}{4}+\frac{i\,t}{2}\right)\right)\right) - \frac{t}{2}\ln\pi = \mathrm{Arg}\left(\Gamma\left(\frac{1}{4}+\frac{i\,t}{2}\right)\right) - \frac{t}{2}\ln\pi \tag{48}$$

Note: for calculation with Mathematica, the normal Γ function should not be used to calculate the term $\ln(\Gamma(\dots))$. This should be done by the function 'LogGamma'. The reason for this is that the branch structure in the complex domain is more complicated for the normal gamma function, and only the main value of the logarithm would be obtained. The LogGamma function overcomes this problem.
If we normalize the phase of $\zeta(\frac{1}{2}+it)$ by the factor $\frac{1}{\pi}$, we get a jump of $+1$ at each zero. If we do the same with $-\vartheta(t)$ and subtract these two functions from each other (and add 1), we get a counting function for the zeros of the zeta function! More details can be found in the Chapter 5.5.

5.2 THE DIFFERENT REPRESENTATIONS OF $\zeta(s)$

A book about primes must include the most important representations of the zeta function.
First of all, the original definition of the zeta function is extremely simple, so people can understand it without necessarily having read mathematics:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\mathrm{Re}(s) > 1) \tag{49}$$

Already Euler proved in the 18th century that $\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$.
For all even positive arguments, there is the simple formula:

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} \quad (B_n\text{: Bernoulli} - \text{numbers}, \mathrm{n} = 1,2,\dots) \tag{50}$$

For the positive odd numbered arguments there are also formulae; these, however, are somewhat more complex, e.g. :

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} \tag{51}$$

For negative integer arguments:

The different representations of $\zeta(s)$

$$\zeta(1-n) = -\frac{B_n}{n} \tag{52}$$

Some examples: $\zeta(0) = -\frac{1}{2}, \zeta(-1) = -\frac{1}{12}, \zeta(-3) = -\frac{1}{120}$

The product formula (from which elementary methods (49) can be deduced) is also easy to understand:

$$\zeta(s) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\dots} = \prod_{p\in\mathbb{P}}^{\infty} \frac{1}{\left(1-\frac{1}{p^s}\right)} \ (Re(s) > 1) \tag{53}$$

Most notable is a theorem proved by the Russian mathematician Voronin that the zeta function can approximate any other function with arbitrary precision (more precisely, every holomorphic complex function within an area with radius $\frac{1}{4}$, without zeros).

Descriptively speaking: every complex-valued function, however chaotic and however complicated its landscape may be, with all the 'hills' and 'valleys', will also appear ‘somewhere’ in the landscape of the complex zeta function, if one only searches far enough in the infinite landscapes of the zeta function …

The derivative of the zeta function is closely connected with the Von Mangoldt function $\Lambda(n)$:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \tag{54}$$

$$\zeta'(0) = -\frac{1}{2}\ln 2\pi \tag{55}$$

$$\zeta'(-2n) = (-1)^n \frac{\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}} \tag{56}$$

More formulae:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ (with Moebius function } \mu(n)) \tag{57}$$

Here a representation using integrals:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \frac{1}{\Gamma(s)} \int_{n=0}^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{58}$$

Further representations with products (Hadamard product):

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} (1 - \frac{s}{\rho}) \tag{59}$$

Because of the conditional convergence of this formula (the evaluation of the product over the terms with the zeros of the zeta function ρ must be done in pairs), one can also write:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{Im(\rho)>0} (1 - \frac{s}{\rho})\,(1 - \frac{s}{1-\rho}) \tag{60}$$

5.3 PRODUCT REPRESENTATION OF $\zeta(s)$ IN THE COMPLEX DOMAIN

Equation (53) describes the product representation of the zeta function. It also holds in the complex domain, but converges only for $Re(s) > 1$. For this reason, it is better to avoid using this formula in the region of the critical line in which the non-trivial zeros lie. What happens if we do the 'forbidden' anyway? Do we then cross a boundary beyond which the serious mathematician should not stray? Do we risk not being taken seriously? In the spirit of mathematical adventure, let us look and see what happens if we do the 'forbidden' anyway:

We use Formula (53) to calculate the values along the critical line:

$$\zeta\left(\frac{1}{2} + t \cdot i\right) = \prod_{p \in \mathbb{P}}^{\infty} \frac{1}{\left(1 - p^{-\frac{1}{2} - t \cdot i}\right)} \quad \text{where } t \geq 0$$

The first thing to notice is that the term $\frac{1}{(\dots)}$ in the infinite product can never be zero. Nevertheless, $\zeta\left(\frac{1}{2} + t \cdot i\right)$ has infinitely often the value 0 along the critical line. How can that happen? Let us look at the real part (black) and the imaginary part (red), as well as the zeros (blue circles) in the range from 2 to 70:

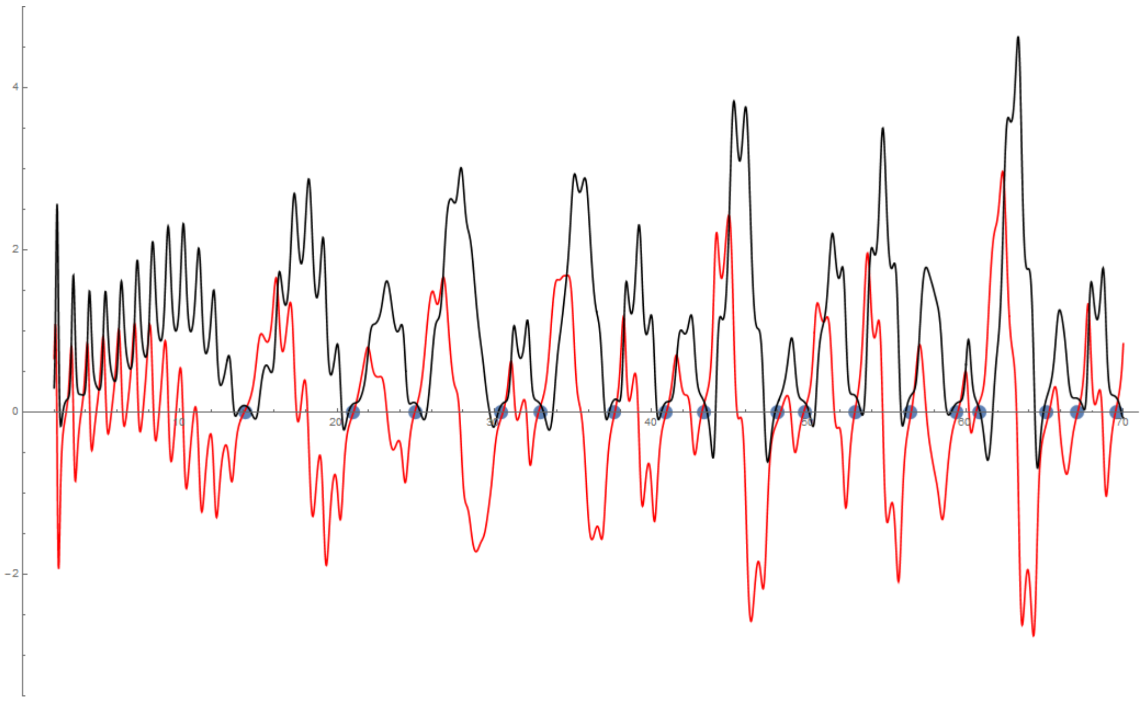


Figure 29. The zeta function (on crit. line, t=0-70, product formula with first 100 primes)

```
Mathematica:
cterm[n_,x_]:=1/(1-Prime[n]^(-1/2-x*I));
myFunc[x_]:=Product[cterm[n,x],{n,1,100}]
xmax=70;
Show[ListPlot[Table[{Im[ZetaZero[i]],0},{i,1,17}],PlotRange-
>{{0,71},{-3.5,5}},ImageSize-
>Large],Plot[{Im[myFunc[x]],Re[myFunc[x]]},{x,2,xmax},PlotStyle-
>{Red,Black},PlotRange->{{0,71},{-3.5,5}},
PlotLegends->{TraditionalForm[Im[Product[1/(1-Prime[n]^(-1/2-
x*I)),{n,1,N}]]],TraditionalForm[Re[Product[1/(1-Prime[n]^(-1/2-
x*I)),{n,1,N}]]]},ImageSize->Large]]
```

Looking at the absolute value of this function, we see clearly that the values calculated with the product formula at the zeros (blue circles) of the zeta function have distinct minima, but never become exactly 0 (which is clear from the formula). Somehow, the infinitely many factors seem to work together in such a way that the infinite product at the zeros nevertheless approaches arbitrarily close to the value 0 if the product is taken over a sufficient number of factors.

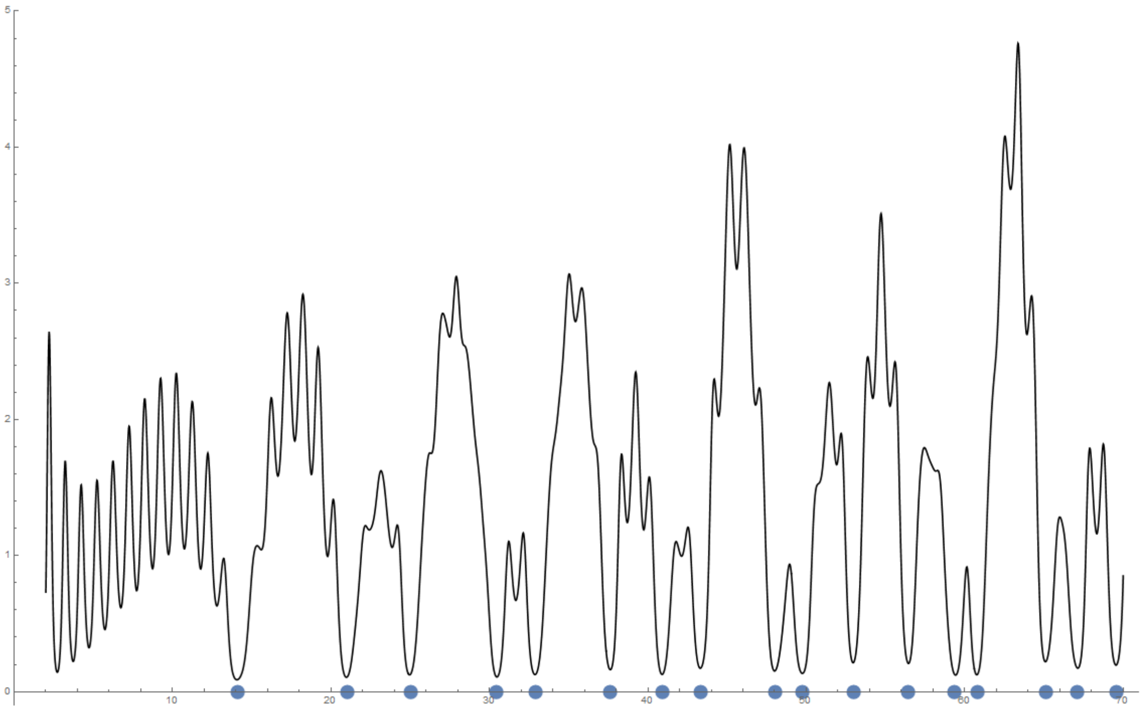


Figure 30. The zeta function (abs. value,crit. line, t=0-70, product formula with first 100 primes)

```
Mathematica:
cterm[n_,x_]:=1/(1-Prime[n]^(-1/2-x*I));
myFunc[x_]:=Product[cterm[n,x],{n,1,100}] xmax=70;
Show[ListPlot[Table[{Im[ZetaZero[i]],0},{i,1,17}],PlotRange-
>{{0,71},{-0.1,5}}],Plot[Abs[myFunc[x]],{x,2,xmax},PlotStyle-
>Black],PlotRange->{{0,71},{-0.1,5}}]
```

Here, by comparison, the 'exact' zeta function:

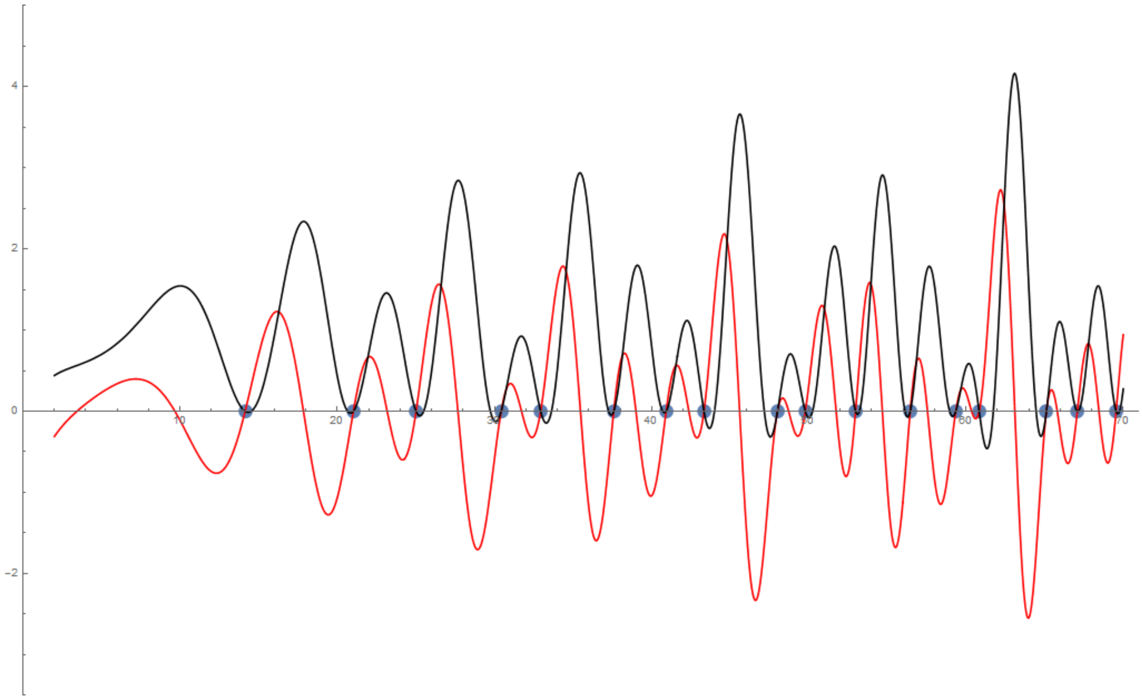


Figure 31. The zeta function (real and imaginary parts, crit. line, t=0-70, exact formula)

```
Mathematica:
xmax=70;
Show[ListPlot[Table[{Im[ZetaZero[i]],0},{i,1,17}],PlotRange-
>{{0,71},{-3.5,5}},ImageSize->Large],
Plot[{Im[Zeta[1/2+x I]],Re[Zeta[1/2+x I]]},{x,2,xmax},
PlotStyle->{Red,Black},PlotRange->{{0,71},{-3.5,5}},
PlotLegends->"Expressions",ImageSize->Large]]
```

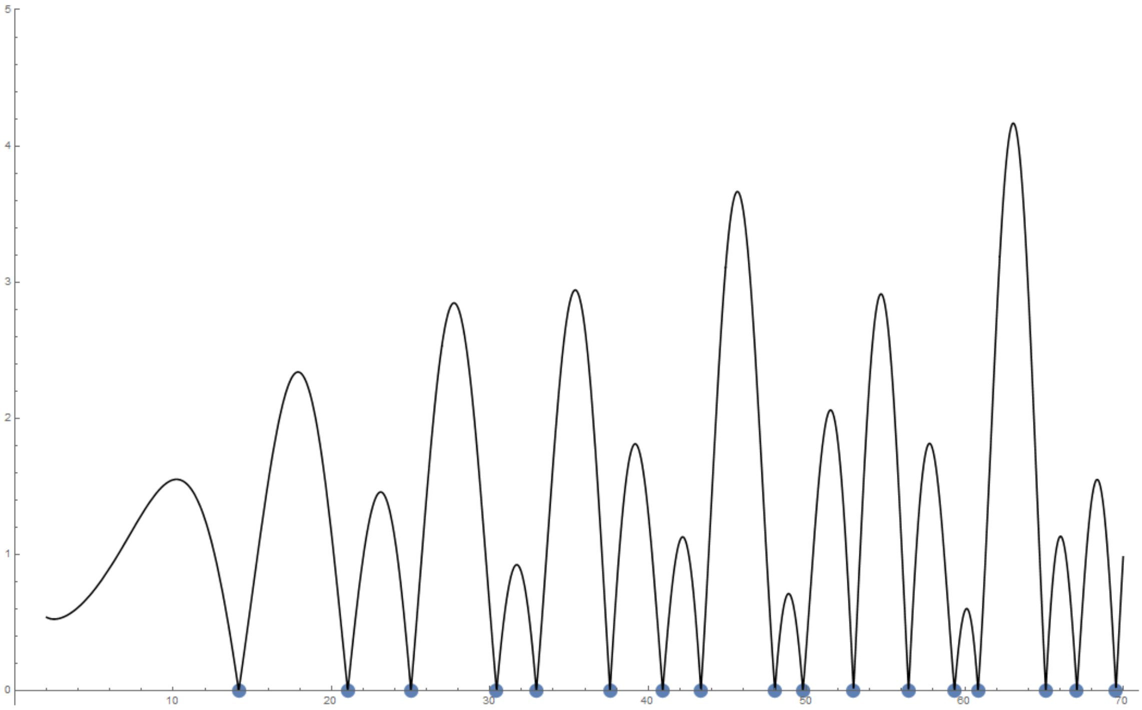


Figure 32. The zeta function (abs. value, crit. line, t=0-70, exact formula)

Comparison of the exact zeta function with the function calculated from the product formula

From a phenomenological point of view, the following statements can be made without claiming to be exact or valid: for the sake of simplicity, let us call the zeta function calculated from the product formula the 'P zeta function'.

At first glance, the P zeta function looks like a somewhat 'broken' zeta function. It becomes 'restless', the more terms $nmax$ in the product formula are added. For small $nmax$, it still looks quite 'restrained', however, it becomes more and more bizarre for large $nmax$, and resembles more and more the pathological 'Weierstrass 𝔓 function'[35], which is everywhere continuous, but nowhere differentiable, and is occasionally also referred to as a 'monster function'. However, it can be observed that the P zeta function in the region of the zeros actually approaches 0 with increasing $nmax$. One could say that the P zeta function converges locally in the neighbourhood of the zeros, in which the convergence range ε for $nmax \to \infty$ is likely to approach 0. Whether the P zeta function actually diverges for all values is not certain. Compared to the exact zeta function, noise (increasingly) appears to be added as $nmax$ grows. The growth of the 'noise and the

[35] https://de.wikipedia.org/wiki/Weierstraß-Funktion

'restlessness' of the P zeta function can be viewed in an animation (as a video on the attached CD) or as a mathematical animation (see the Appendix under "Riemann's zeta function").

It is worth taking a closer look at the product term:

$$\prod_{p\in\mathbb{P}}^{\infty}\frac{1}{\left(1-p^{-\frac{1}{2}-t\cdot i}\right)} \tag{61}$$

By splitting up the real and imaginary part we get:

$$\prod_{n=1}^{\infty}\frac{p_n-\sqrt{p_n}\cos(t\cdot\ln(p_n))-i\sqrt{p_n}\sin\left(t\cdot\ln\left(p_n\right)\right)}{-2\sqrt{p_n}\cos\left(t\cdot\ln\left(p_n\right)\right)+p_n+1}$$

Let us treat the product formation recursively:

$x_{n+1}+i\,y_{n+1}=(x_n+i\,y_n)\cdot Product\ term(n)$, then we get:

$$x_{n+1}=\frac{\sqrt{p_n}\left(\mathrm{x}_n\left(-\cos\left(t\cdot\ln\left(p_n\right)\right)\right)+y_n\sin\left(t\cdot\ln\left(p_n\right)\right)+\mathrm{x}_n\sqrt{p_n}\right)}{-2\sqrt{p_n}\cos\left(t\cdot\ln\left(p_n\right)\right)+p_n+1}$$

$$y_{n+1}=\frac{\sqrt{p_n}\left(-\mathrm{x}_n\sin\left(t\cdot\ln\left(p_n\right)\right)+y_n\left(-\cos\left(t\cdot\ln\left(p_n\right)\right)\right)+y_n\sqrt{p_n}\right)}{-2\sqrt{p_n}\cos\left(t\cdot\ln\left(p_n\right)\right)+p_n+1}$$

If we simply leave the cos terms in the numerator, we get an iteration that converges far faster (at least in the range of the zeros):

$$\begin{aligned} x_{n+1}&=\frac{x_n+\dfrac{y_n\sin(t\ln(p_n))}{\sqrt{p_n}}}{-\dfrac{2\cos\left(t\cdot\ln\left(p_n\right)\right)}{\sqrt{p_n}}+\dfrac{1}{p_n}+1}\\ y_{n+1}&=\frac{y_n-\dfrac{\mathrm{x}_n\sin\left(t\cdot\ln\left(p_n\right)\right)}{\sqrt{p_n}}}{-\dfrac{2\cos\left(t\cdot\ln\left(p_n\right)\right)}{\sqrt{p_n}}+\dfrac{1}{p_n}+1}\end{aligned} \tag{62}$$

The absolute value of the function shows clear, absolute minima at the zeros, which are very close to 0:

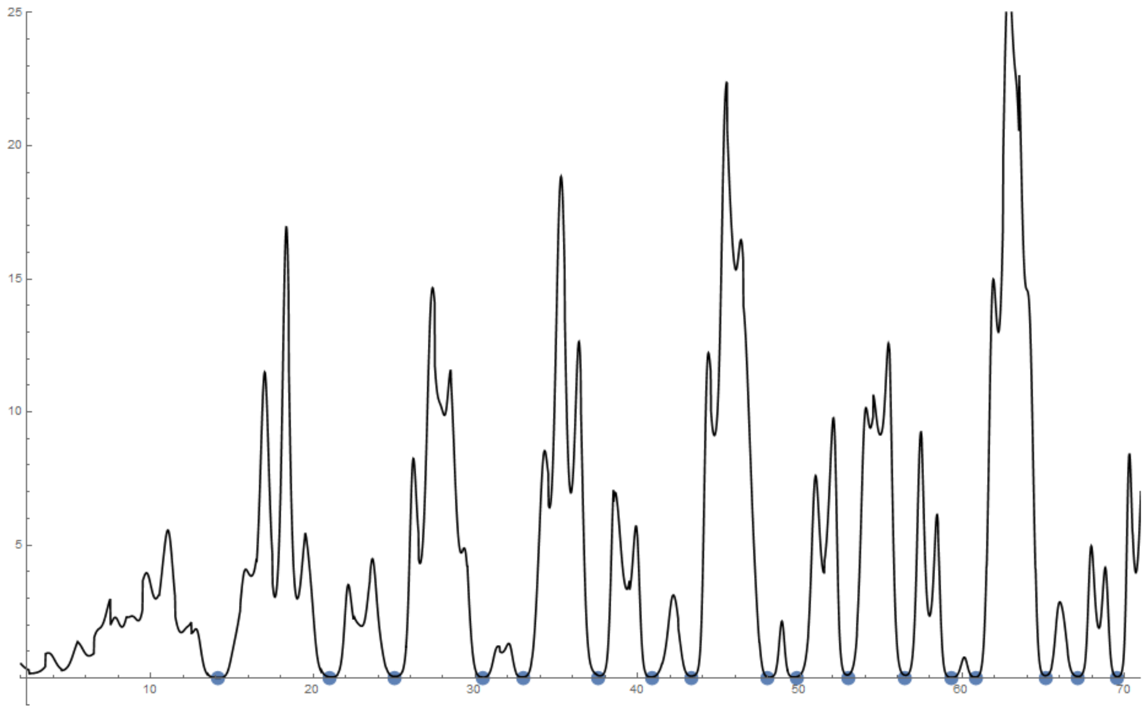


Figure 33. Plot using Formula (62) where $x_0 = 1, y_0 = 0$, zeros: blue circles

The corresponding Mathematica program can be found in the Appendix under "Riemann's zeta function".

It is interesting that for the calculation of the position of the zeros of the zeta function (the position of the resulting absolute minima), an exact knowledge of all primes is not that important. Just taking the first five prime numbers (2,3,5,7,11) in Formula (62) gives the approximate position of the first 15 zero positions of the zeta function:

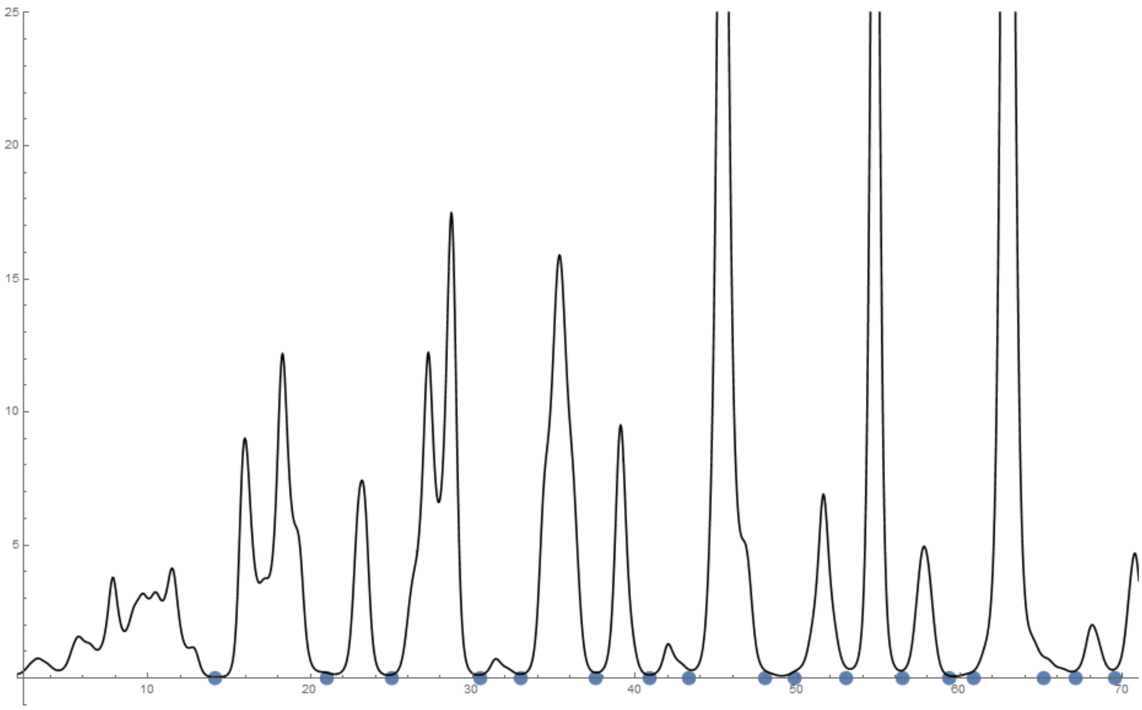


Figure 34. Zeta(s) calculated with (62) by using the first 5 prime numbers

5.4 AN UNEXPECTED PRODUCT REPRESENTATION OF A SLIGHTLY DIFFERENT $\zeta(s)$

Staying with (53), let us be still more adventurous.
What happens if (concerning the infinite product) we do not consider t as a variable and let the product run over all the primes, but consider p as a variable and let the product run over all the zeros of the zeta function?

Well, instead of:

$$\zeta\left(\frac{1}{2}+t\cdot i\right)=\prod_{n=1}^{\infty}\frac{1}{\left(1-p_n{}^{-\frac{1}{2}-t\cdot i}\right)}$$

$$\mathfrak{Z}(p)=\prod_{n=1}^{\infty}\frac{1}{\left(1-p^{-\frac{1}{2}-\rho_n\cdot i}\right)}\quad \text{where } \rho_n\text{: zeros of } \zeta(x), p\in\mathbb{R} \tag{63}$$

Following the zeta function, we call this the Z function.

Here are the results:

Considering the absolute value of this function, we see clearly that the values calculated with the product formula **have clear, absolute minima at the prime number positions (blue circles)**, but never become exactly 0 (which is clear from the formula).

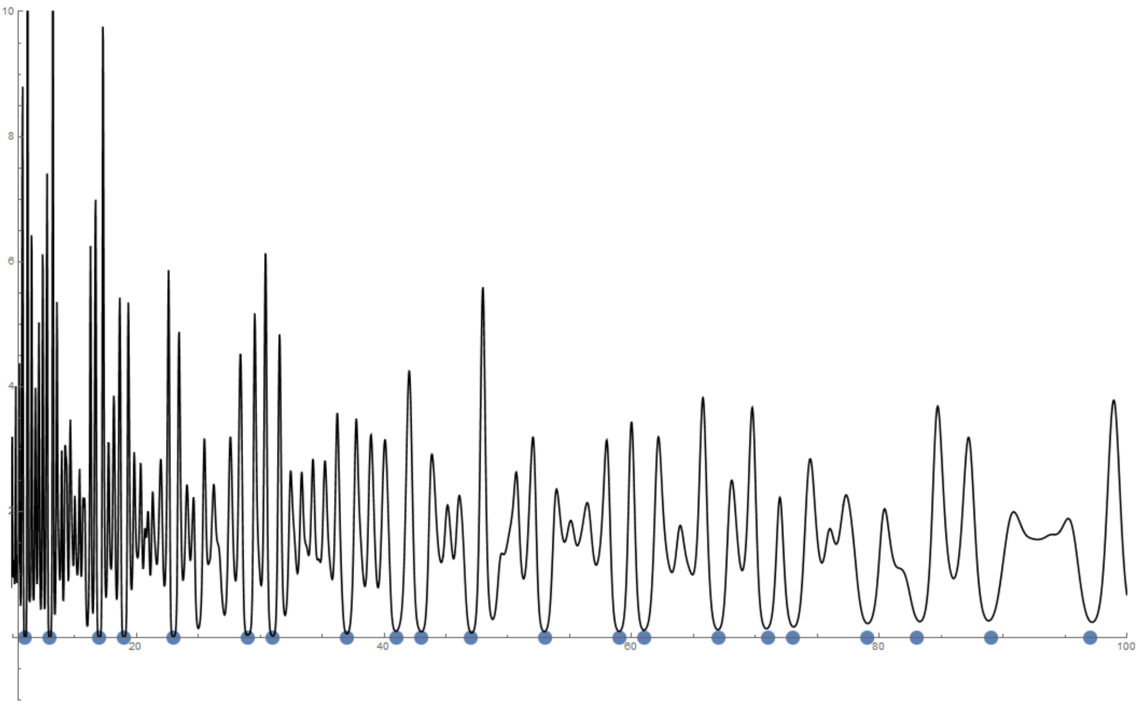


Figure 35. Formula (63) (ABS(), x=10-100, product over 100 zeros of the zeta function)

```
cterm[n_,p_]:=1/(1-p^(-ZetaZero[n]));
myFunc[p_]:=Product[cterm[n,p],{n,1,100}]
xmin=10;xmax=100;
Show[ListPlot[Table[{Prime[i],0},{i,5,25}],PlotRange->{{xmin,xmax},{-
1,10}}],Plot[Abs[myFunc[x]],{x,2,xmax},PlotStyle->Black,PlotRange-
>{{xmin,xmax},{-5,10}},MaxRecursion->6]]
```

It would be interesting to know the exact, explicit formula for $\mathfrak{Z}(p)$!

The same holds for the convergence properties of $\mathfrak{Z}(p)$ as for the infinite product of Formula (61). Here also we have only 'local' convergence in the vicinity of the prime numbers. The more terms with zeta zeros are included in the product, the more the function becomes ‘restless’ and ‘noisy’. If we also use the same method of convergence acceleration according to (62), then we get:

$$\mathfrak{Z}(p) = \prod_{n=1}^{\infty} \frac{\mathrm{p} - \sqrt{p}\cos(z_n \cdot \ln(\mathrm{p})) - i\sqrt{\mathrm{p}} \cdot \sin(z_n \cdot \ln(\mathrm{p}))}{-2\sqrt{\mathrm{p}} \cdot \cos(z_n \cdot \ln(\mathrm{p})) + \mathrm{p} + 1}$$

z_n being here the imaginary parts of the zeta function zeros, and $p \in \mathbb{R}$.

If we consider the product computation recursively:

$x_{n+1} + i\, y_{n+1} = (x_n + i\, y_n) \cdot \text{product\_term}(n)$, then we get:

$$x_{n+1} = \frac{\sqrt{\mathrm{p}}(\mathrm{x}_n(-\cos(z_n \cdot \ln(\mathrm{p}))) + y_n \sin(z_n \cdot \ln(\mathrm{p})) + x_n\sqrt{\mathrm{p}})}{-2\sqrt{\mathrm{p}} \cdot \cos(z_n \cdot \ln(\mathrm{p})) + \mathrm{p} + 1}$$

$$y_{n+1} = \frac{\sqrt{\mathrm{p}}(-\mathrm{x}_n \sin(z_n \cdot \ln(\mathrm{p})) + y_n(-\cos(z_n \cdot \ln(\mathrm{p}))) + y_n\sqrt{\mathrm{p}})}{-2\sqrt{\mathrm{p}} \cdot \cos(z_n \cdot \ln(\mathrm{p})) + \mathrm{p} + 1}$$

If we omit the cos terms in the numerator, then we again get the more quickly converging iteration formula:

$$x_{n+1} = \frac{x_n + \frac{\mathrm{y}_n \sin(z_n \ln(\mathrm{p}))}{\sqrt{\mathrm{p}}}}{-\frac{2\cos(z_n \cdot \ln(\mathrm{p}))}{\sqrt{\mathrm{p}}} + \frac{1}{\mathrm{p}} + 1}$$
$$y_{n+1} = \frac{y_n - \frac{\mathrm{x}_n \sin(z_n \cdot \ln(\mathrm{p}))}{\sqrt{\mathrm{p}}}}{-\frac{2\cos(z_n \cdot \ln(\mathrm{p}))}{\sqrt{\mathrm{p}}} + \frac{1}{\mathrm{p}} + 1} \tag{64}$$

The graph of this 'convergence-accelerated' function looks like this:

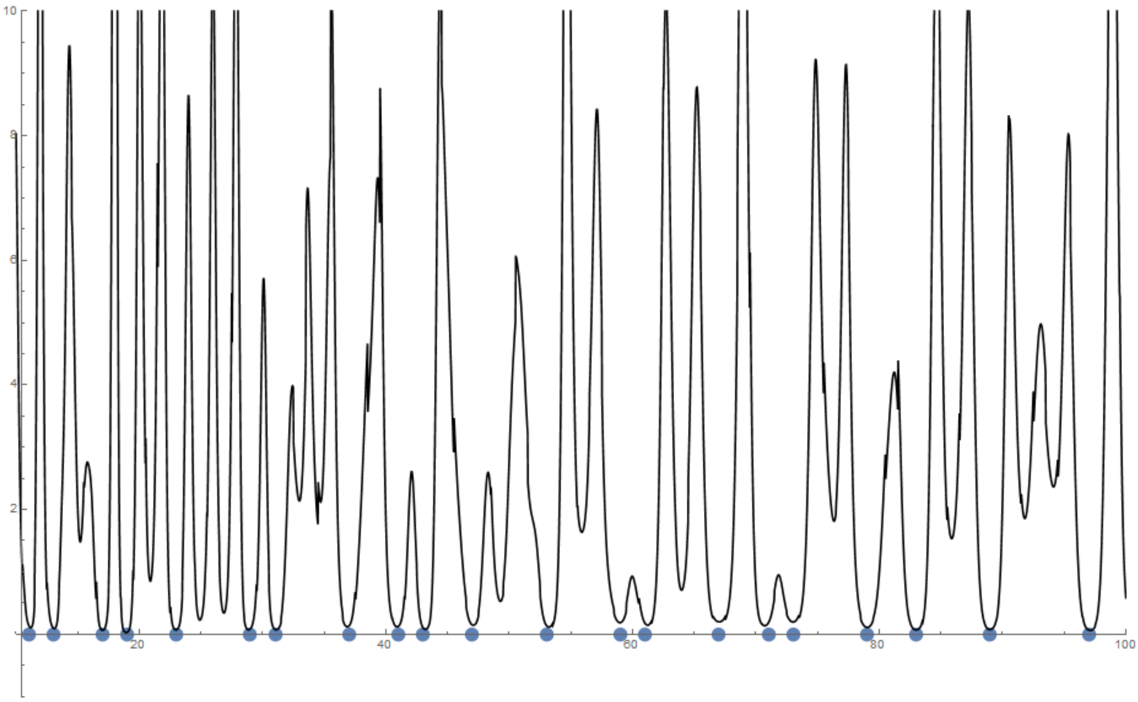


Figure 36. Graph generated by (64) with absolute minima at primes, $x_0 = 1, y_0 = 0$

The Mathematica program with which the graph was created can be found in the Appendix under "Riemann's zeta function".

5.5 A COUNTING FUNCTION FOR THE NUMBER OF ZEROS

For the number of primes up to a given limit N, there are asymptotic and exact formulae, e.g. (133):

$$\boldsymbol{\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})}$$

The sum runs over all non trivial zeros of the zeta function. We have the heuristic assumption that the number of non-trivial zeta zeros up to a given limit can be represented in a similar way by an infinite sum (this time over a prime number term).

The known asymptotic approximation is:

$$\boldsymbol{N(t) = \frac{t}{2\pi}\left(\ln\frac{t}{2\pi} - 1\right)} \tag{65}$$

Here, first of all, is a graphic representation of the function $N(t)$ that indicates the number of zeros up to an upper limit t (in comparison with the exact values):

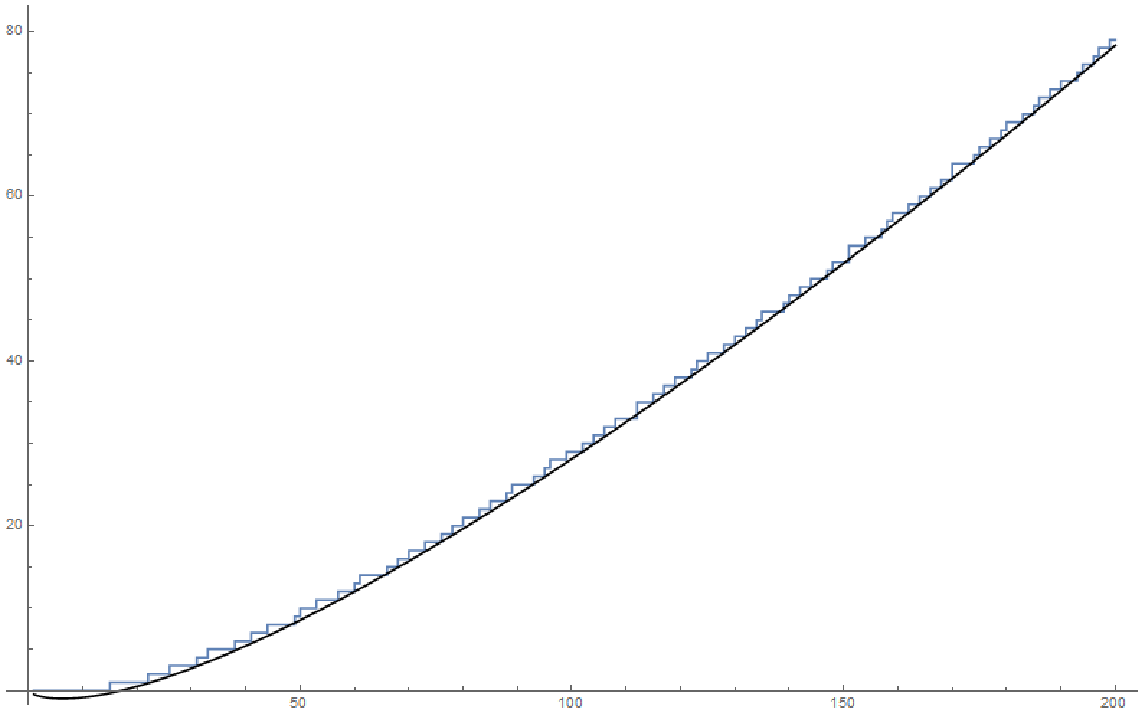


Figure 37. Number of zeros of the zeta function. Range 0-200, (exact and approximation)

```
Mathematica:
nn=200;temp=Table[0,{nn}];k=1;While[z=Im[ZetaZero[k]];
z<nn,k++;temp[[Ceiling[z];;nn]]++]
NExact[t_]:=temp[[Round[t]]]
NApprox[t_]:=t/(2*Pi)*(Log[t/(2Pi)]-1)
Show[ListLinePlot[Table[NExact[x],{x,1,nn}],InterpolationOrder-
>0,ImageSize->Large],Plot[NApprox[x],{x,1,nn},PlotStyle-
>Black,ImageSize->Large]]
```

However, exact formulae can be also found in the literature (for derivation, see above):

$$N(t) = \frac{1}{\pi} Im(\ln(\Gamma(\frac{1}{4} + \frac{it}{2})) - \frac{t}{2\pi} \ln(\pi) + \frac{1}{\pi} Im(\ln(\zeta(\frac{1}{2} + it)) + 1 \quad (66)$$

Graph:

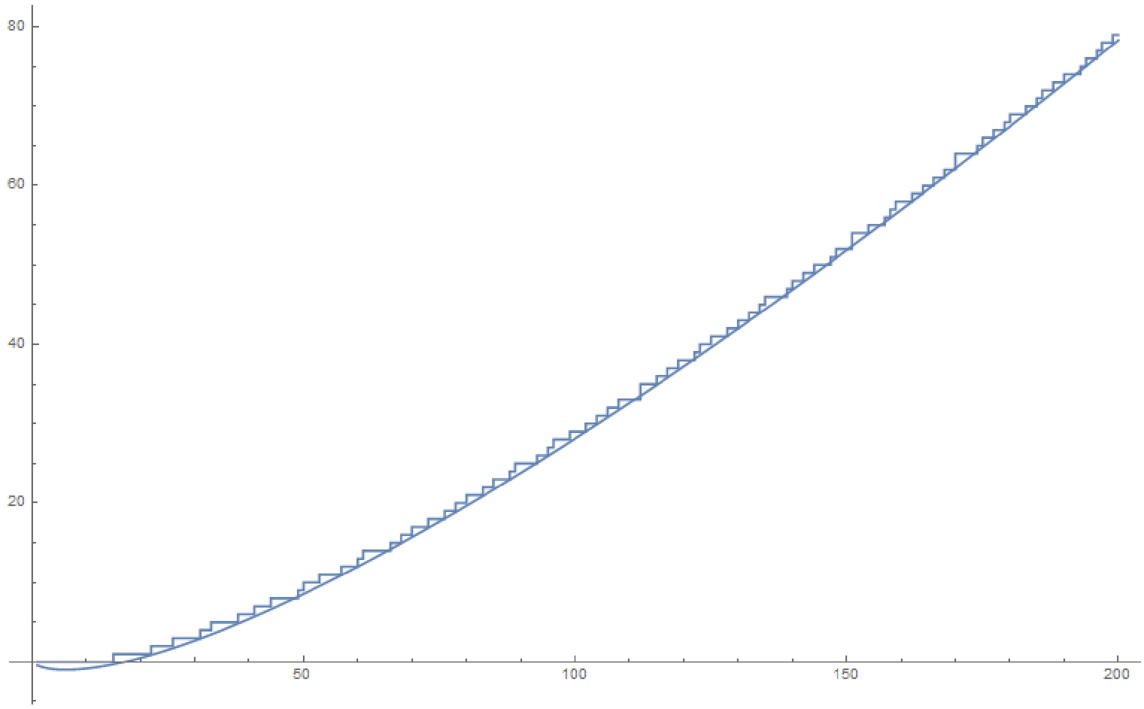


Figure 38. Number of zeros of the zeta function. Range: 0-200 (Formula (66) and approximation)

```
Mathematica:
countZero[t_]:=1/Pi*Im[LogGamma[1/4+I*t/2]]-
t/(2*Pi)Log[Pi]+1/Pi*Im[Log[Zeta[1/2+I*t]]]+1
NApprox[t_]:=t/(2*Pi)*(Log[t/(2Pi)]-1)
Show[ListLinePlot[Table[NApprox[n],{n,1,200}],
PlotRange->All],ListLinePlot[Table[countZero[n],{n,1,200}],
InterpolationOrder->0, PlotRange->All]]
```

Formula (66) seems to produce exactly the number of the zeros. (Verified by the author up to 100000).

The duality between prime numbers and zeros of the zeta function is an essential theme of this book. Of course, there is a temptation to find a representation for the number of zeros, which consists of an approximation term (which describes the asymptotic development), and an additional sum term, which takes into account all the details and local nuances the more terms we include.

Thus the duality between prime numbers and zeros of the zeta function would be perfect. We think, for example, of Formula (133) which gives the exact number of prime numbers:

A counting function for the number of zeros

$$\pi(x) = R(x) - \frac{1}{\ln(x)} + \frac{1}{\pi}\arctan\left(\frac{\pi}{\ln(x)}\right) + \sum_{\rho} R(x^{\rho})$$

If we replace in (66) the zeta term by the product representation with prime numbers, then we have found such a formula:

$$N_c(t) = \frac{1}{\pi}\ \ln\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{it}{2\pi}\ln\pi - \frac{1}{\pi}\sum_{n}\ln\left(1 - p_n^{-\frac{1}{2}-it}\right)$$
$$N(t) = Im\ N_c(t) \tag{67}$$

Here is a plot of this function from t = 0 to 60 (using the first 1000 primes for the sum term, the asymptotic part in blue):

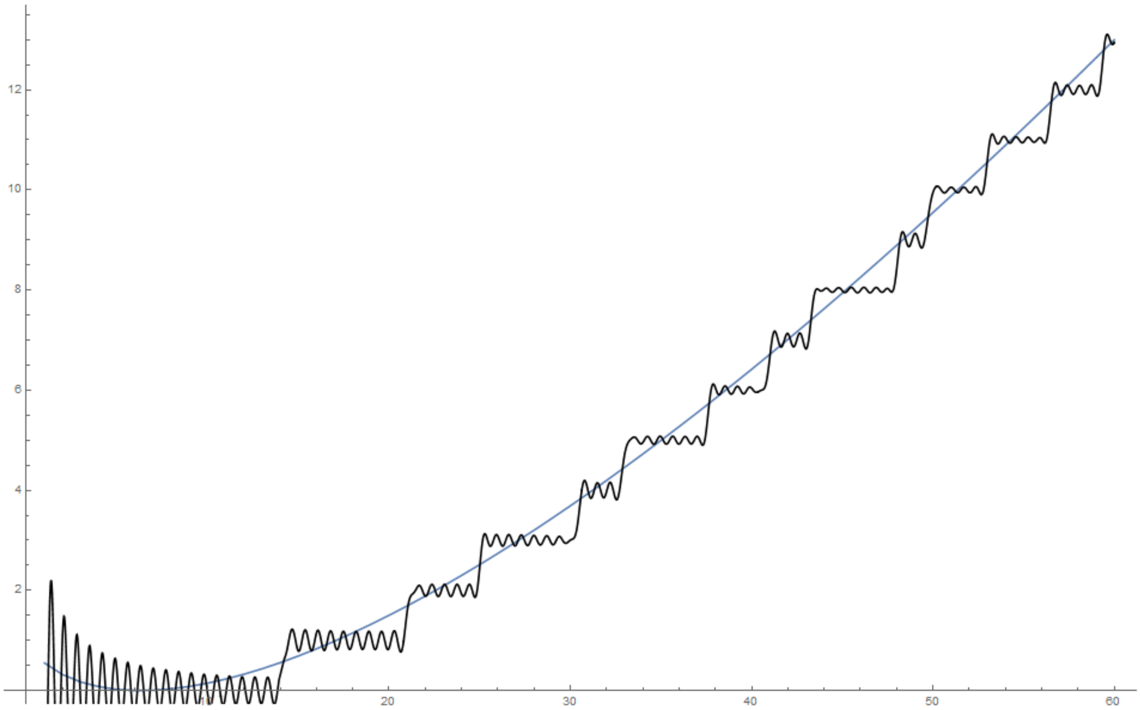


Figure 39. Zero-counting function of the zeta function with prime number sum term

```
Mathematica:
NApprox[t_]:=t/(2*Pi)*(Log[t/(2Pi)]-1)+1
countZeroComplex[t_]:=1/Pi*LogGamma[1/4+I*t/2]-I*t/(2*Pi)Log[Pi]-
1/Pi*Sum[Log[1-Prime[n]^(-1/2-I*t)],{n,1,1000}]+I
Show[ListLinePlot[Table[NApprox[n],{n,1,60}],PlotRange-
>All],Plot[Im[countZeroComplex[n]],{n,1,60},PlotStyle-
>Black,PlotRange->All]]
```

Note: unfortunately, this representation does not converge absolutely. The amplitudes of the oscillations become bigger as more prime terms are added.

5.6 THE ZETA FUNCTION AND QUANTUM CHAOS: A GANGWAY TO PHYSICS

Occasionally random coincidence lends a hand in mathematical discoveries. This was probably the case in the 70s of the last century. By chance the mathematician H. Montgomery and the physicist F. Dyson met and casually told each other about their current research projects – poring, one assumes, over the odd diagram or formula at the same time.

Montgomery was a mathematician, specializing in number theory and the zeta function in particular, who had investigated the relationship between the complex zeros and prime numbers. Dyson was one of the leading nuclear physicists in the field of the so-called random matrices (a special mathematical field used in the treatment of the properties of large and heavy atomic nuclei).

Perhaps Montgomery showed Dyson a diagram of the position of the zeros, because Dyson quickly realized that a striking similarity existed between the distribution of the zeros of the zeta function and certain physical spectra. Physically, these spectra describe energy levels in heavy atomic nuclei; mathematically, such spectra are calculated from the eigenvalues of so-called Gaussian random matrices (i.e. matrices occupied by random values corresponding to a Gaussian normal distribution). Expressed in the language of the physicists, this means that the spectral values are the 'eigenvalues' of a 'Hermitian' operator[36]. Hermitian (also called ‘self-adjoint’) operators play an important role in quantum mechanics. Due to their symmetry properties, they always have real eigenvalues.

Now, the conjecture is that the complex zeros of the zeta function are nothing other than the (real) eigenvalues of a mysterious Hermitian operator. This hypothesis is referred to in the literature as the 'GUE' hypothesis ('GUE': Gaussian Unitary Ensemble). Unfortunately this operator has not yet been found. There are, however, overwhelming numerical indications that such an operator does exist (see below).

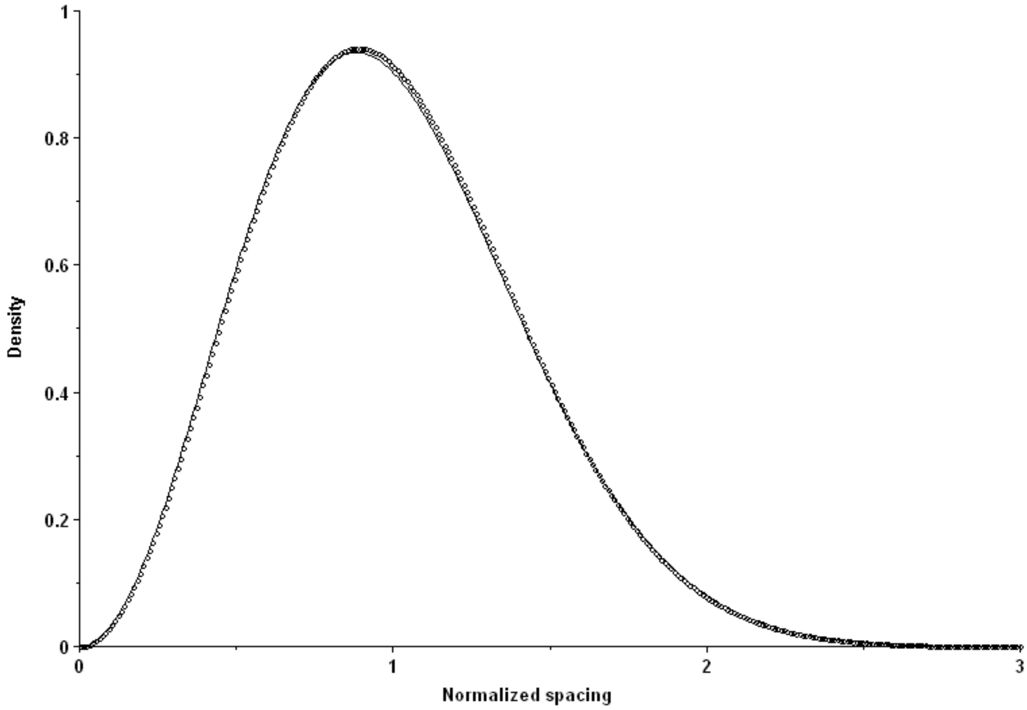
In fact, this conjecture goes much further back in time: the Hungarian mathematician George Pólya expressed this conjecture more than 100 years ago (Hilbert-Pólya conjecture).[37]

Consider the differences of the complex zeros occurring along the critical line and normalize these differences:

$$\delta_n = \frac{z_{n+1} - z_n}{2\pi} \ln \frac{z_n}{2\pi} \tag{68}$$

From the theory, we know for sure that these δ_n have the mean value of 1.

[36] https://de.wikipedia.org/wiki/Hermitescher_Operator
[37] https://en.wikipedia.org/wiki/Hilbert-Pólya_conjecture



The smallest known value of δ_n is located at 1034741742903.353 (this is the 4.088.664.936.217th zero) and has the normalized value of 0.00007025 . This corresponds to an actual difference of 0.00001709! (as of Jan. 2016).

However, such small zero differences are very rare. In general, the zeros display a 'repulsive' tendency and avoid coming too close to each other. This behaviour is also known by the energy levels of quantum mechanics. Here is a comparison of the statistical distribution from the prediction provided by the GUE theory (solid line) and the actual values of the zeros of the zeta function. For the calculation of the statistical distribution, 2 billion zeros in the range up to 10^{13} have been evaluated. The graphic was taken from:

http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeroscompute.html

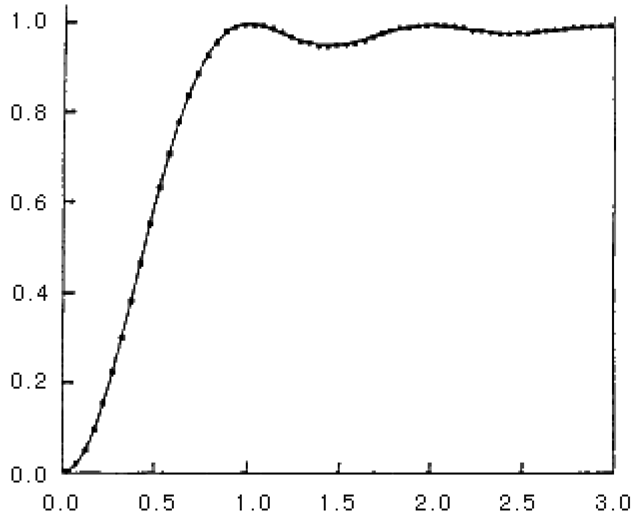
Most of the results and calculations we owe to Andrew Odlyzko:

http://www.dtc.umn.edu/~odlyzko/

As can be seen, the evidence that the zeros of the zeta function have their origin actually in a (still unknown) operator is obviously true.
Further information about this current research area can be found at:

http://www.dartmouth.edu/~chance/chance_news/recent_news/primes_part3/part3.html

The pair correlation between two arbitrary zeros also appears to follow the theoretical prediction from the GUE theory:



However there are still areas of darkness regarding the nature of this unknown operator. Investigations by Odlyzko (Fourier analysis of the critical zeros) show a somewhat different behaviour than is to be expected from physical GUE eigenvalues (e.g. peaks in prime number powers). Therefore the conjecture also exists that the underlying operator does not come from the eigenvalues of a GUE operator, but from the eigenvalues of a more general chaotic system. The interested reader is encouraged to search the Internet using the keywords "quantum chaology".

A relatively unknown method uses connections between quantum oscillators and the zeta function. Without going into too much detail, the method of Crandall (Richard Crandall, 2001) is described here:

There is a temporal solution of a 'smooth' wave function $\psi(x,t)$, which is described by the Schrödinger equation and is known to have no zeros on the X axis at time $t = 0$. However, after a time t in which the wave function evolves according to the Schrödinger equation, this wave function becomes 'noisy' and 'fuzzy' and acquires infinitely many zeros on the X axis that are identical to the critical zeros of the zeta function. This wave function can be represented as follows:

$$\psi(x,t) = f\left(\frac{1}{2} + ix\right)\zeta\left(\frac{1}{2} + ix\right) = e^{\frac{x^2}{2a^2}} \sum_{n=0}^{\infty} c_n (-1)^n H_{2n}\left(\frac{x}{a}\right) \tag{69}$$

Where a is real and c_n are constants still to be determined (depending on a). H_n is the Hermitian polynomial of order n. Let $f(s)$ be an analytic function which has no zeros. If we restrict the infinite sum to a finite number of terms, we can use numerical methods to calculate the finite number of zeros. Borwein (Borwein, 2000) was able to use this method to calculate the first seven critical zeros of the zeta function with an accuracy of 10 decimal digits using the first 27 sum terms. In principle, this method can be used to calculate all critical zeros. The calculation is based on the calculation of the eigenvalues of a Hessenberg matrix, which ultimately leads to the calculation of the zeros of a characteristic polynomial.

Here, further highly interesting contexts arise that are related to Riemann's conjecture.

6 DIGRESSION: THE RIEMANN FUNCTION $R(s)$

The Riemann function $R(x)$ (not to be confused with Riemann's zeta function $\zeta(s)$) plays an important role in the theory of prime numbers. It gives the best simple approximation for the calculation of $\pi(x)$, the number of primes up to the limit x. In the following, the most important properties and computational methods are briefly described. The most frequently used representations are the summation using the Moebius function $\mu(n)$ and the integral logarithm li(x),

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(\mathrm{x}^{\frac{1}{n})} \; for \; x > 1 \tag{70}$$

as well as the (very rapidly converging) summation by means of powers of $\ln(x)$ and values of $\zeta(n)$ with integral arguments n, which is generally referred to in the literature as the ‘Gram’ function or series:

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n!\, n\, \zeta(n+1)} \; for \; x > 0 \tag{71}$$

The following plot shows how well the Riemann function $R(x)$ approximates to the function $\pi(x)$ (see also Table 29):

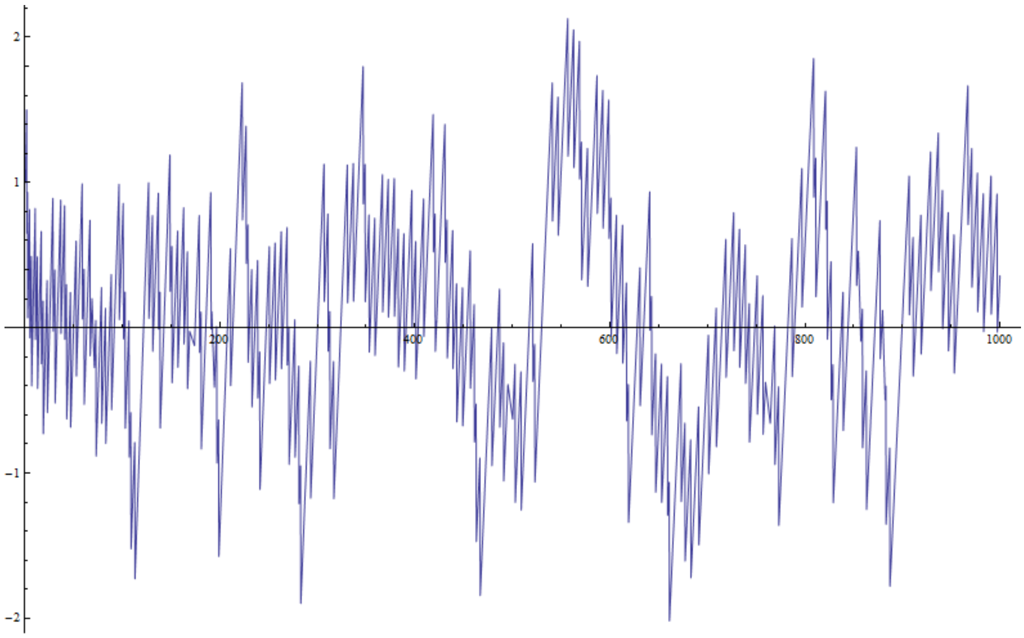


Figure 40. $\mathrm{r(x)}-\pi(\mathrm{x})$, values from x = 1 to 1000

7 A FEW IMPORTANT ARITHMETICAL FUNCTIONS

7.1 OMEGA FUNCTIONS: NUMBER OF PRIME FACTORS

$\omega(n)$ denotes the number of different prime factors of a positive integer n. Thus, $\omega(n)$ is defined by the factorization of an integer as:

$$n = {p_1}^{e_1}{p_2}^{e_2} \ldots {p_{\omega(n)}}^{e_{\omega(n)}}$$

In contrast, $\Omega(n)$ denotes the total number of prime factors of an integer n:

$$\Omega(n) = \sum_{k=1}^{e_{\omega(n)}} e_i \tag{72}$$

Clearly, $\Omega(n)$ is simply the sum of the prime powers of n.

g $\omega(n)$ is defined by `PrimeNu[n]` and $\Omega(n)$ by `PrimeOmega[n]`.

Numbers that are composed only of different factors are identical to square-free numbers.

The asymptotic behaviour of $\omega(n)$ is given by:

$$\omega(n) \sim \ln\ln n + B_1 + \sum_{k=1}^{\infty}\left(-1 + \sum_{j=0}^{k-1}\frac{\gamma_j}{j!}\right)\frac{(k-1)!}{(\ln n)^k}$$

where $B_1$is the Mertens constant and γ_j are the Stieltjes constants.

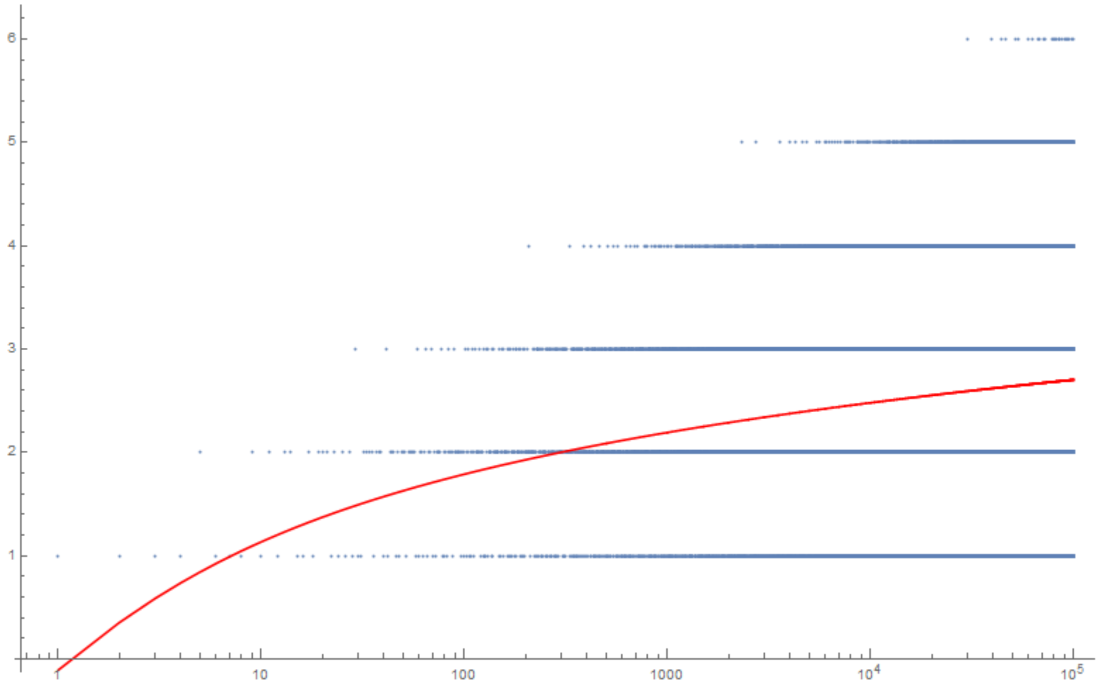


Figure 41. Function ω(n), number of different prime factors (red: asymptotic)

```
Mathematica:
mertensB1=0.2614972128;
nmax=100000;
Show[ListLogLinearPlot[Table[PrimeNu[n],{n,2,nmax}],
PlotRange->All],ListLogLinearPlot[Table[Log[Log[n]]+mertensB1,
{n,2,nmax}],PlotRange->All,Joined->True,PlotStyle->Red]]
```

The asymptotic behaviour of $\Omega(n)$ is also approximate:

$\Omega(n) \sim \ln \ln n + B_2$, with $B_2 = 0.494906$

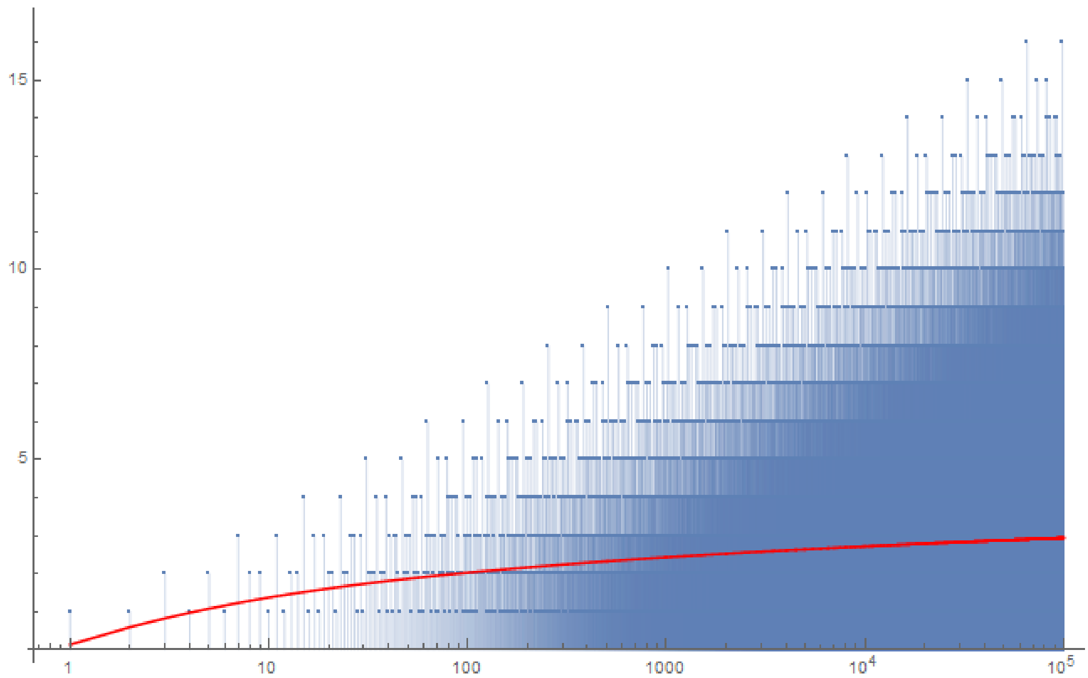


Figure 42. Function Ω(n), total number of prime factors (red: asymptotic)

```
Mathematica:
mertensB2=0.494906;
nmax=100000;
Show[ListLogLinearPlot[Table[PrimeOmega[n],{n,2,nmax}],
PlotRange->All,Filling->Axis],ListLogLinearPlot[Table[Log[Log[n]]+
mertensB2,{n,2,nmax}],PlotRange->All,Joined->True,PlotStyle->Red]]
```

Note:
$\Omega(n)$ is closely related to the *Gradus Suavitatis* of Leonhard Euler (see Chapter 15.1)

The following relationships apply to other arithmetical functions:
Liouville's function:
$\lambda(n) = (-1)^{\Omega(\text{n})}$

Instead of calculating the number of all prime factors ($\Omega(n)$) or the number of different prime numbers $\omega(n)$ of the prime factor decomposition of a number, the sum of all prime factors, sopfr (n) (sum of all the different primes (s)) can be calculated. The former is also referred to as an integer log

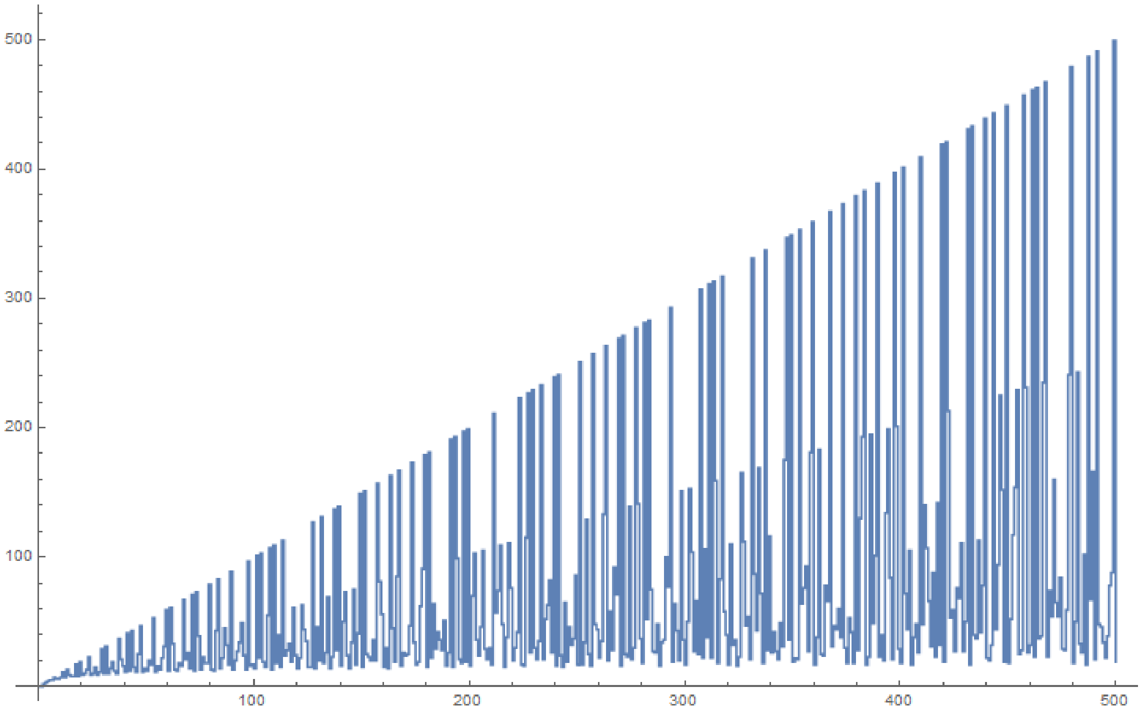


Figure 43. Integer logarithm: sum of all primes of the decomposition for n: sopfr(n)

```
Mathematica:
f[n_]:=Plus@@Times@@@FactorInteger@n;f[1]=0;
ListLinePlot[Table[f[n],{n,1,500}],InterpolationOrder->0,PlotRange->All]
```

7.2 THE LIOUVILLE FUNCTION

The Liouville functions λ and L are defined as:

$$\lambda(n) = (-1)^{\Omega(\text{n})}, L(n) = \sum_{k=1}^{n} \lambda(k) \tag{73}$$

$\lambda(n)$ is -1, if n has an odd number of prime factors and +1 if n has an even number of prime factors.
$\lambda(n)$ is closely related to Riemann's ζ function:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\lambda(k)}{k^s} \tag{74}$$

$L(n)$ is the summatory function of $\lambda(n)$.
For $L(n)$ exist the following formulae:[38]

$$L(x) = \sum_{m=1}^{\frac{x}{w}} \mu(m) \left\{ \left\lfloor \sqrt{\frac{x}{m}} \right\rfloor - \sum_{k=1}^{v-1} \lambda(k) \left(\left\lfloor \frac{x}{km} \right\rfloor - \left\lfloor \frac{x}{mv} \right\rfloor \right) \right\} - \sum_{l=\frac{x}{w}-1}^{\frac{x}{v}} L\left(\frac{x}{l}\right) \sum_{\substack{m \mid l \\ m=1}}^{\frac{x}{w}} \mu(m) \tag{75}$$

The graph of $L(x)$ looks like this:

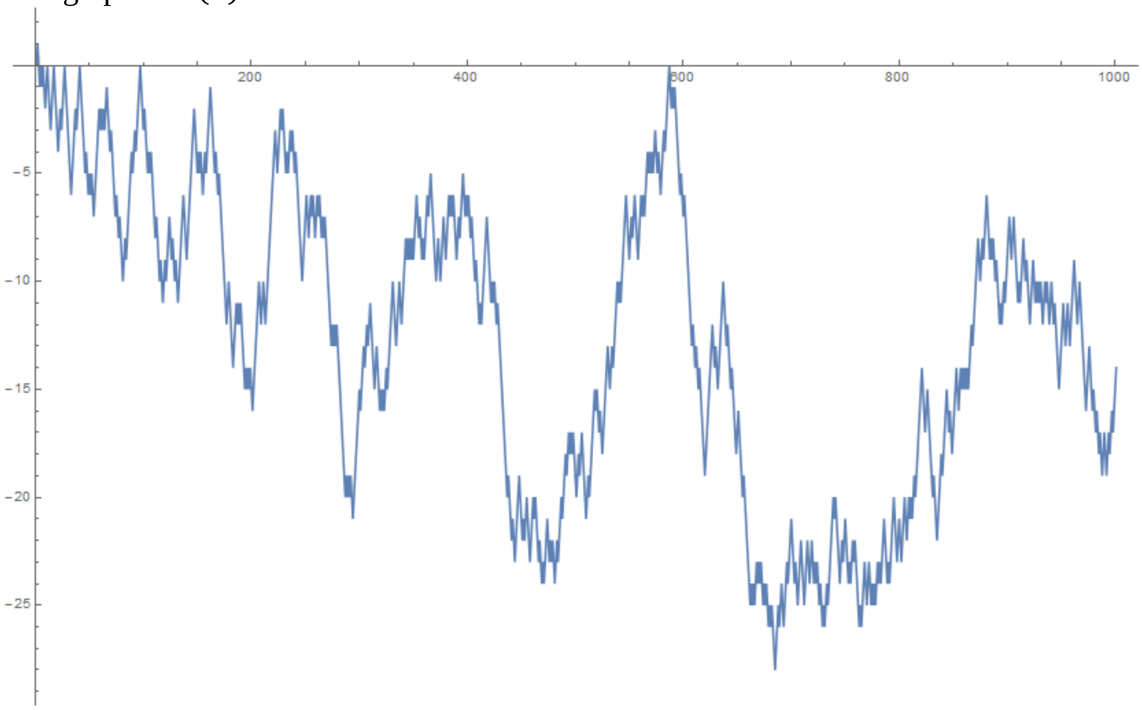


Figure 44. Liouville lambda function, from 1 to 1000

```
Mathematica:
lTab=Accumulate[Join[{0},LiouvilleLambda[Range[1000]]]];
ListLinePlot[lTab]
```

$L(x)$ has a strong tendency to be negative. It was assumed until the 1950s that $L(x) \leq 0$ is always true. In fact, however, the first counterexample was found in 1962: $L(906180359) = 1$. The smallest counterexample is $L(906150257)$. It is still unclear whether there are only finitely many counterexamples or infinitely many (as of Jan. 2016).

[38] http://mathworld.wolfram.com/LiouvilleFunction.html

Like many arithmetical functions, $L(x)$ can also be approximated by means of a sum over the complex zeros of the zeta function (using the first N zeros):

$$L(x) = 1 + \frac{\sqrt{x}}{\zeta\left(\frac{1}{2}\right)} + 2\mathrm{Re}\left(\sum_{k=1}^{N} \frac{x^{\rho_k}\zeta(2\rho_k)}{\rho_k \zeta'(\rho_k)}\right) \tag{76}$$

```
Mathematica program: please contact the author.
```

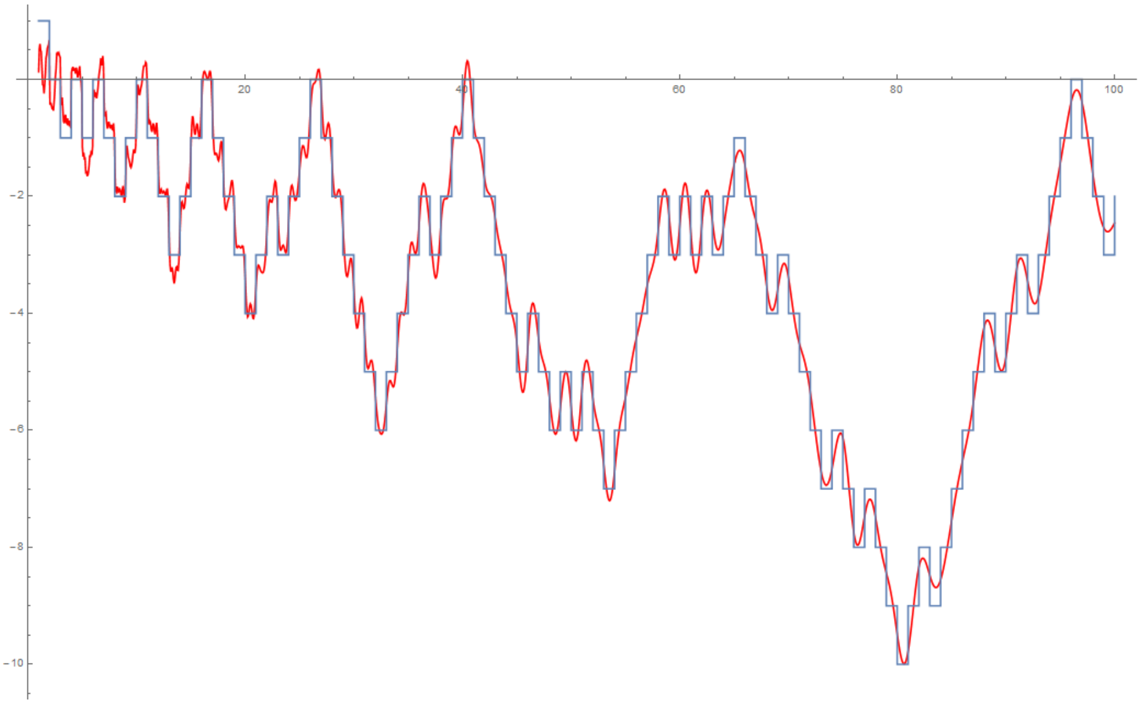


Figure 45. Liouville function, from 1 to 100: exact and computed analytically

7.3 THE CHEBYSHEV FUNCTION

The summatory function of the Mangoldt function $\Lambda(n)$ is called Chebyshev function of the 2nd kind (psi function):

$$\psi(x) = \sum_{p^k \le x} \ln(p) = \sum_{n \le x} \Lambda(n)$$

```
Mathematica:
ListLinePlot[Table[{n,Sum[MangoldtLambda[k],{k,1,n}]},{n,1,100}],Inter
polationOrder->0]
```

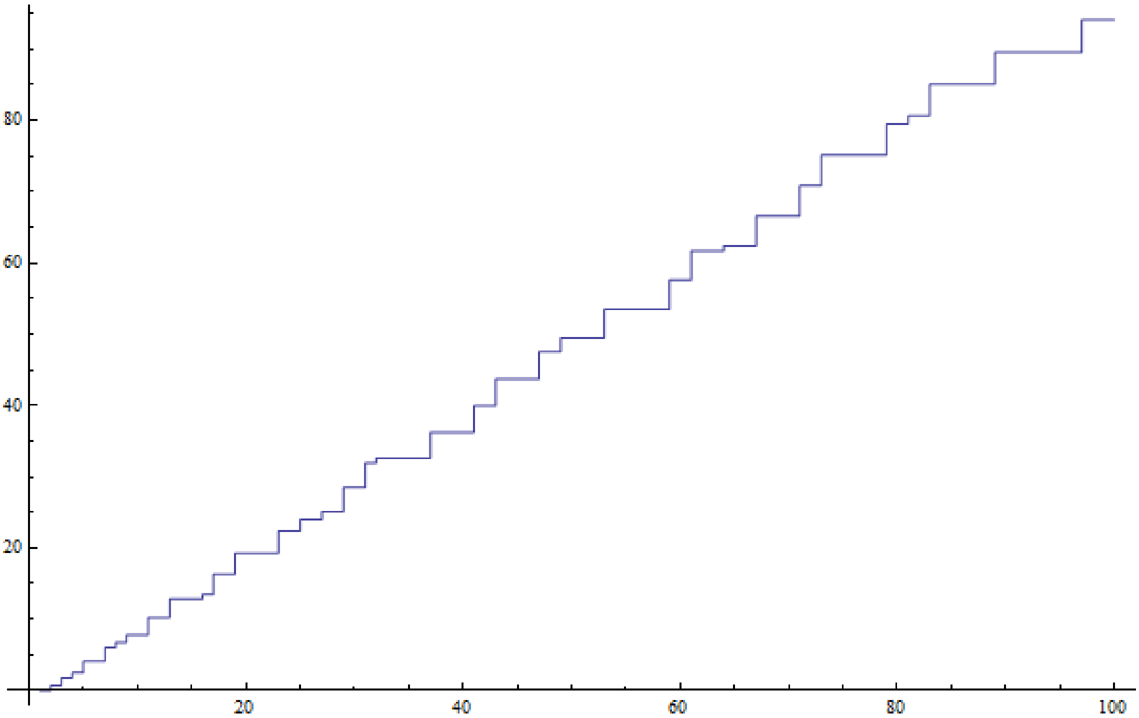


Figure 46. Chebyshev psi function, going from 0 to 100

The Chebyshev psi function can be represented analytically as an explicit function:

$$\boldsymbol{\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln(2\pi) - \frac{1}{2}\ln(1 - x^{-2})} \qquad (77)$$

The summation runs over the non-trivial zeros of the zeta function (here, over the first 50 zero pairs).

Mathematica:

$$\text{myPsi}\ (\text{x\_}, \text{y\_}) := -\sum_{i=-50}^{50} \text{If}[i \neq 0, \frac{(x+iy)^{\rho_i}}{\rho_i}, 0] - 0.5\log\left(1 - \frac{1}{x^2}\right) + x - \log(2\pi)$$

```
Plot[Re[myPsi[x,0]],{x,1,100}] (*real part*)
```

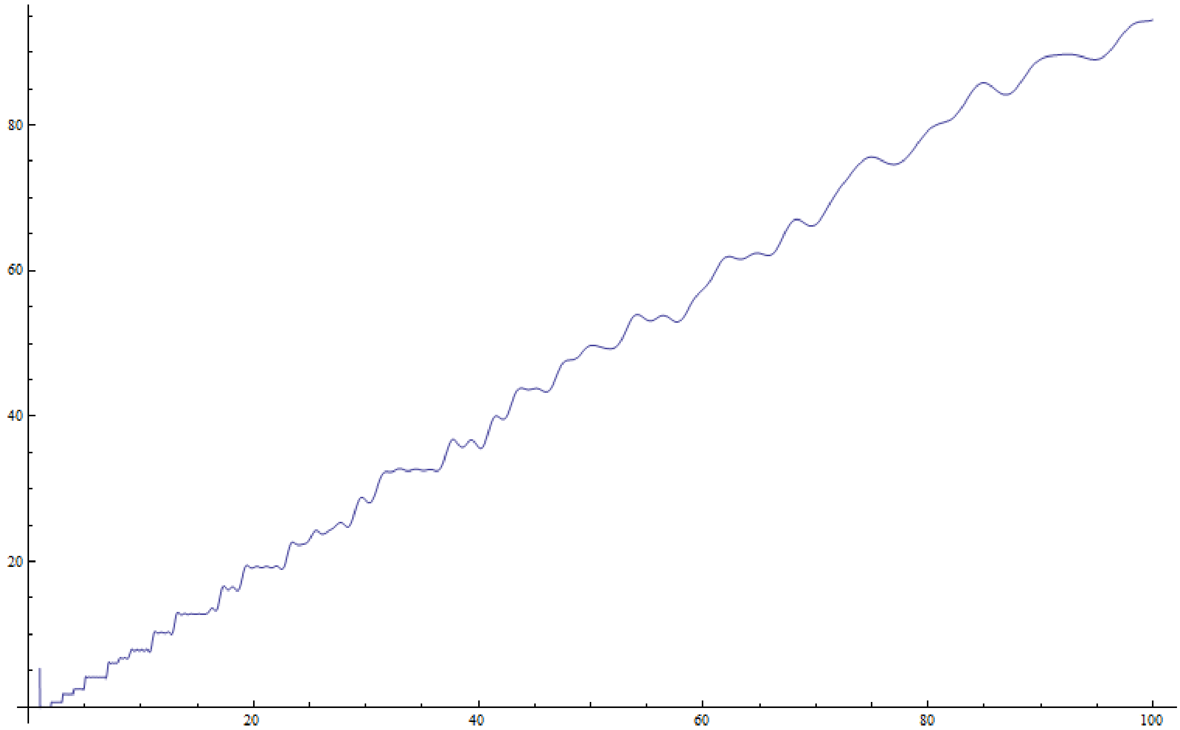


Figure 47. Chebyshev psi function, computed analytically, plot from 0 to 100

Both functions in comparison (sum over the first 75 zero pairs):

```
Mathematica program: please contact the author.
Show[Plot[Re[myPsi[x, 0]], {x, 1, 100},PlotPoints->400,
PlotStyle->Red], ListLinePlot[Table[{n, Sum[MangoldtLambda[k], {k, 1,
n}]},{n, 1, 100}], InterpolationOrder->0]]
```

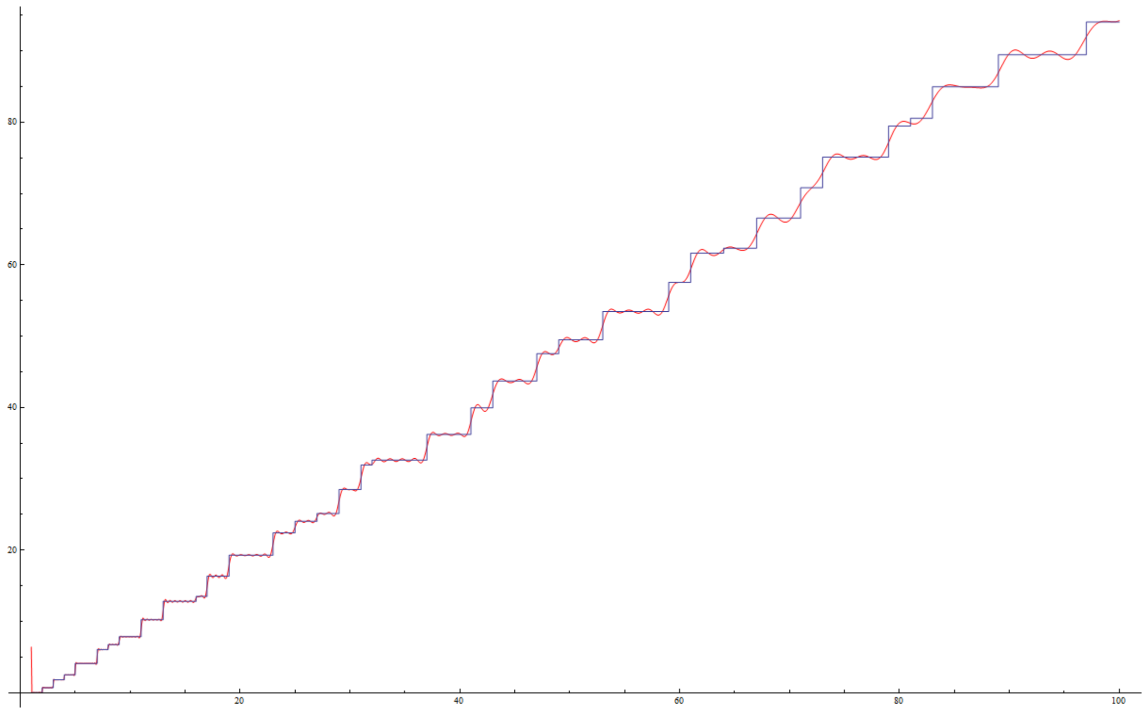


Figure 48. Chebyshev psi function; comparison of analytic and arithmetical methods of calculation

7.4 THE EULER PHI FUNCTION (TOTIENT FUNCTION)

The Euler phi function $\boldsymbol{\varphi(n)}$ (totient function) indicates how many numbers exist that are coprime to n and less than or equal to $\boldsymbol{n}$.

Definition of $\boldsymbol{\varphi(n)}$:

$$\boldsymbol{\varphi(n) = |\{k \in \mathbb{N} | 1 \leq k \leq n \wedge ggT(k,n) = 1\}|} \tag{78}$$

7.4.1 CALCULATION AND GRAPHIC REPRESENTATION OF THE PHI FUNCTION

```
Mathematica:
ListLinePlot[Table[{n,EulerPhi[n]},{n,1,100}],InterpolationOrder->0]
```

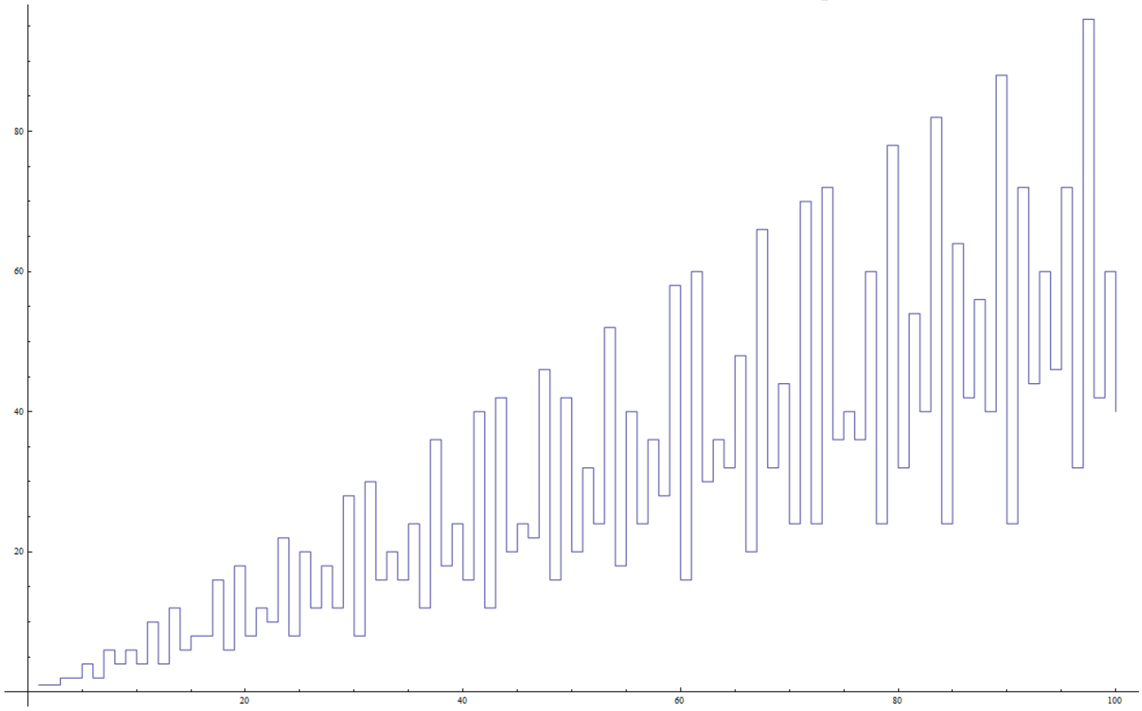


Figure 49. Euler phi function, depicted from 1 to 100

Calculation of $\boldsymbol{\varphi(n)}$
(Let a_i be the powers of the prime decomposition of $n = \prod_{i=1}^{r} p_i^{a_i}$)

$$\boldsymbol{\varphi(n) = \prod_{p|n} p^{a_i-1}(p-1) = n\prod_{p|n}(1-\frac{1}{p})} \tag{79}$$

$$\boldsymbol{\varphi(n) = n \lim_{s\to 1} \zeta(s) \sum_{d|n} \mu(d)(e^{\frac{1}{d}})^{(s-1)}} \tag{80}$$

Its summatory function $\boldsymbol{\Phi(n)}$ calculates the sum up to n:

$$\boldsymbol{\Phi(n) = \sum_{k=1}^{n} \varphi(n)} \tag{81}$$

```
Mathematica:
PhiSum[n_]:=Sum[EulerPhi[k],{k,1,n}]
```

Applying Perron's formula, we get an analytic expression for $\phi(x)$:

$$\boldsymbol{\Phi}(\boldsymbol{x})\sim\frac{\mathbf{1}}{\mathbf{6}}+\frac{\mathbf{3}\boldsymbol{x}^{\mathbf{2}}}{\boldsymbol{\pi}^{\mathbf{2}}}+\boldsymbol{Re}\left(\sum_{k=1}^{N}\frac{x^{\rho_k}\zeta(\rho_k-1)}{\rho_k\zeta'(\rho_k)}\right)+\sum_{k=1}^{N}\frac{x^{-2k}\zeta(-2k-1)}{(-2k)\zeta'(-2k)} \tag{82}$$

```
Mathematica program: please contact the author.
```

```
More useful Mathematica commands:
DirichletTransform[EulerPhi[n],n,s]
```

Comparison of the analytic $\Phi(n)$ with the arithmetical version of the function $\Phi(x)$ (the sum taken over the first 50 non trivial and 50 trivial zeros):

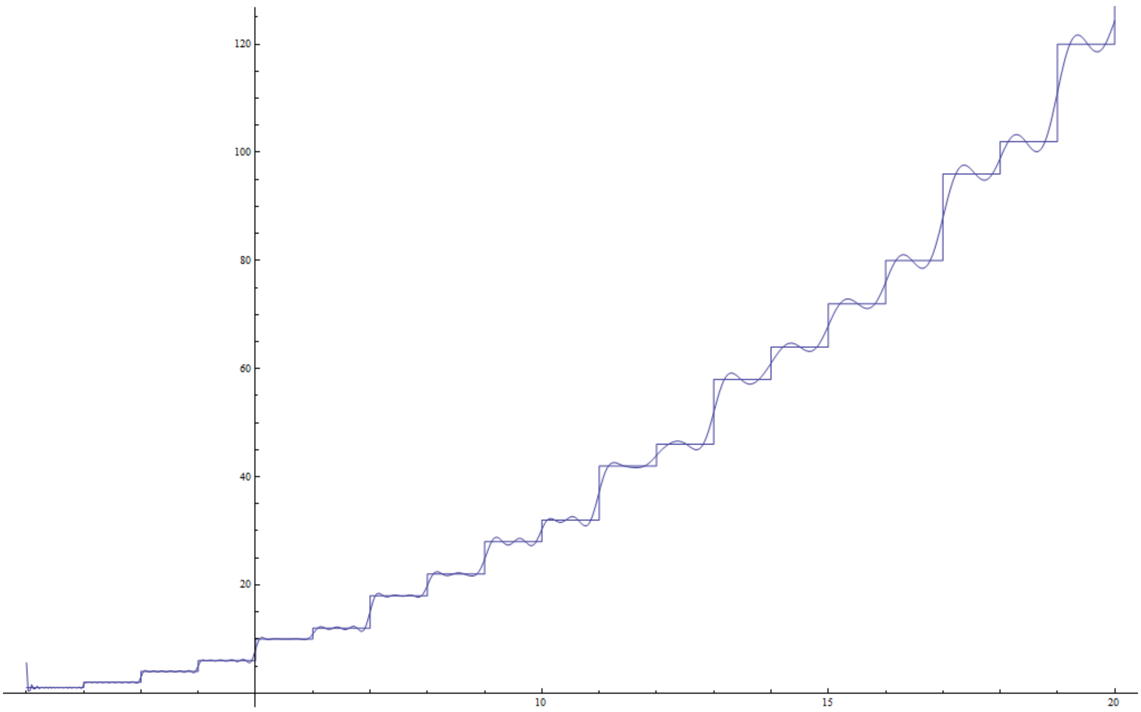


Figure 50. Summatory function Φ(n) of the phi function, plotted from 1 to 100 (comparison of the arithmetical and analytic method of calculation)

```
Mathematica:
Show[Plot[myPhi[x],{x,1,20},MaxRecursion->2,PlotPoints->150],
ListLinePlot[Table[{n,PhiSum[n]},{n,1,40}],InterpolationOrder->0]]
```

The analytic version $\varphi(x)$ of $\varphi(n)$ denotes:

$$\boldsymbol{\varphi}(\boldsymbol{x})=\boldsymbol{\Phi}(\mathbf{x})-\boldsymbol{\Phi}(\mathbf{x}-\mathbf{1}) \tag{83}$$

Comparison of the analytic $\varphi(x)$ with the arithmetical function $\varphi(n)$ (with sums over the first 50 non trivial and 50 trivial zeros):

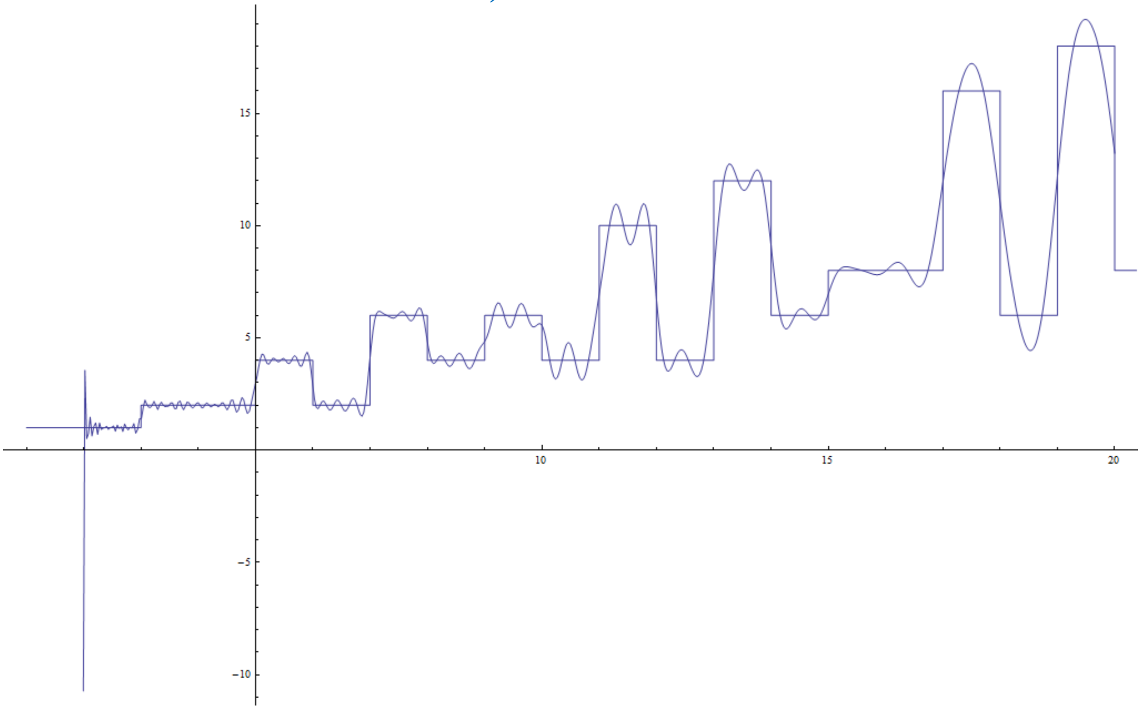


Figure 51. Euler phi function (comparison analytical and arithmetical calculation)

```
Mathematica:
myEulerPhi[x_]:=myPhi[x]-myPhi[x-1] (*definition see above *)
Show[Plot[myEulerPhi[x],{x,1,20},MaxRecursion->2,
PlotPoints->150],ListLinePlot[Table[{n,EulerPhi[n]},{n,1,40}],
InterpolationOrder->0] ]
```

7.4.2 PROPERTIES OF THE PHI FUNCTION

Properties of $\boldsymbol{\varphi(n)}$:

$\varphi(mn) = \varphi(m)\varphi(n)$ (if $\gcd(m,n) = 1$)

$\varphi(p) = p - 1$ (if p is a prime number)

$\varphi(p^k) = p^{k-1}(p-1) = p^k(1 - \frac{1}{p})$ (powers of prime numbers)

$$\varphi(n) = \frac{2}{n} \sum_{\substack{1 \le j \le n-1 \\ ggT(n,j)=1}} j$$

$\boldsymbol{gcd(a,m) = 1 \Rightarrow a^{\varphi(n)} \equiv 1 (\mathrm{mod}\ m)}$ **(theorem of Fermat-Euler)**

$\boldsymbol{p \nmid a \Rightarrow a^{p-1} \equiv 1 (\mathrm{mod}\ p)}$ **(special case for prime numbers, 'little Fermat theorem')**

$\varphi(mn) = \varphi(m)\varphi(n)\frac{d}{\varphi(d)}$, where $d = gcd(m, n)$

$\varphi(n^m) = n^{m-1}\varphi(n)$

$$\boldsymbol{\varphi(n)\sigma_0(n) = \sum_{\substack{1 \le k \le 1 \\ ggT(k,n)=1}} \mathbf{ggT}(k-1, n)} \tag{84}$$

$$\boldsymbol{\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}} \tag{85}$$

The Euler φ function has been generalized by Ramanujan ($\varphi_1(n) = \varphi(n)$):

$$\boldsymbol{\varphi_s(n) = n^s \prod_{p|n} (1 - \frac{1}{p^s})} \tag{86}$$

Ramanujan determined $\varphi_s(n)$ to be:

$$\boldsymbol{\varphi_s(n) = \frac{\mu(n)n^s}{\zeta(s)\sum_{k=1}^{\infty} \frac{\mu(nk)}{k^s}}} \tag{87}$$

$$\boldsymbol{\varphi(n) = \frac{\mu(n)n}{\zeta(s)\sum_{k=1}^{\infty} \frac{\mu(nk)}{k}}} \tag{88}$$

$\varphi(n)$ can also be calculated with a Ramanujan expansion:

$$\boldsymbol{\varphi(n) = \frac{n}{\zeta(s+1)} \sum_{q=1}^{\infty} \frac{\mu(q)c_q(n)}{\varphi_2(q)}}$$

This formula is, however, not practicable, because for the computation of $\varphi(n)$, one also needs $\varphi_2(k)$ ($k = 1 \ldots \infty$).

7.5 THE SUM-OF-DIVISORS FUNCTION (SIGMA FUNCTION)

7.5.1 DEFINITION, PROPERTIES

The sum-of-divisors function $\boldsymbol{\sigma_k(n)}$ gives the sum of the kth powers of the positive divisors of n (including n).

Definition of $\boldsymbol{\sigma_k(n)}$:

$$\boldsymbol{\sigma_k(n) = \sum_{d|n} d^k} \tag{89}$$

Calculation of $\boldsymbol{\sigma_k(n)}$:

(let a_i be the powers of the prime factor decomposition of $n = \prod_{i=1}^{r} p_i^{a_i}$)

$$\boldsymbol{\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} = \prod_{i=1}^{r} \sum_{j=0}^{a_i} p_i^{jk}} \tag{90}$$

If $\sigma_1(n)$ is a prime number, so also is $\sigma_0(n)$. Here are the first 23 pairs:

```
(2,3) (3,7) (3,13) (5,31) (3,31) (7,127) (3,307) (7,1093)
(3,1723) (5,2801) (3,3541) (13,8191) (3,5113) (3,8011)
(3,10303) (7,19531) (3,17293) (3,28057) (5,30941) (3,30103)
(17,131071) (5,88741) (3,86143)
```

```
Mathematica:
For [i=1,i<100000,i++,If[PrimeQ[DivisorSigma[0,i]]==True&&
OddQ[DivisorSigma[0,i]],Print[DivisorSigma[0,i],",",
FactorInteger[DivisorSigma[1,i]]]]]
```

Odd prime values of $\sigma_0(n)$ are rare. Among the first 100000, 79 values (in ascending order) can be found:

```
{3,3,5,3,3,7,5,3,3,3,3,3,5,7,3,3,11,3,3,3,3,5,3,3,3,13,3,3,
3,3,3,3,3,3,3,3,3,3,5,7,3,3,3,3,3,3,3,3,3,5,3,3,3,3,3,3,3,3
,3,3,3,3,3,3,11,3,17,3,3,3,3,3,3,3,5,3,3,3,3}
```

```
Mathematica:
Select[Select[DivisorSigma[0,Range[100000]],OddQ],PrimeQ]
```

Here are a few plots of $\sigma_k(n)$ for different values of k:

```
Mathematica:
k=0;ListLinePlot[Table[{n,DivisorSigma[k,n]},{n,1,50}],
InterpolationOrder->0]
```

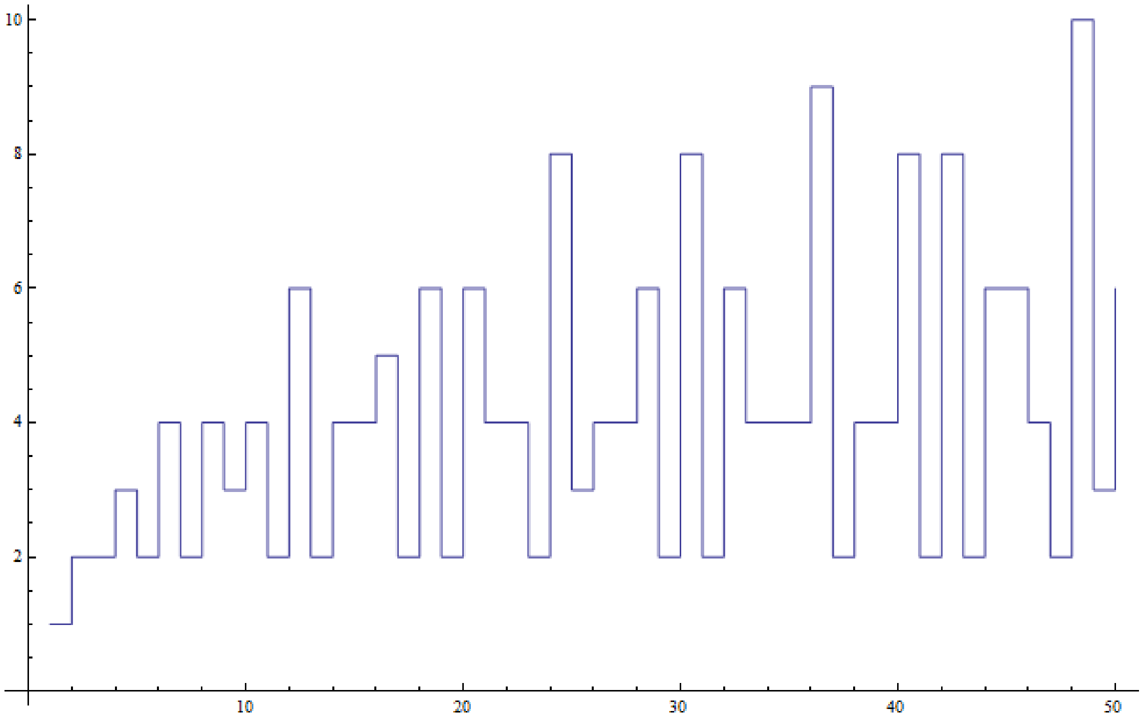


Figure 52. Number of divisors function σ_0(n), plotted from 0 to 50

```
Mathematica:
k=1;ListLinePlot[Table[{n,DivisorSigma[k,n]},{n,1,100}],
InterpolationOrder->0]
```

The first 100 values of $\sigma_0(n)$ read:

{1,2,2,3,2,4,2,4,3,4,2,6,2,4,4,5,2,6,2,6,4,4,2,8,3,4,4,6,2,
8,2,6,4,4,4,9,2,4,4,8,2,8,2,6,6,4,2,10,3,6,4,6,2,8,4,8,4,4,
2,12,2,4,6,7,4,8,2,6,4,8,2,12,2,4,6,6,4,8,2,10,5,4,2,12,4,4
,4,8,2,12,4,6,4,4,4,12,2,6,6,9}

The first 100 values of $\sigma_1(n)$ read:

{1,3,4,7,6,12,8,15,13,18,12,28,14,24,24,31,18,39,20,42,32,3
6,24,60,31,42,40,56,30,72,32,63,48,54,48,91,38,60,56,90,42,
96,44,84,78,72,48,124,57,93,72,98,54,120,72,120,80,90,60,16
8,62,96,104,127,84,144,68,126,96,144,72,195,74,114,124,140,
96,168,80,186,121,126,84,224,108,132,120,180,90,234,112,168
,128,144,120,252,98,171,156,217}

Prime values of $\sigma_1(n)$ are rare, the first 37 values (ascending) in the range up to 1m read:

{3,7,13,31,31,127,307,1093,1723,2801,3541,8191,5113,8011,10
303,19531,17293,28057,30941,30103,131071,88741,86143,147073
,524287,292561,459007,492103,797161,552793,579883,598303,68
4757,704761,732541,735307,830833}

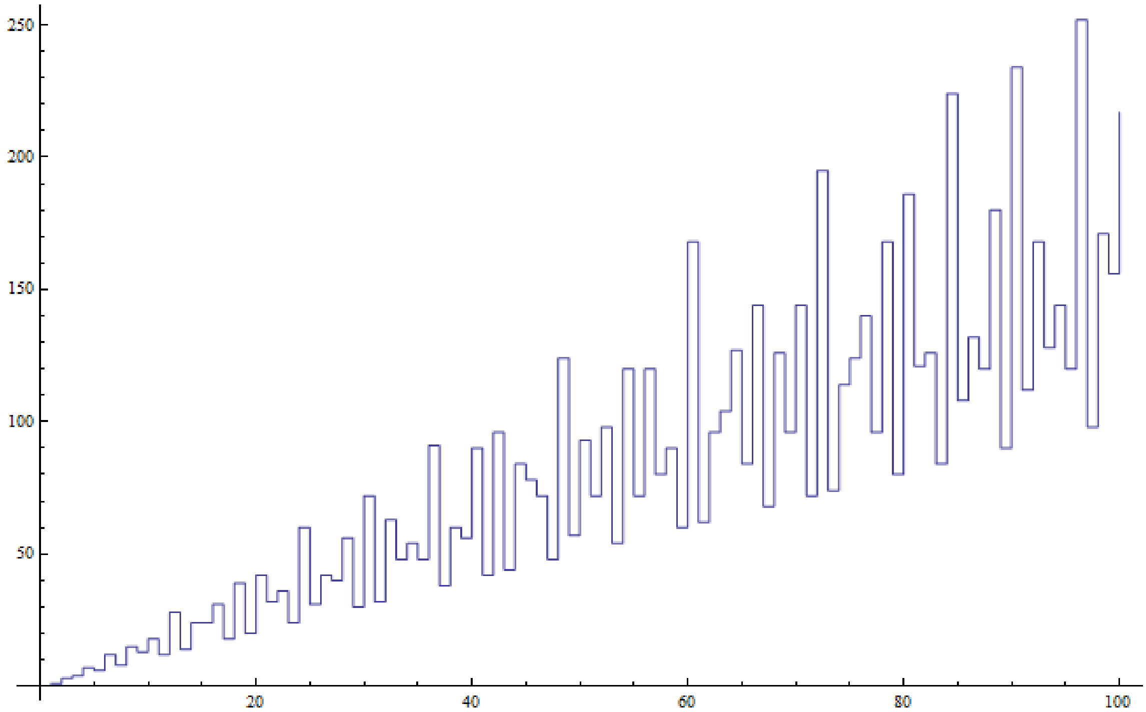


Figure 53. Sum of divisors function $\sigma_1(n)$, plotted from 0 to 100

The sigma function can be expanded in a Ramanujan series with the Ramanujan sums $c_q(n)$ as coefficients (note that s and n need not to be integers):

$$\boldsymbol{\sigma_s(n) = n^s \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}}} \tag{91}$$

as well as:

$$\boldsymbol{\sigma_0(n) = -\sum_{q=1}^{\infty} \frac{ln(q)}{q} c_q(n)} \tag{92}$$

As shown in Chapter 7.9.3, it is possible to extend the Ramanujan sums $c_q(n)$ to real or complex values. Instead of (91) we get a 'Ramanujan sum function'

$$\boldsymbol{\sigma_s(x) = x^s \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(x)}{q^{s+1}}} \tag{93}$$

whose two real and imaginary parts oscillate fairly quickly. This complex function is a wonderful extension of the Ramanujan sums defined only for integer values to $\mathbb{R}$ bzw. $\mathbb{C}$. We can see this clearly when we look at the absolute value of this function: it can be clearly seen that function values at integer arguments are exactly the same as for the arithmetical version. It would be interesting to take a closer look at the information hidden in the ‘phase’ of this function. It looks as if the phase 'rotates', sometimes faster, sometimes more slowly (see Figure 55).

Asymptotic behaviour of $\sigma_k(n)$

$$\sigma_1(n) < e^{\gamma} n \ln(\ln(n)) + \frac{0.6483}{\ln(\ln(n))}, n > 3$$

Properties of $\sigma_k(n)$

$\sigma_0(p) = 2$ (Each prime number has only two divisors: itself and the 1)
$\sigma_0(p^n) = n + 1$
$\sigma_0(n) = \prod_{i=1}^{r}(a_i + 1)$ (a_i see Formula (90))
$\sigma_1(p) = p + 1$
There are infinitely many n such that $\sigma_0(n) = \sigma_0(n+1)$

Conjectures:
The only integer number n for which $\sigma_2(n)$ is prime is 2, where $\sigma_2(2) = 5$.

7.5.2 GRAPHIC REPRESENTATIONS OF THE SIGMA FUNCTION

Here are a few graphs in which the values of $\sigma(n)$ (calculated analytically with the Ramanujan series in red, exact values from number theory in blue), are compared. It can be seen that the red curve is exactly the same as the arithmetical value for integer values.

```
Mathematica code for the following illustration:

cnqx[q_,n_]:= Sum[If[GCD[a,q]==1,Exp[2.0*Pi*I*a*(n/q)],0],{a,1,q}];
s=1.0;
Show[Plot[n^s*Zeta[s+1]*Abs[Sum[cnqx[q,n]/q^(s+1),{q,1,1000}]],
{n,1,12},PlotStyle->Red],ListLinePlot[Table[{k,DivisorSigma[s,k]},
{k,1,12}],InterpolationOrder->0]]
```

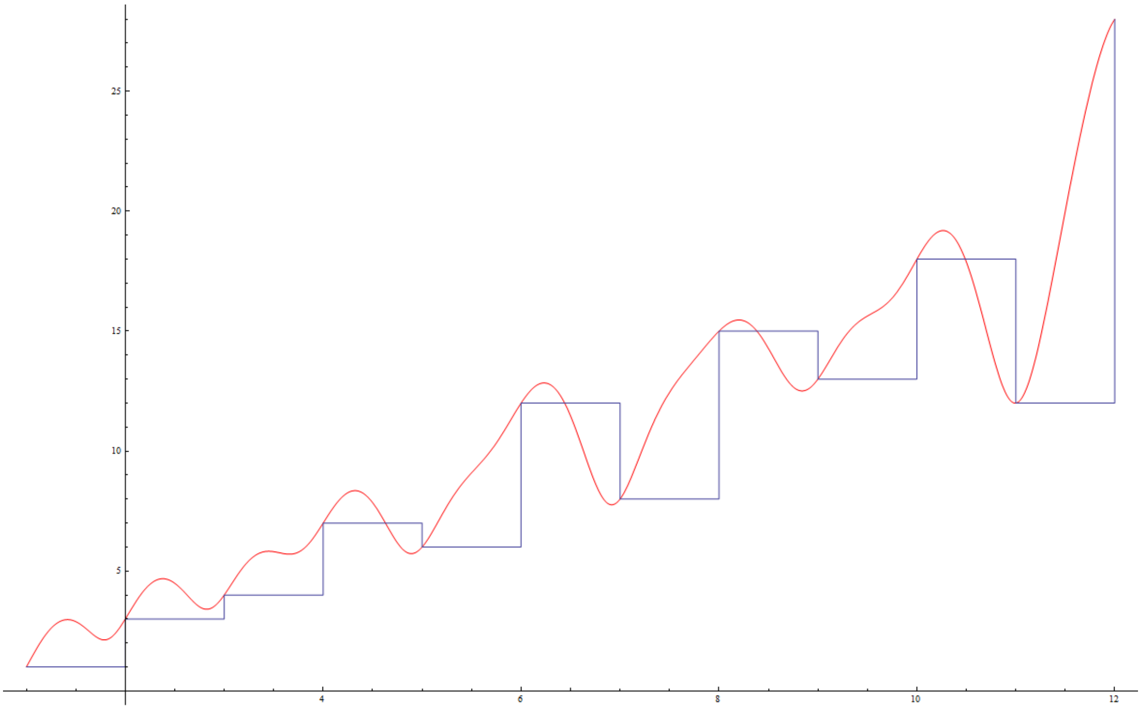


Figure 54. $|\sigma_1(x)|$: comparison of the sigma values calculated analytically with the exact values. Ramanujan sums c_q(n) up to q=1000 have been evaluated, n = 0 to 12

```
Mathematica code for the following illustration:
cnqx[q_,n_]:=Sum[If[GCD[a,q]==1,Exp[2.0*Pi*I*a*(n/q)],0],{a,1,q}];
s=1.0;
Show[Plot[Arg[Sum[cnqx[q,n]/q^(s+1),{q,1,100}]],{n,1,12},
PlotStyle->Red],ListLinePlot[Table[{k,DivisorSigma[s,k]},
{k,1,12}],InterpolationOrder->0]]
```

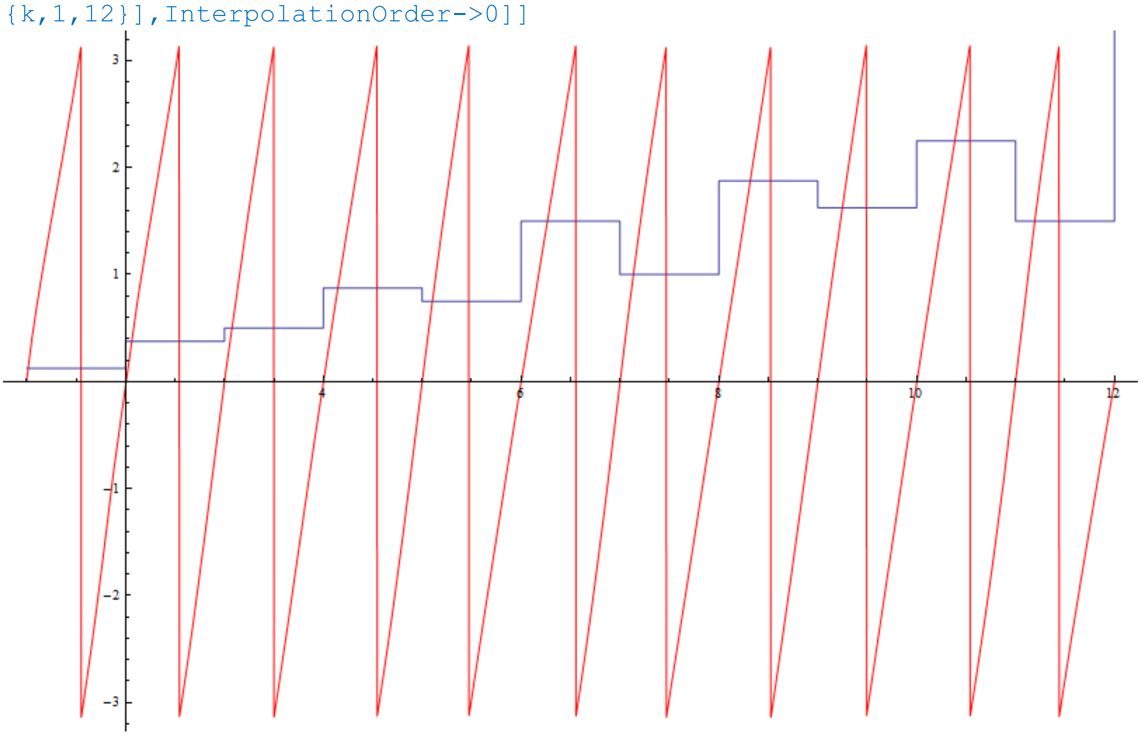


Figure 55. arg($\sigma_1(x)$): argument of the extended sigma function. Ramanujan sums c_q(n) up to q=100 have been evaluated, n goes from 0 to 12. The graph in blue has been rescaled.

```
Mathematica code for the following illustration:
cnqx[q_,n_]:= Sum[If[GCD[a,q]==1,Exp[2.0*Pi*I*a*(n/q)],0],{a,1,q}];
s=1.0;
Plot[n^s*Zeta[s+1]*Abs[Sum[cnqx[q,n]/q^(s+1),{q,1,50}]],{n,1,5000},
MaxRecursion->2,PlotPoints->3000]
```

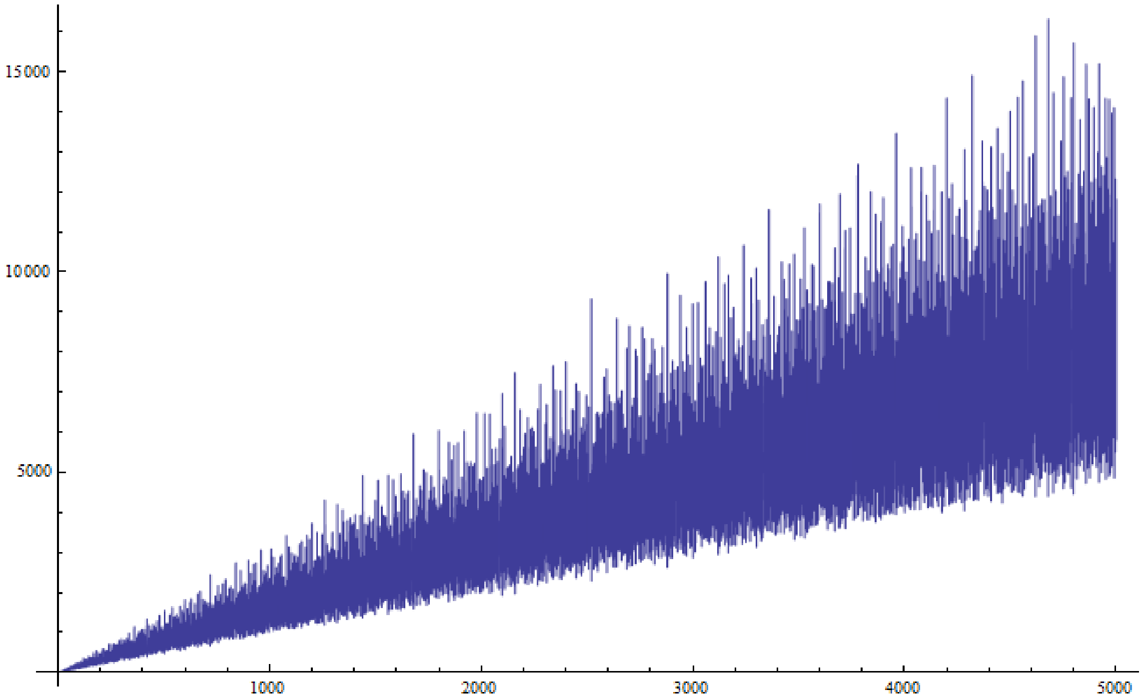


Figure 56. $|\sigma_1(x)|$: values of sigma, calculated analytically. Ramanujan sums $c_q(n)$ up to q=50 have been evaluated

```
Mathematica code for the following illustration:
Show[Monitor[Plot[n^s*Zeta[s+1]*Abs[Sum[cnqx[q,n]/q^(s+1),{q,1,1000}]]
,{n,100,150},PlotStyle-
>Red],n],ListLinePlot[Table[{k,DivisorSigma[s,k]},{k,100,150}],Interpo
lationOrder->0]]
```

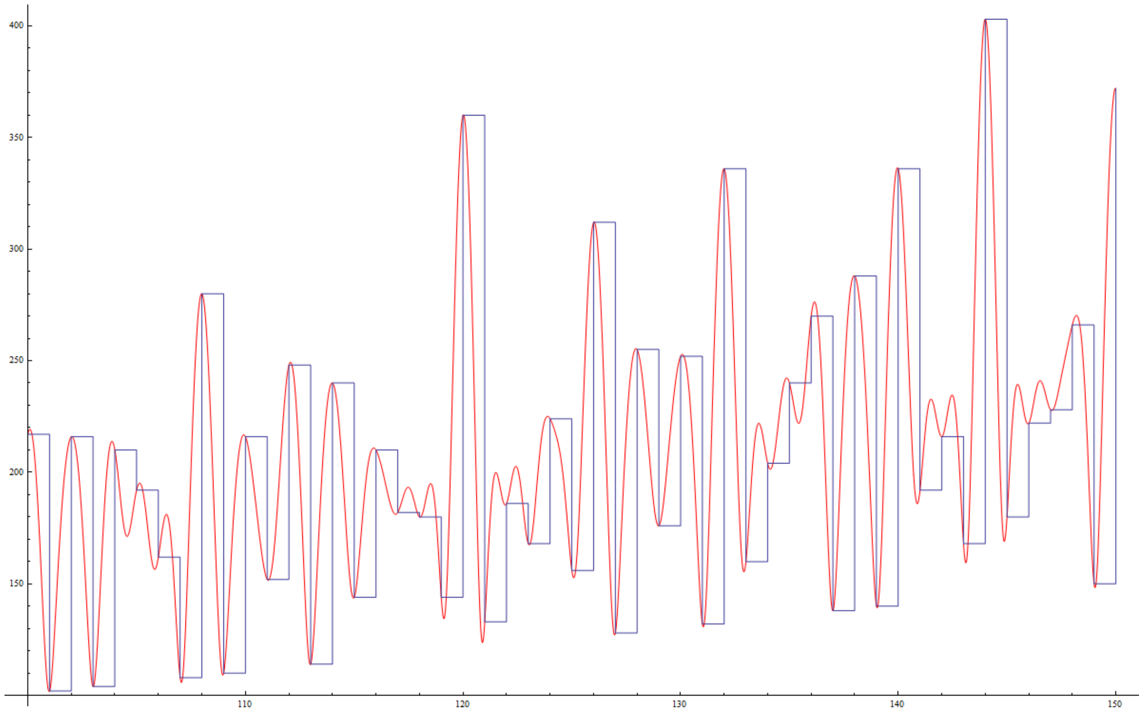


Figure 57. $|\sigma_1(x))$: comparison of the sigma values calculated analytically with the exact values. Ramanujan sums c_q(n) up to q=1000 have been evaluated, n goes from 100 to 150

```
Mathematica code for the following illustration:
Show[Plot[n^s*Zeta[s+1]*Abs[Sum[cnqx[q,n]/q^(s+1),{q,1,3000}]]-n-
1,{n,1000000000,1000000100},PlotStyle->Red],ListLinePlot[
Table[{k,DivisorSigma[s,k]-k-
1},{k,1000000000,1000000100}],InterpolationOrder->0]]
```

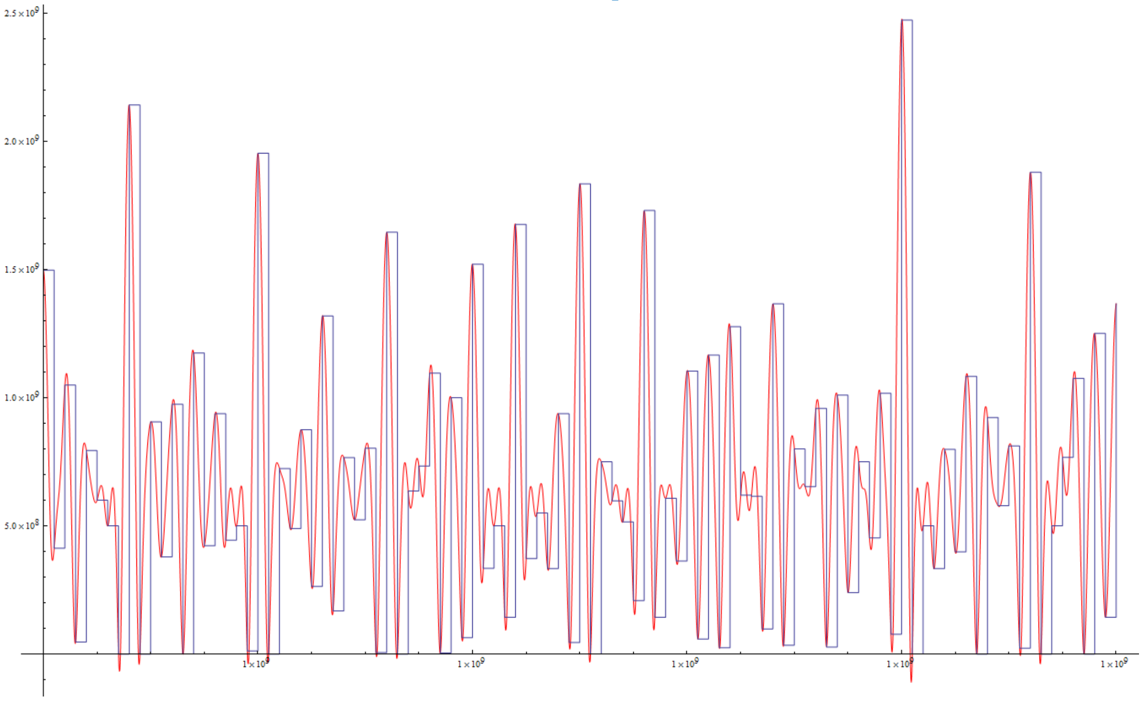


Figure 58. $|\sigma_1$(x) |-x-1: comparison of the sigma values calculated analytically with the exact values. Zeros are at prime number positions

More formulae concerning the sigma function:

$$\sigma_k(n) = \sum_{m=1}^{n} m^{k-1} \sum_{j=1}^{m} \cos(\frac{2\pi jn}{m}) \tag{94}$$

```
Mathematica:
myDivisorSigma[k_,n_]:=Sum[m^(k-1)
Sum[Cos[(2 Pi j n)/m],{j,1,m}],{m,1,n}]
```

Graph following Formula (94):

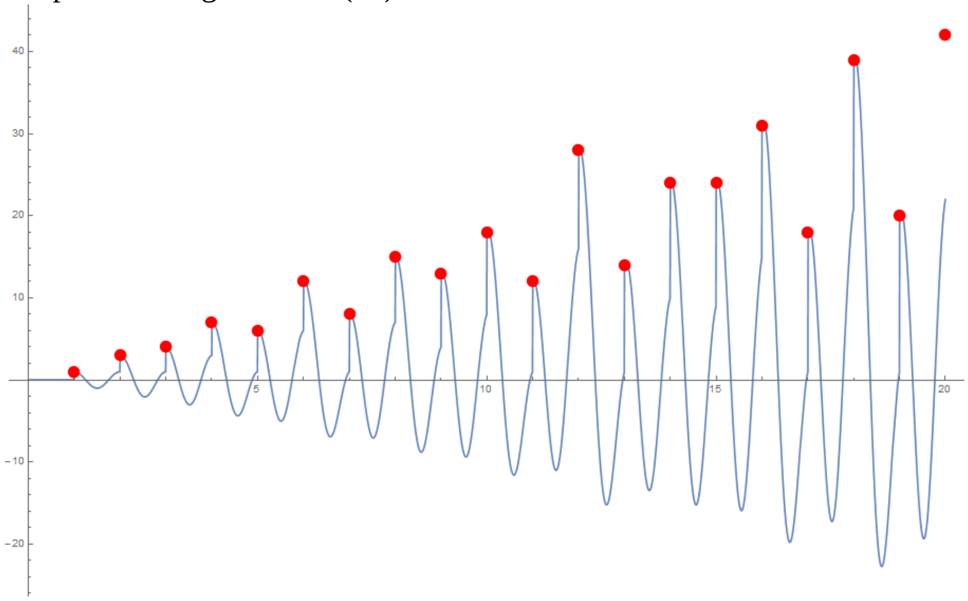


Figure 59. The sigma function calculated analytically using Cos() terms

```
Mathematica:
myDivisorSigma[k_,n_]:=Sum[m^(k-1) Sum[Cos[(2 pi j
n)/m],{j,1,m}],{m,1,n}]
xmin=0; xmax=20;
Show[Plot[N[myDivisorSigma[1,x]],{x,xmin,xmax},
PlotRange->All,AxesOrigin-{0,0}],ListPlot[Table[{n,DivisorSigma[1,n]},
{n,xmin,xmax}],PlotStyle->Red]]
```

$$\sigma_0(n) = \sum_{m=1}^{\infty} (-1)^{m+1} (2\pi n)^{2m} \sum_{j=1}^{m} \frac{(-1)^j 2^{2j-1} \pi^{2j} (B_{2j})^2}{((2j)!)^2 (-2j+2m+1)!} \tag{95}$$

```
Mathematica:
myDivisorSigma0[n_] == Sum[(-1)^(m + 1) (2 Pi n)^(2 m)
Sum[((-1)^j (2 Pi)^(2 j) BernoulliB[2 j]^2)/(2 (2 j)!^2 (2 m + 1 - 2
j)!), {j, 1, m}], {m, 1, Infinity}]
```

7.6 THE RAMANUJAN TAU FUNCTION

In the mathematical literature four different Ramanujan tau functions can be found (the arguments indicate to the most frequently used number field, n: integer, s: complex, t: real).

$\tau(n)$: Ramanujan tau function, Mathematica: RamanujanTau[n]

$L(s)$: Ramanujan tau Dirichlet L function, Mathematica: RamanujanTauL[s]

$Z(t)$: Ramanujan tau Z function, Mathematica: RamanujanTauZ[n]

$\Theta(t)$: Ramanujan tau theta function, Mathematica: RamanujanTauTheta[n]

The graph of $\tau(n)$ looks like:

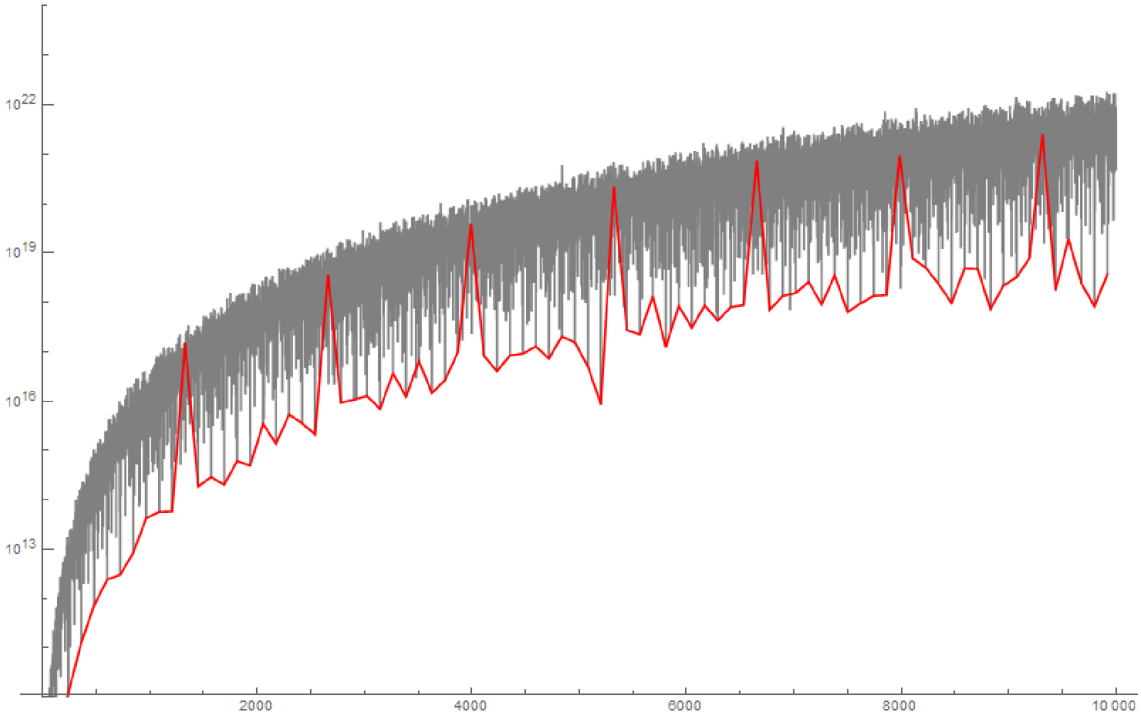


Figure 60. Ramanujan $\tau(\mathrm{N})$ (gray), in red: only n for $Mod(n, 11^2) = 0$

```
Mathematica:
Show[ListLogPlot[Table[{n,Abs[RamanujanTau[n]]},{n,1,10000}],Joined->True,PlotRange->{10^10,10^24},PlotStyle->Gray],
ListLogPlot[Table[{n,Abs[RamanujanTau[n]]},{n,121,10000,121}],Joined->True,PlotRange->{10^10,10^24},PlotStyle->Red,InterpolationOrder->1]]
```

The Ramanujan tau function has many representations (or definitions):
By its generating function [39] $G(x)$

$$G(x) = x\prod_{n=1}^{\infty}(1-x^n)^{24} = \sum_{n=1}^{\infty}\tau(n)x^n \\ = x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 \\ - 6048x^6 + \cdots = x(1 - 3x + 5x^3 - 7x^6 + \cdots)^8 \tag{96}$$

```
(*Mathematica (the first 50 values):*)
CoefficientList[Take[Expand[Product[(1-x^k)^24,{k,1,50}]],50],x]:
```

[39] A sequence a(n) can be defined by the coefficients of a power series expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$. $f(x)$ is called the 'generating function' of $a(n)$

```
{1,-24,252,-1472,4830,-6048,-16744,84480,-113643,-115920,…}

(*From theory of modular forms:*)
max = 28; g[k_] := -BernoulliB[k]/(2k) + Sum[ DivisorSigma[k - 1, n -
1]*q^(n - 1), {n, 2, max + 1}]; CoefficientList[ Series[ 8000*g[4]^3 -
147*g[6]^2, {q, 0, max}], q] // Rest
```

Properties of $\boldsymbol{\tau(n)}$

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}), \text{if } p \in \mathbb{P}\ and\ r > 0 \tag{97}$$

$$|\tau(p)| \le 2p^{\frac{11}{2}}, \text{if } p \in \mathbb{P} \tag{98}$$

$\tau(n)$ is a multiplicative function: $\tau(mn) = \tau(m)\tau(n), \text{if } \gcd(m,n) = 1$

There are very many relationships between τ (n) and the sum-of-divisors functions $\sigma_k(n)$.[40]
Here an example:

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{k=1}^{n-1}\sigma_5(k)\sigma_5(n-k) \tag{99}$$

Ramanujan discovered the following recursive identities:

$$(n-1)\tau(n) = \sum_{m=1}^{b_n}(-1)^{m+1}(2m+1) \times \left(n-1-\frac{9}{2}m(m+1)\right)\tau\left(n-\frac{1}{2}m(m+1)\right), \text{where } b_n = \frac{1}{2}\left(\sqrt{8n+1}-1\right) \tag{100}$$

$$\tau(p^n) = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^j\binom{n-j}{n-2j}p^{11j}\left(\tau(p)\right)^{n-2j} \tag{101}$$

The Ramanujan tau L, tau theta and tau Z functions

$$Z(t) = e^{i\theta(t)}\,L(it+6) \tag{102}$$

[40] https://en.wikipedia.org/wiki/Ramanujan_Tau_function or: http://mathworld.wolfram.com/TauFunction.html

Alternatively (similar to the decomposition of the zeta function by means of the Riemann-Siegel function):

$$L(it+6) = e^{-i\theta(t)} Z(t)$$

$$\boldsymbol{L(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}} \tag{103}$$

where $\theta(t)$ is the Ramanujan tau theta function and $L(s)$ the Ramanujan tau L function. The function $L(s)$ is also known as ‘Ramanujan's Dirichlet L series’. It has similar properties to those of Riemann's zeta function $\zeta(z)$. In fact, it belongs to the type of generalized zeta functions. Ramanujan conjectured that all non-trivial zeros of $L(s)$ lie on the 'critical' line $Re[s] = 6$.
Similar to the zeta function, $L(s)$ also has an Euler product representation:

$$\boldsymbol{L(s) = \prod_{p\epsilon\mathbb{P}}^{\infty} \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}} \tag{104}$$

More formulae and identities can be found on the Internet.[41]
Graphic illustrations (black: real part, red: imaginary part):

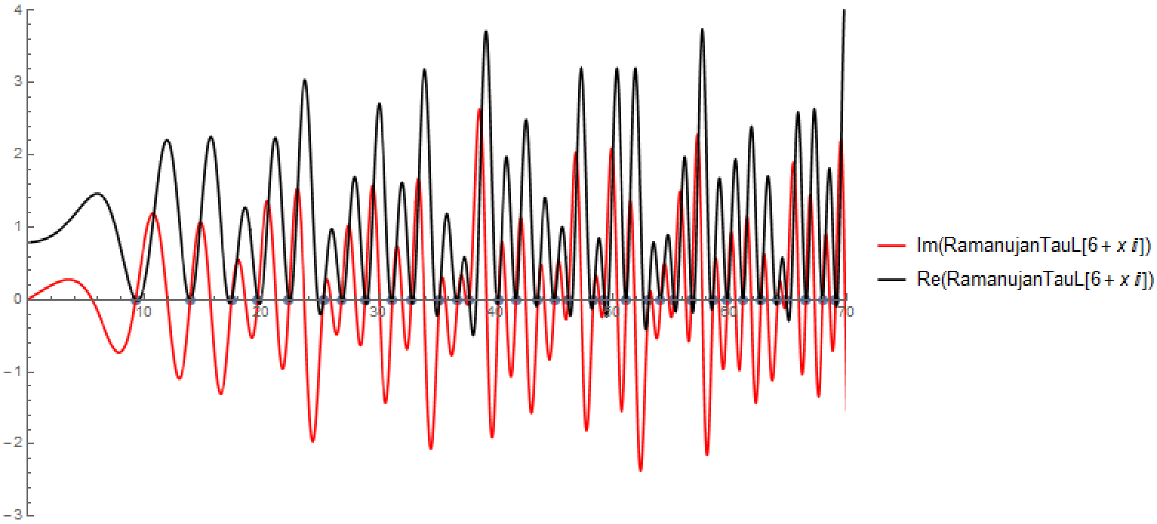


Figure 61. Ramanujan tau L function (Dirichlet-L-series) 0-70, having 34 zeros along the critical line

```
Mathematica:
Show[Plot[{Im[RamanujanTauL[6+x I]],Re[RamanujanTauL[6+x
I]]},{x,0,xmax},PlotStyle->{Red,Black},PlotLegends-
>"Expressions",PlotRange->{{0,70},{-3.,4}},ImageSize->Large]]
```

[41] http://mathworld.wolfram.com/TauDirichletSeries.html

The density of the zeros of the Ramanujan tau L function $L(s)$ is about twice that of the zeta function $\zeta(s)$. In the range up to 70, the ζ function 17 has zeros, while the tau L function has 34 zeros.

A table with the first zeros of Ramanujan's tau L function is given in the Appendix 'Zeros of Ramanujan's tau L function'.

7.7 THE MERTENS FUNCTION

The Mertens function $M(n)$ is the summatory function of the Moebius function $\mu(n)$:

$$M(n) = \sum_{k=1}^{n} \mu(k) \tag{105}$$

Definition of $\mu(n)$ (for $n > 0$):

$$\mu(n) = \begin{Bmatrix} 1, \text{if } n \text{ square} - \text{free and having an even number of prime factors} \\ -1, \text{if } n \text{ square} - \text{free and having an odd number of prime factors} \\ 0, \text{if } n \text{ has a square prime factor} \end{Bmatrix}$$

$\mu(n)$ can be calculated without knowledge of the prime factor decomposition of n (however with the same complexity):

$$\mu(n) = \sum_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} e^{-2\pi i \frac{k}{n}} \tag{106}$$

Properties of the μ function:

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \tag{107}$$

Also interesting is the representation as a sum over Farey sequences:

$$M(n) = \sum_{a \in \mathcal{F}_n} e^{-2\pi i a} \tag{108}$$

Representation of the Moebius function:

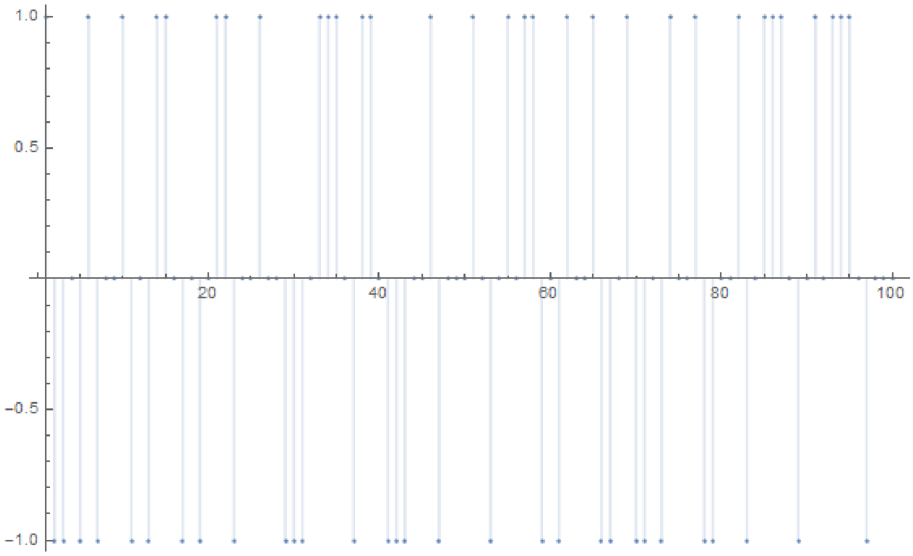


Figure 62. Moebius function μ(n), from 1 to 100

```
Mathematica:
DiscretePlot[MoebiusMu[k],{k,100}]
```

Note: the Moebius function $\mu(n)$ is also defined for negative n or whole complex numbers.

Formulae

$\mu(n)$ is closely related to the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \tag{109}$$

Graph of the Mertens function:

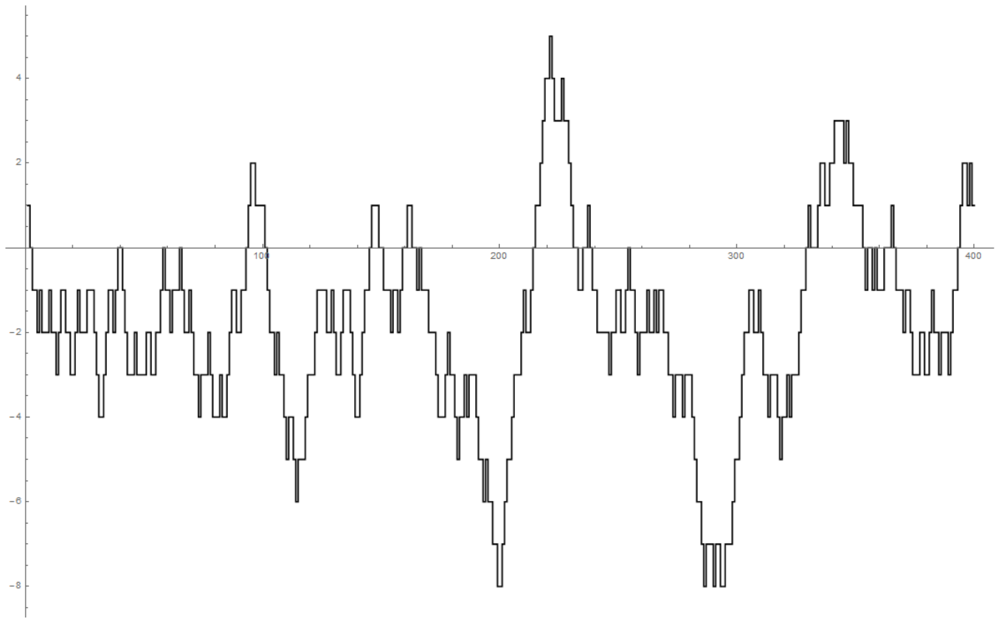


Figure 63. Mertens function M(n) from 1 to 400

```
Mathematica:
m[n_]:=Sum[MoebiusMu[k],{k,1,n}]
ListLinePlot[Table[m[n],{n,400}],InterpolationOrder->0,
PlotStyle->Black]
```

The Mertens function has zeros at:

2,39,40,58,65,93,101,145,149,150,159,160,163,164,166,214,231,232,235,
236,238,254,329,331,332,333,353,355,356,358,362,363,364,366,393…

There is also a recursive representation of the Mertens function (here calculated using the following Mathematica program):

```
Mathematica:
(*Conjectured recurrence (two combined recurrences):*)
t[n_,k_]:=t[n,k]=If[And[n==1,k==1],3,If[Or[And[n==1,k==2],And[n==2,k==
1]],2,If[n==1,(-t[n,k-1]-Sum[t[i,k],{i,2,k-1}])/(k+1)+t[n,k-
1],If[k==1,(-t[n-1,k]-Sum[t[n,i],{i,2,n-1}])/(n+1)+t[n-1,k],If[n>=k,-
Sum[t[n-i,k],{i,1,k-1}],-Sum[t[k-i,n],{i,1,n-1}]]]]]];
nn=100;
MatrixForm[Table[Table[t[n,k],{k,1,nn}],{n,1,nn}]];
Table[t[1,k],{k,1,nn}]-2 (*Mats Granvik,Jul 10,2011*)
```

Further interesting arithmetical relations to the zeta function and other functions can be found on the Internet[42].

7.8 THE RADICAL

The radical $\mathrm{rad}(n)$ is defined as the product of different prime factors of n:

$$\mathrm{rad}(n) = \prod_{\substack{p|n \\ p\in\mathbb{P}}} p \tag{110}$$

The calculation using Mathematica is very simple:

```
Table[Last[Select[Divisors[n], SquareFreeQ]], {n, 100}]

rad[n_] := Times @@ (First@# & /@ FactorInteger@ n); Array[rad, 100]
```

The first 50 values read:

{1,2,3,2,5,6,7,2,3,10,11,6,13,14,15,2,17,6,19,10,21,22,23,6,5,26,3,14,
29,30,31,2,33,34,35,6,37,38,39,10,41,42,43,22,15,46,47,6,7,10}

[42] https://en.wikipedia.org/wiki/Mertens_function

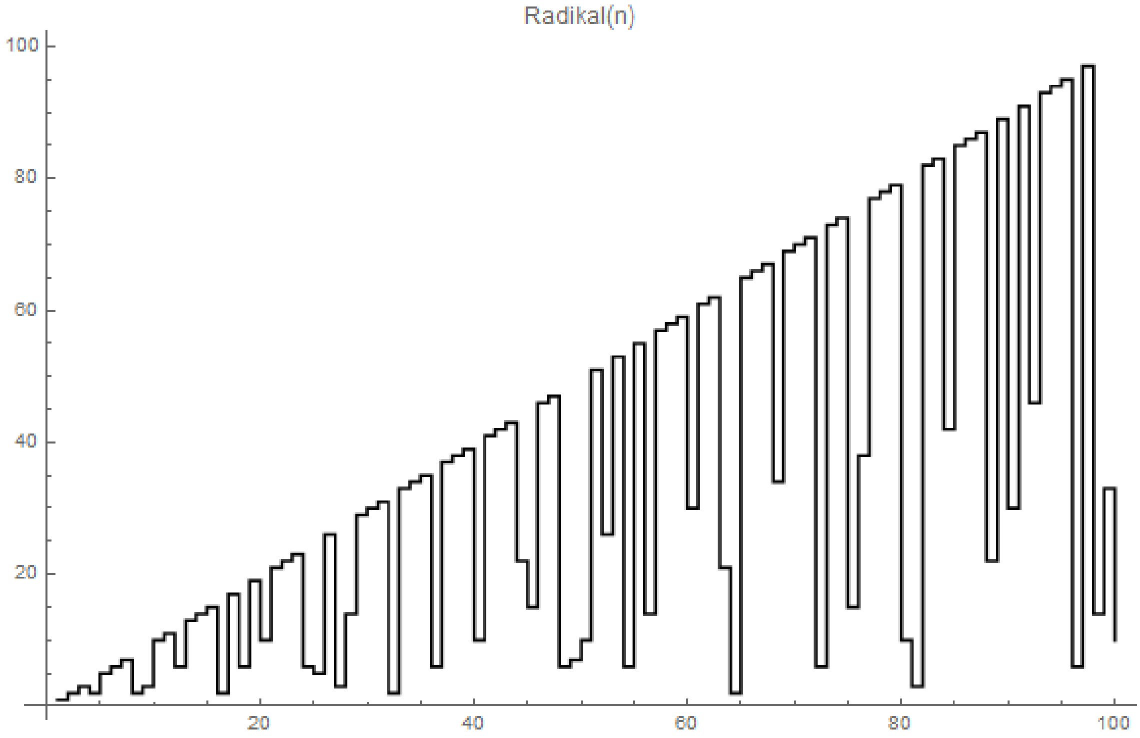


Figure 64. Radical(n) (n=1,100)

Properties

An important application of the function $\mathrm{rad}(n)$ can be found in the examination of the 'abc conjecture' (Chapter 11.1).

Note:
The Moebius transformation of rad(n) gives the absolute values of $\mu(n)\varphi(n)$.

7.9 RAMANUJAN SUMS

Ramanujan series

Using the Ramanujan sums $c_q(n)$, many arithmetical functions can be represented by a so-called Ramanujan series expansion:

$\mathbf{0 = \sum_{k=1}^{\infty} \frac{1}{k} c_q(n)}$ (Ramanujan series of the null function)

$\boldsymbol{\sigma_s(n) = n^s \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}}}$ (Ramanujan series of the sigma function)

$\boldsymbol{\sigma_0(n) = -\sum_{q=1}^{\infty} \frac{ln(q)}{q} c_q(n)}$ (number-of-divisors function)

7.9.1 DEFINITION

A Ramanujan sum is a function depending on two integers n and q:

$$\boldsymbol{c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{2\pi i \frac{a}{q} n}, \; n = 0, 1, 2, \ldots} \text{ let } \boldsymbol{(a, 0)} \text{ be defined as } \boldsymbol{a} \tag{111}$$

$(a, q) = 1$ means that $\gcd(a, q)$ shall be 1, i.e. a and q shall be 'coprime'; i.e. the sum includes all a that are coprime to q ($\gcd(a, q) = 1$).

Example: the Ramanujan sums for q = 1 to 15 (and $n = 0$ to 20) are (the periodicity is in each case q, printed in red):

$c_1(n)$:{1,**1**,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,…}
$c_2(n)$:{1,**-1**,**1**,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,…}
$c_3(n)$:{2,**-1,-1,2**,-1,-1,2,-1,-1,2,-1,-1,2,-1,-1,2,-1,-1,2,-1,-1,…}
$c_4(n)$:{2,**0,-2,0,2**,0,-2,0,2,0,-2,0,2,0,-2,0,2,0,-2,0,2,…}
$c_5(n)$:{4,**-1,-1,-1,-1,4**,-1,-1,-1,-1,4,-1,-1,-1,-1,4,-1,-1,-1,-1,4,…}
$c_6(n)$:{2,**1,-1,-2,-1,1,2**,1,-1,-2,-1,1,2,1,-1,-2,-1,1,2,1,-1,…}
$c_7(n)$:{6,**-1,-1,-1,-1,-1,-1,6**,-1,-1,-1,-1,-1,-1,6,-1,-1,-1,-1,-1,-1,…}
$c_8(n)$:{4,**0,0,0,-4,0,0,0,4**,0,0,0,-4,0,0,0,4,0,0,0,-4,…}
$c_9(n)$:{6,**0,0,-3,0,0,-3,0,0,6**,0,0,-3,0,0,-3,0,0,6,0,0,…}
$c_{10}(n)$:{4,**1,-1,1,-1,-4,-1,1,-1,1,4**,1,-1,1,-1,-4,-1,1,-1,1,4,…}
$c_{11}(n)$:{10,**-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,10**,-1,-1,-1,-1,-1,-1,-1,-1,-1,…}(11)
$c_{12}(n)$:{4,**0,2,0,-2,0,-4,0,-2,0,2,0,4**,0,2,0,-2,0,-4,0,-2,…}
$c_{13}(n)$:{12,**-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,12**,-1,-1,-1,-1,-1,-1,-1,…}
$c_{14}(n)$:{6,**1,-1,1,-1,1,-1,-6,-1,1,-1,1,-1,1,6**,1,-1,1,-1,1,-1,…}
$c_{15}(n)$:{8,**1,1,-2,1,-4,-2,1,1,-2,-4,1,-2,1,1,8**,1,1,-2,1,-4,…}
$c_{16}(n)$:{{8,0,0,0,0,0,0,0,-8,0,0,0,0,0,0,0,8,0,0,0,0}
$c_{17}(n)$:{{16,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,16,-1,-1,-1}

Mathematica program for the table above (please note that the table has not been created by numerical calculations, but only by means of symbolic computation …):

```
Clear[q]; Clear[n];
Column[Table[FullSimplify[Sum[If[GCD[a,q]==1,Exp[2*Pi*I*a*(n/q)],0],{a
,1,q}]],{q,1,17},{n,0,20}]]
```

or (if n is limited up to the period):

```
Column[Table[FullSimplify[Sum[If[GCD[a,q]==1,Exp[2*Pi*I*a*(n/q)],0],{a
,1,q}]],{q,1,15},{n,0,q}]]
```

The following representation with real trigonometric functions can be extended to ℝ (below the $c_q(n)$ going from $n = 1$ to 17), prime indices are shown in red:

$$\cos[2n\pi]$$
$$\cos[n\pi]$$
$$\cos[\frac{2n\pi}{3}] + \cos[\frac{4n\pi}{3}]$$
$$\cos[\frac{n\pi}{2}] + \cos[\frac{3n\pi}{2}]$$
$$\cos[\frac{2n\pi}{5}] + \cos[\frac{4n\pi}{5}] + \cos[\frac{6n\pi}{5}] + \cos[\frac{8n\pi}{5}]$$
$$\cos[\frac{n\pi}{3}] + \cos[\frac{5n\pi}{3}]$$
$$\cos[\frac{2n\pi}{7}] + \cos[\frac{4n\pi}{7}] + \cos[\frac{6n\pi}{7}] + \cos[\frac{8n\pi}{7}] + \cos[\frac{10n\pi}{7}] + \cos[\frac{12n\pi}{7}]$$
$$\cos[\frac{n\pi}{4}] + \cos[\frac{3n\pi}{4}] + \cos[\frac{5n\pi}{4}] + \cos[\frac{7n\pi}{4}]$$
$$\cos[\frac{2n\pi}{9}] + \cos[\frac{4n\pi}{9}] + \cos[\frac{8n\pi}{9}] + \cos[\frac{10n\pi}{9}] + \cos[\frac{14n\pi}{9}] + \cos[\frac{16n\pi}{9}]$$
$$\cos[\frac{n\pi}{5}] + \cos[\frac{3n\pi}{5}] + \cos[\frac{7n\pi}{5}] + \cos[\frac{9n\pi}{5}]$$
$$\cos\left[\frac{2n\pi}{11}\right] + \cos\left[\frac{4n\pi}{11}\right] + \cos\left[\frac{6n\pi}{11}\right] + \cos\left[\frac{8n\pi}{11}\right] + \cos\left[\frac{10n\pi}{11}\right] +$$
$$\cos[\frac{12n\pi}{11}] + \cos[\frac{14n\pi}{11}] + \cos[\frac{16n\pi}{11}] + \cos[\frac{18n\pi}{11}] + \cos[\frac{20n\pi}{11}]$$
$$\cos[\frac{n\pi}{6}] + \cos[\frac{5n\pi}{6}] + \cos[\frac{7n\pi}{6}] + \cos[\frac{11n\pi}{6}]$$
$$\cos\left[\frac{2n\pi}{13}\right] + \cos\left[\frac{4n\pi}{13}\right] + \cos\left[\frac{6n\pi}{13}\right] + \cos\left[\frac{8n\pi}{13}\right] + \cos\left[\frac{10n\pi}{13}\right] + \cos\left[\frac{12n\pi}{13}\right] +$$
$$\cos[\frac{14n\pi}{13}] + \cos[\frac{16n\pi}{13}] + \cos[\frac{18n\pi}{13}] + \cos[\frac{20n\pi}{13}] + \cos[\frac{22n\pi}{13}] + \cos[\frac{24n\pi}{13}]$$
$$\cos[\frac{n\pi}{7}] + \cos[\frac{3n\pi}{7}] + \cos[\frac{5n\pi}{7}] + \cos[\frac{9n\pi}{7}] + \cos[\frac{11n\pi}{7}] + \cos[\frac{13n\pi}{7}]$$
$$\cos[\frac{2n\pi}{15}] + \cos[\frac{4n\pi}{15}] + \cos[\frac{8n\pi}{15}] + \cos[\frac{14n\pi}{15}] + \cos[\frac{16n\pi}{15}] + \cos[\frac{22n\pi}{15}] + \cos[\frac{26n\pi}{15}] + \cos[\frac{28n\pi}{15}]$$
$$\cos[\frac{n\pi}{8}] + \cos[\frac{3n\pi}{8}] + \cos[\frac{5n\pi}{8}] + \cos[\frac{7n\pi}{8}] + \cos[\frac{9n\pi}{8}] + \cos[\frac{11n\pi}{8}] + \cos[\frac{13n\pi}{8}] + \cos[\frac{15n\pi}{8}]$$
$$\cos\left[\frac{2n\pi}{17}\right] + \cos\left[\frac{4n\pi}{17}\right] + \cos\left[\frac{6n\pi}{17}\right] + \cos\left[\frac{8n\pi}{17}\right] + \cos\left[\frac{10n\pi}{17}\right] + \cos\left[\frac{12n\pi}{17}\right] + \cos\left[\frac{14n\pi}{17}\right] + \cos\left[\frac{16n\pi}{17}\right] +$$
$$\cos[\frac{18n\pi}{17}] + \cos[\frac{20n\pi}{17}] + \cos[\frac{22n\pi}{17}] + \cos[\frac{24n\pi}{17}] + \cos[\frac{26n\pi}{17}] + \cos[\frac{28n\pi}{17}] + \cos[\frac{30n\pi}{17}] + \cos[\frac{32n\pi}{17}]$$

Table: Ramanujan sums $\boldsymbol{c_q}(n)$ represented by cos() terms. This representation can be extended from ℕ to ℝ or ℂ (see illustration below).

```
Mathematica program for the table above:
Clear[n];
Column[Table[FullSimplify[Sum[If[GCD[a,q]==1,Cos[2*Pi*a*(n/q)],0],{a,1
,q}]],{q,1,17}]]
```

For integers, the representation can be simplified (in each case the second half of a term is the same as the first half, prime indices in red):

$$
\begin{gathered}
1,\\
\mathrm{Cos}[n\pi],\\
2\mathrm{Cos}\left[\frac{2n\pi}{3}\right],\\
2\mathrm{Cos}\left[\frac{n\pi}{2}\right],\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{5}\right]+\mathrm{Cos}\left[\frac{4n\pi}{5}\right]\right),\\
\mathrm{Cos}\left[\frac{n\pi}{3}\right],\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{7}\right]+\mathrm{Cos}\left[\frac{4n\pi}{7}\right]+\mathrm{Cos}\left[\frac{6n\pi}{7}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{n\pi}{4}\right]+\mathrm{Cos}\left[\frac{3n\pi}{4}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{9}\right]+\mathrm{Cos}\left[\frac{4n\pi}{9}\right]+\mathrm{Cos}\left[\frac{8n\pi}{9}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{n\pi}{5}\right]+\mathrm{Cos}\left[\frac{3n\pi}{5}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{11}\right]+\mathrm{Cos}\left[\frac{4n\pi}{11}\right]+\mathrm{Cos}\left[\frac{6n\pi}{11}\right]+\mathrm{Cos}\left[\frac{8n\pi}{11}\right]+\mathrm{Cos}\left[\frac{10n\pi}{11}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{n\pi}{6}\right]+\mathrm{Cos}\left[\frac{5n\pi}{6}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{13}\right]+\mathrm{Cos}\left[\frac{4n\pi}{13}\right]+\mathrm{Cos}\left[\frac{6n\pi}{13}\right]+\mathrm{Cos}\left[\frac{8n\pi}{13}\right]+\mathrm{Cos}\left[\frac{10n\pi}{13}\right]+\mathrm{Cos}\left[\frac{12n\pi}{13}\right]\right)\\
2\left(\mathrm{Cos}\left[\frac{n\pi}{7}\right]+\mathrm{Cos}\left[\frac{3n\pi}{7}\right]+\mathrm{Cos}\left[\frac{5n\pi}{7}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{2n\pi}{15}\right]+\mathrm{Cos}\left[\frac{4n\pi}{15}\right]+\mathrm{Cos}\left[\frac{8n\pi}{15}\right]+\mathrm{Cos}\left[\frac{14n\pi}{15}\right]\right),\\
2\left(\mathrm{Cos}\left[\frac{n\pi}{8}\right]+\mathrm{Cos}\left[\frac{3n\pi}{8}\right]+\mathrm{Cos}\left[\frac{5n\pi}{8}\right]+\mathrm{Cos}\left[\frac{7n\pi}{8}\right]\right),\\
2\left(\begin{gathered}\mathrm{Cos}\left[\frac{2n\pi}{17}\right]+\mathrm{Cos}\left[\frac{4n\pi}{17}\right]+\mathrm{Cos}\left[\frac{6n\pi}{17}\right]+\mathrm{Cos}\left[\frac{8n\pi}{17}\right]+\\ \mathrm{Cos}\left[\frac{10n\pi}{17}\right]+\mathrm{Cos}\left[\frac{12n\pi}{17}\right]+\mathrm{Cos}\left[\frac{14n\pi}{17}\right]+\mathrm{Cos}\left[\frac{16n\pi}{17}\right]\end{gathered}\right)
\end{gathered}
$$

Table: Ramanujan sums $c_q(n)$ represented by cos() terms, for integer numbers n.

There is an alternative method of calculating the Ramanujan sum, using the Moebius function $\mu(n)$ and the Euler totient function $\varphi(q)$:

$$\boldsymbol{c_q(n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)}} \tag{112}$$

Mathematica:
```
nmax=14;cnq[q_,n_]:=EulerPhi[q]*(MoebiusMu[q/GCD[q,n]]/
EulerPhi[q/GCD[q,n]]);Column[Table[cnq[q,n],{q,1,nmax},{n,1,nmax}]]
```

Here are a few graphic illustrations of Ramanujan sums:

```
Mathematica program: please contact the author.
```

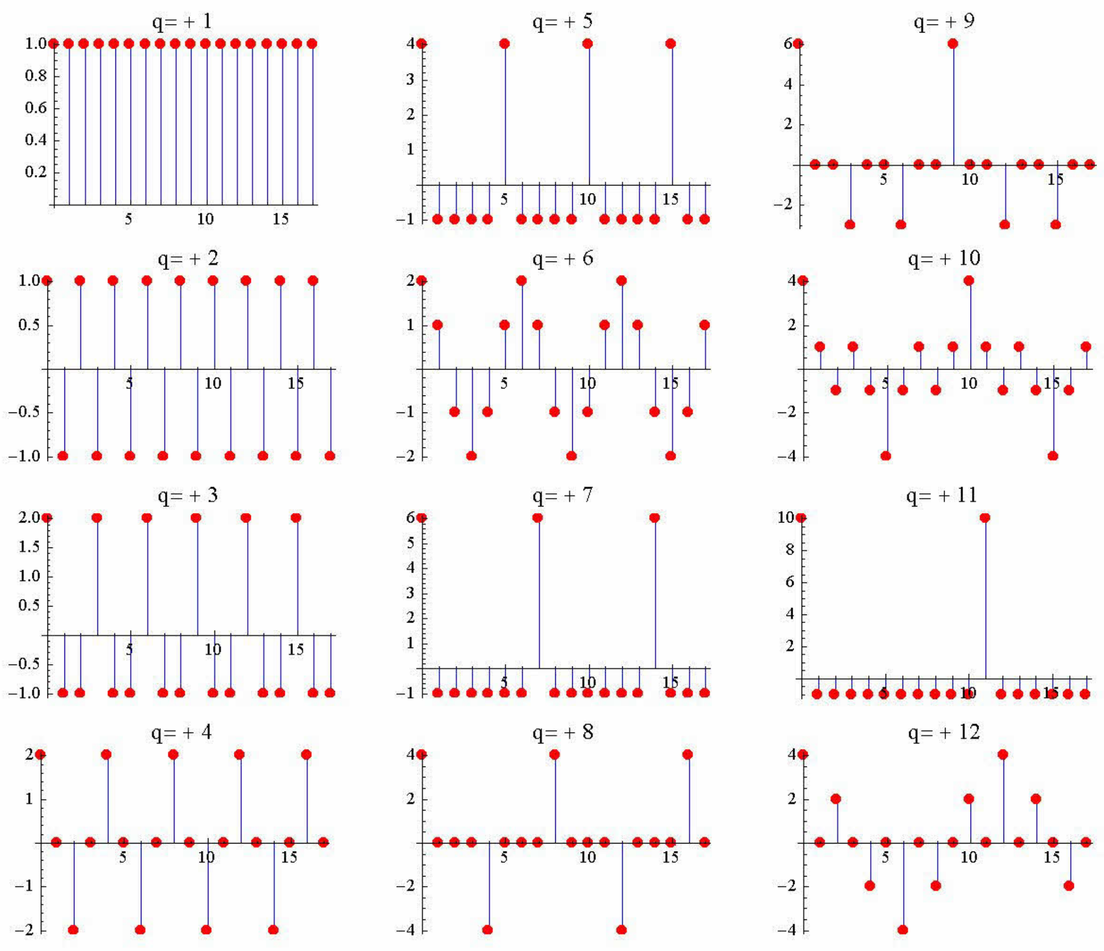


Figure 65. Ramanujan sums $c_q(\mathrm{n})$ from $q\ = 1$ to 12 and n going from 0 to 17

```
Mathematica programm: please contact the author.
```

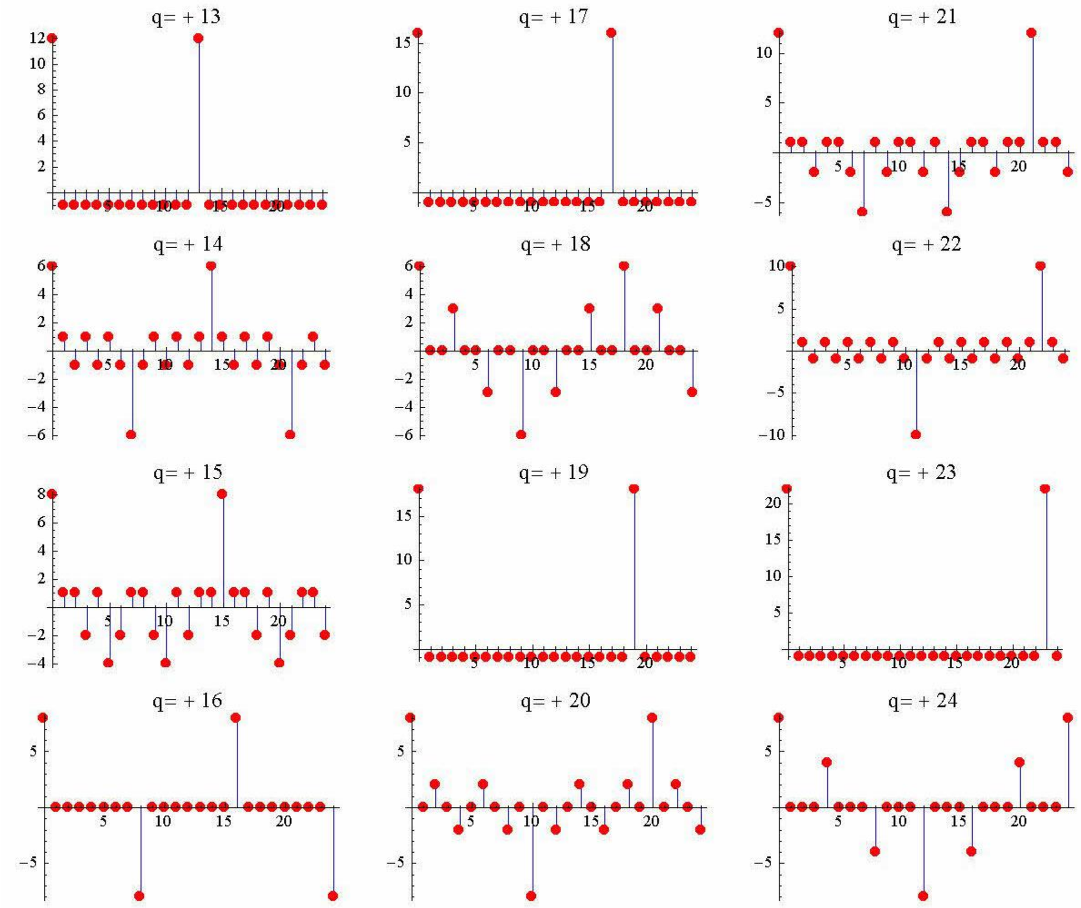


Figure 66. Ramanujan sums $c_q(\mathrm{n})$ from $q\ = 1$ to 24 and n from 0 to 24

7.9.2 PROPERTIES

$\boldsymbol{c_q(n)}$ has a number of remarkable properties. The following ones can easily be checked using the table above:

$\boldsymbol{c_q(n)}$ is always real and integer despite its complex definition.

$$\boldsymbol{c_q(n) = c_q(-n)}$$

$\boldsymbol{c_q(0) = \varphi(q)}$

$\boldsymbol{c_q(1) = \mu(q)}$

$\boldsymbol{c_{qr}(n) = c_q(n)c_r(n)}$, **if** $\boldsymbol{(q,r) = 1}$ **(multiplicativity)**

$\boldsymbol{c_q(n) = c_q\big((q,n)\big)}$

$\boldsymbol{|c_q(n)|}$ **never becomes larger than** $\boldsymbol{\varphi(q)}$, **if** $\boldsymbol{q}$ **is fixed)**

$\boldsymbol{|c_q(n)|}$ **never becomes larger than** $\boldsymbol{n}$ **(if** $\boldsymbol{n}$ **is fixed)**

$c_q(n)$ **= 0, if the natural number** $\frac{q}{(q,n)}$ **has** p^2 **as a divisor,** p **being prime**

$$c_q(q) = c_q(kq) = \varphi(q), k = 0, 1, 2, \ldots$$

$$c_p(n) = \begin{cases} -1, \text{if } p \nmid n \\ \varphi(p), \text{if } p \mid n \end{cases}$$

$$c_{p^k}(n) = \begin{cases} 0, \text{if } p^{k-1} \nmid n \\ -1p^{k-1}, \text{if } p^{k-1} \mid n \text{ and } p^k \nmid n \\ \varphi(p^k), \text{if } p^k \mid n \end{cases}$$

$\sum_{n=a}^{a+q-1} c_q(n) = 0$ **(the sum over all terms of a period results in 0!)**

$\frac{1}{m}\sum_{k=1}^{m} c_{m_1}(k) c_{m_2}(k) = \begin{cases} \varphi(m), \text{if } m_1 = m_2 = m \\ 0, \text{otherwise} \end{cases}$ **(m = lcm(**m_1, m_2**))**

(orthogonality)

7.9.3 EXTENSION TO ℝ

If we allow n to have any real values x, then we get a real function that depends on an integer parameter q:

$$c_q(x) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{2\pi i \frac{a}{q} x}, \tag{113}$$

Here are a few graphic illustrations of Ramanujan sums:

```
Mathematica program: please contact the author.
```

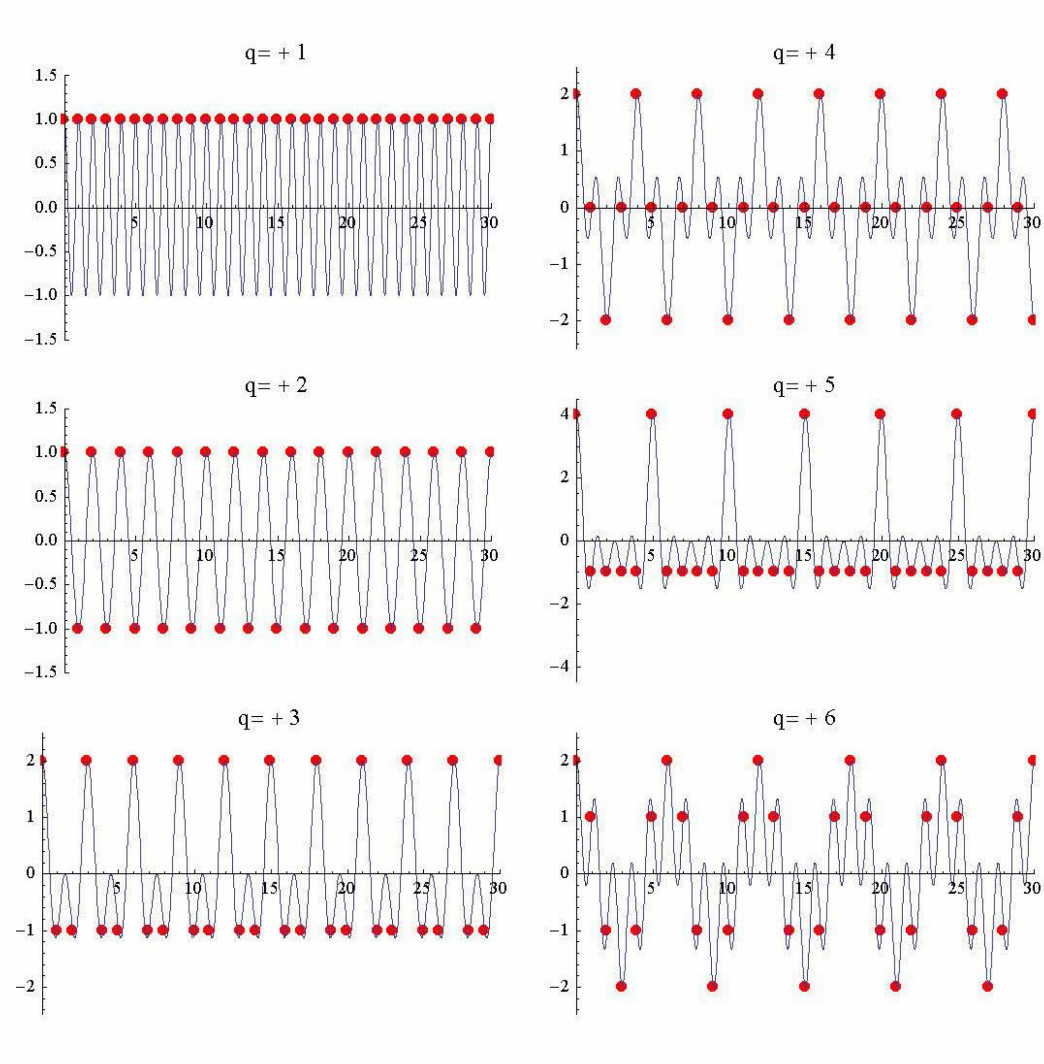


Figure 67. Ramanujan sums $c_q(x)$, analytically extended ($q = 1$ to 6 and $x = 0$ to 30)

`Mathematica program: please contact the author.`

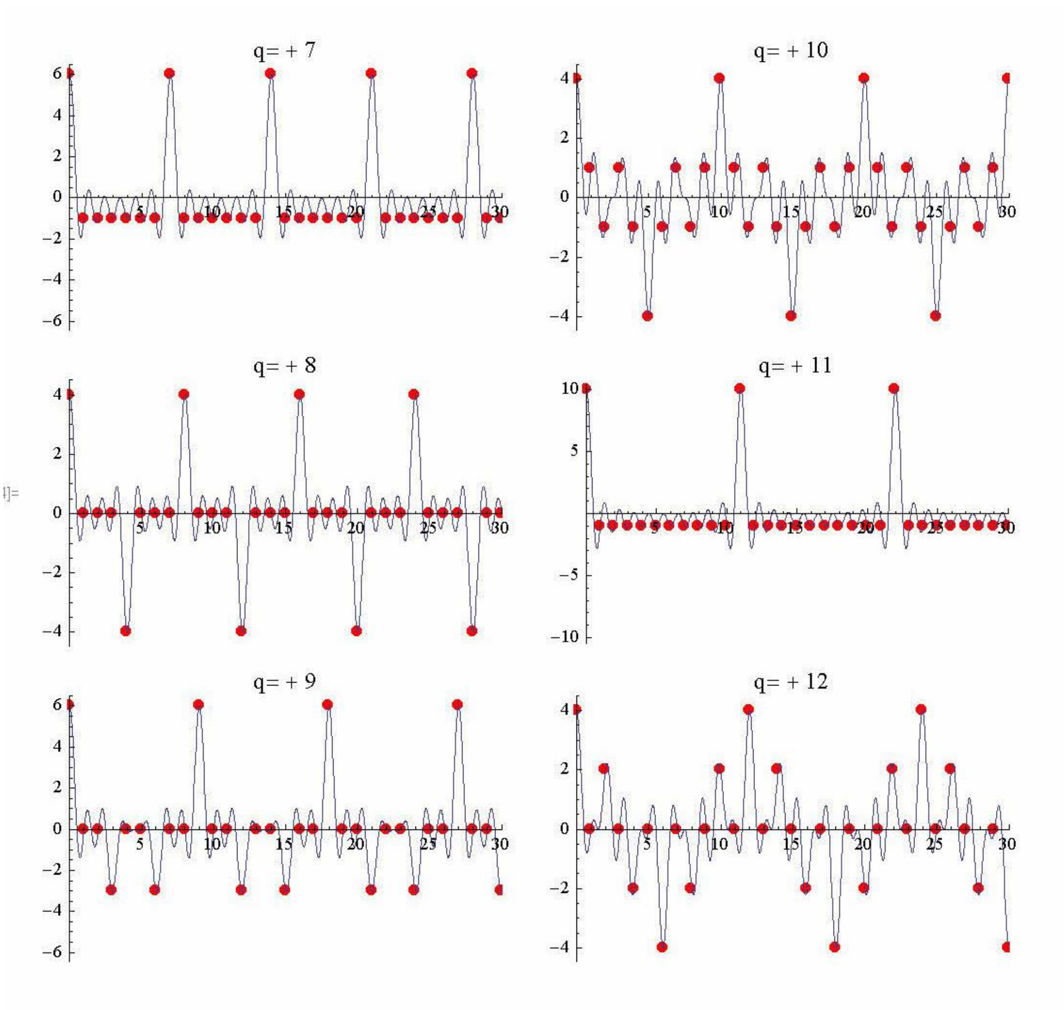


Figure 68. Ramanujan sums $c_q(\mathrm{x})$, analytically extended ($q = 7$ to 12 and $x = 0$ to 30)

8 FUNCTIONS FOR THE CALCULATION OF PRIME NUMBERS

We distinguish between the following types of functions that can be used to compute prime numbers:

- functions $f(n)$ that provide exactly all p_n
- functions $f(n)$ that always return a prime number
- functions $f(n)$ whose positive sets of integer values assumed by the function are identical to the set of prime numbers
- functions that calculate the number of primes up to a given limit

8.1 FUNCTIONS THAT PROVIDE EXACTLY ALL PRIME NUMBERS

$$p_n = \left\lfloor 1 - \log_2\left(-\frac{1}{2} + \sum_{d \mid P_{n-1}} \frac{\mu(d)}{2^d - 1}\right)\right\rfloor \tag{114}$$

With P_n being the primorial function, which means $p_1 p_2 p_3 \ldots p_n$ (aka as $P\#$). The identity was discovered by J.M. Gandhi (1971).

The next formula comes from Williams (1964). For this he needs the prime counting function $\pi(n)$ or the following function $F(j)$, which is defined as follows:

$$F(j) = \left\lfloor \cos^2\left(\pi \frac{(j-1)! + 1}{j}\right)\right\rfloor$$

$$p_n = 1 + \sum_{m=1}^{2^n} \left\lfloor \left\lfloor \frac{n}{\sum_{j=1}^{m} F(j)} \right\rfloor^{\frac{1}{n}} \right\rfloor \tag{115}$$

or:

$$p_n = 1 + \sum_{m=1}^{2^n} \left\lfloor \left\lfloor \frac{n}{1 + \pi(m)} \right\rfloor^{\frac{1}{n}} \right\rfloor \tag{116}$$

$$p_n = \lfloor 10^{2^n} \alpha \rfloor - 10^{2^{n-1}} \lfloor 10^{2^{n-1}} \alpha \rfloor, \text{where } \alpha = \sum_{m=1}^{\infty} \frac{p_m}{10^{2^m}} \tag{117}$$

All these formulae are very interesting theoretically, but they are not suitable for the practical calculation of prime numbers.

8.2 FUNCTIONS THAT ALWAYS RETURN A PRIME NUMBER

$$p_n = \left\lfloor A^{3^n} \right\rfloor \tag{118}$$

A is called the 'Mills constant' and has approximately a value of 1.3063778838. The first six prime numbers generated by this formula ('Mills' prime numbers) read:

2, 11, 1361, 2521008887, 1602223620400981813183132O183, 41131011492151048000305295379159531704861396235397599331359 4999488277040407483256849 9

So far, the first 11 'Mill' primes of the form $\left\lfloor A^{3^n} \right\rfloor$ have been calculated, the largest one having more than 20000 decimal digits. The Mills constant A has been calculated up to a precision of 6850 decimal digits (as of Nov. 2015).

Wright (1951) found the following formula:

$$p_n = \left\lfloor 2^{2^{2^{\cdot^{\cdot^{2^{\omega}}}}}} \right\rfloor \text{ (power tower from n exponents}, \omega = 1.9287800) \tag{119}$$

The first primes of this sequence read:

3,13,16381, … (the fourth already has more than 5000 decimal digits)

8.3 FUNCTIONS WHOSE SET OF POSITIVE INTEGERS EQUATES TO THE SET OF PRIME NUMBERS

Since the year 1976, a polynomial of degree 25 with 26 variables has been known (Jones, Sato, Wada & Wies)[43] whose positive set of values coincides with the set of primes, provided the 26 variables are integers.

Let us define the following constants:

$$\begin{aligned}
C0 &= wz + h + j - q \\
C1 &= (gk + 2g + k + 1)(h + j) + h - z \\
C2 &= 2n + p + q + z - e \\
C3 &= 16(k+1)^3(k+2)(n+1)^2 + 1 - f^2 \\
C4 &= e^3(e+2)(a+1)^2 + 1 - o^2 \\
C5 &= (a^2 - 1)y^2 + 1 - x^2 \\
C6 &= 16r^2y^4(a^2 - 1) + 1 - u^2 \\
C7 &= \left(\left(a + u^2(u^2 - a)\right)^2 - 1\right)(n + 4dy)^2 + 1 - (x + cu)^2 \\
C8 &= n + l + v - y \\
C9 &= (a^2 - 1)l^2 + 1 - m^2
\end{aligned}$$

43 https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/JonesSatoWadaWiens.pdf

$$C10 = ai + k + 1 - l - i$$
$$C11 = p + l(a - n - 1) + b(2an + 2a - n^2 - 2n - 2) - m$$
$$C12 = q + y(a - p - 1) + s(2ap + 2a - p^2 - 2p - 2) - x$$
$$C13 = z + pl(a - p) + t(2ap - p^2 - 1) - pm$$

Then there exists the following inequality whose positive integer solutions exactly coincide with the set of all prime numbers:

$$(\boldsymbol{k+2})(\boldsymbol{1 - C0^2 - C1^2 - \cdots - C13^2}) > \boldsymbol{0} \tag{120}$$

You can write a Mathematica program using this formula to search for prime numbers (see Appendix). You can also run the program to find positive solutions for this polynomial. But you will need a lot of patience: a Quad Core Pentium 3 GHz once devoted an entire week to the search without finding a solution!

8.4 RECURSIVE FORMULAE

$$\boldsymbol{p_n = p_{n-1} + ggT(n, p_{n-1})}, \textit{where } \boldsymbol{p_1 = 7} \tag{121}$$

This sequence contains only primes or 1's[44]. If we discard the 1's we get:

```
{5,3,11,3,23,3,47,3,5,3,101,3,7,11,3,13,233,3,467,3,5,3,941
,3,7,1889,3,3779,3,7559,3,13,15131,3,53,3,7,30323,3,60647,3
,5,3,101,3,121403,3,242807,3,5,3,19,7,5,3,47,3,37,5,3,17,3,
199,53,3,29,3,486041,3,7,421,23,3,972533,3,577,7}
```

```
f[1] = 7; f[n_] := f[n] = f[n - 1] + GCD[n, f[n - 1]];
DeleteCases[Differences[Table[f[n], {n, 10^6}]], 1]
```

[44] Eric S. Rowland, A simple prime-generating recurrence, Abstracts Amer. Math. Soc., 29 (No. 1, 2008), p. 50

8.5 FUNCTIONS HAVING ZEROS OR MINIMA AT PRIME NUMBER POSITIONS

8.5.1 A VARIANT OF THE $\mathfrak{Z}$ FUNCTION

In Chapter 5.4, we introduced a function the minima of which are the prime numbers:

$$\mathfrak{Z}(s) = \prod_{n=1}^{\infty} \frac{1}{\left(1 - s^{-\frac{1}{2} - \rho_n \cdot i}\right)} \text{ where } \rho_n\text{: zeros of } \zeta\,(s)$$

With this we have an infinite product that runs over all non-trivial zeros of the zeta function. This function is unsuitable, however, for the practical calculation of primes, since it is very expensive in terms of computing time and there are also problems with the convergence properties of the infinite product, because this infinite product converges only locally at the prime positions. If we break off the infinite product at a finite value (e.g. 100 or 1000), we in fact obtain a function-graph in which minima close to zero appear at the location of the prime numbers. The accuracy of the position of the zeros is the higher, the more product terms are taken into account. The disadvantage is that the 'prime' regions having larger differences to prime positions will diverge and become very large. A small modification results in a 'smoother' function graph:

$$\mathfrak{Z}^*(x) = \ln\left(1 + Abs \prod_{n=1}^{[3x]} \frac{1}{\left(1 - x^{-\frac{1}{2} - \rho_n \cdot i}\right)}\right) \tag{122}$$

Here is a function graph of $\mathfrak{Z}^*(x)$:

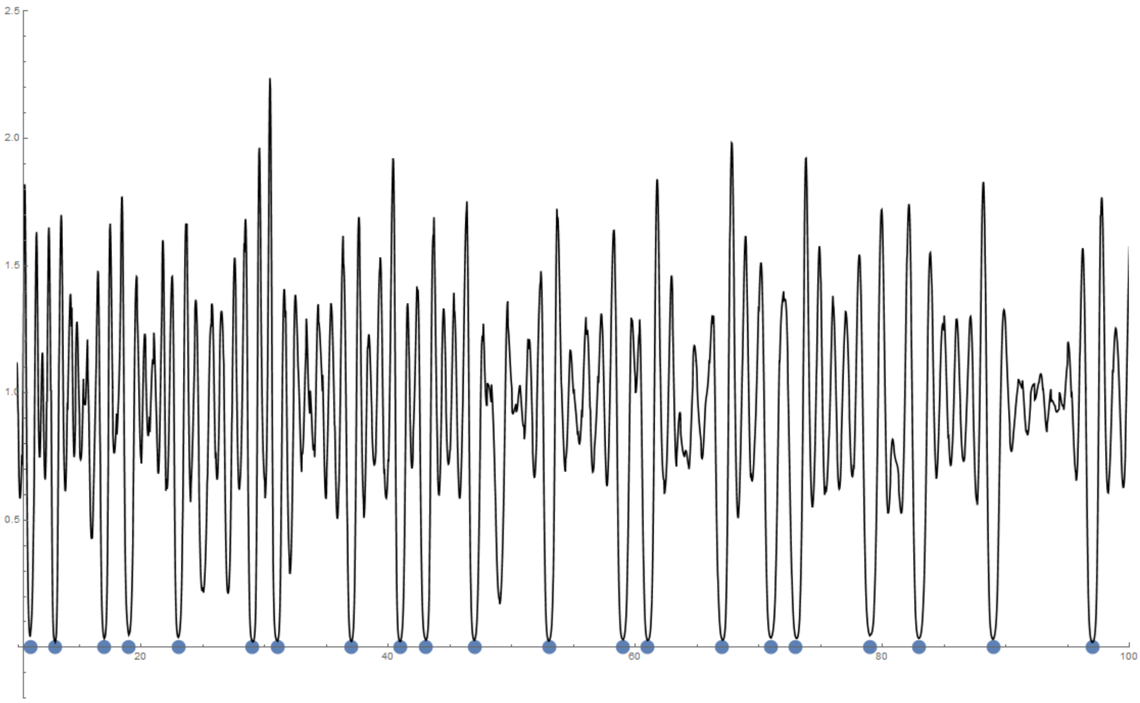


Figure 69. Variant of $\mathfrak{Z}(x)$ after (122) from 10 to 100, with zeros (minima) at prime numbers

```
Mathematica:
cterm[n_,p_]:=N[1/(1-p^(-ZetaZero[n]))];
myFunc[p_]:=Product[cterm[n,p],{n,1,3*p}]
xmin=10;xmax=100;Show[ListPlot[Table[{Prime[i],0},{i,5,PrimePi[xmax]}]
,PlotRange->{{xmin,xmax},{-0.2,2.5}}],Plot[Log[1+Abs[myFunc[x]]],
{x,2,xmax},PlotStyle->Black,PlotRange->{{xmin,xmax},{-0.2,2.5}}]]
```

Note: there are also minima at positions belonging to powers of primes (less strong).

8.5.2 THE REED JAMESON FUNCTION

In Chapter 4.10.1.4, we discussed the recursive Reed Jameson sequence. The sum of the modulus values belonging to negative and positive indices is 0 if the corresponding index n is a prime number.

The Reed Jameson sequence is recursively defined by:

$$a_n = a_{n-5} + a_{n-2}, \text{where } a_0 = 5, a_1 = 0, a_2 = 2, a_3 = 0, a_4 = 2$$

The inverse Reed Jameson sequence is recursively defined by:

$$b_n = b_{n-5} - b_{n-3}, \text{where } b_0 = 5, b_1 = 0, b_2 = 0, b_3 = -3, b_4 = 0$$

$$\boldsymbol{R_n = (a_n \bmod n) + (b_n \bmod n)} \qquad (123)$$

The assumption that R_n is 0 (and only then) if n is a prime number was numerical disproven in 2018 by Peter Danzeglocke. He found Reed Jameson pseudoprimes in

the range $> 10^{15}$ (see Appendix). Remarkable is the fact that the method works in the range up to 10^{10} (as of Dec. 2020).[45]

Graph of the Reed Jameson function:

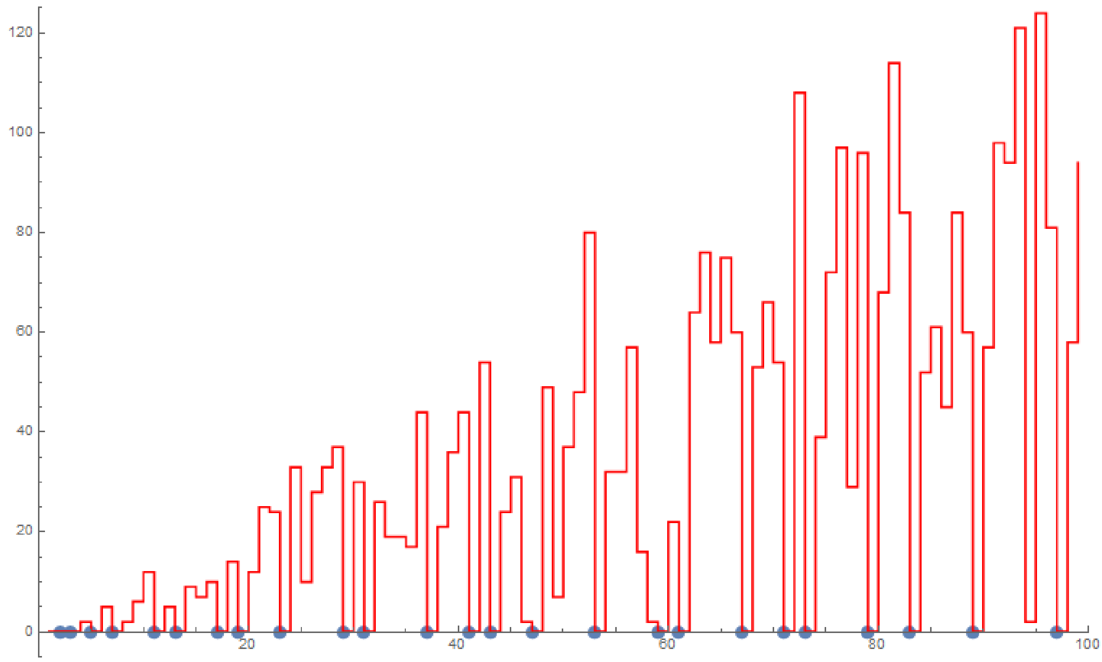


Figure 70. Reed Jameson function from 0 to 100

`Mathematica program: please contact the author.`

8.5.3 OTHER ARITHMETICAL FUNCTIONS HAVING ZEROS AT PRIME NUMBER POSITIONS

Using the Euler totient function $\varphi(n)$:

$f(n) = \varphi(n) - n + 1$ (is 0, if n is a prime number)

Using the sum of divisors function $\sigma_k(n)$:

$f(n) = \sigma_1(n) - n - 1$ (is 0, if n is a prime number)

$f(n) = \sigma_0(n) - 2$ (is 0, if n is a prime number)

8.6 FORMULAE FOR CALCULATING THE NUMBER OF PRIMES

Let us take a closer look at the prime counting function $\pi(n)$:

$$\pi: \mathbb{N} \to \mathbb{N}, n \mapsto \pi(n): \pi(n) = |\{p \in \mathbb{P} \,|p \le n\}|$$

Here, $\mathbb{P}$ is the set of the prime numbers and $|\ldots|$ denotes the number of elements of the set. The pi function is usually extended to the field of the real numbers: $\pi(x)$, $x \in \mathbb{R}$

[45] It is known, that many linear recurrent sequences always contain infinitely many pseudo primes.

Here are a few exact formulae:

Hardy and Wright (1979):

$$\pi(n) = -1 + \sum_{j=3}^{n} \left[(j-2)! - j \left\lfloor \frac{(j-2!)}{j} \right\rfloor \right] \tag{124}$$

Williams (1964):

$$\pi(n) = -1 + \sum_{j=1}^{n} F(j), \text{where } F(j) = \left\lfloor \cos^2 \left(\pi \frac{(j-1)! + 1}{j} \right) \right\rfloor \tag{125}$$

or

$$\pi(n) = -1 + \sum_{j=2}^{n} H(j), \text{where } H(j) = \frac{\sin^2(\pi \frac{\left((j-1)!\right)^2}{j})}{\sin^2 \frac{\pi}{j}} \tag{126}$$

A similar formula originates from Mini:

$$\pi(n) = \sum_{j=2}^{n} \left[\frac{(j-1)! + 1}{j} - \left\lfloor \frac{(j-1)!}{j} \right\rfloor \right] \tag{127}$$

The simplest formula has been well known since the 18th century (Legendre and Gauss, 1798):

$$\pi(x) \approx \frac{x}{\ln(x) - 1.08366} \tag{128}$$

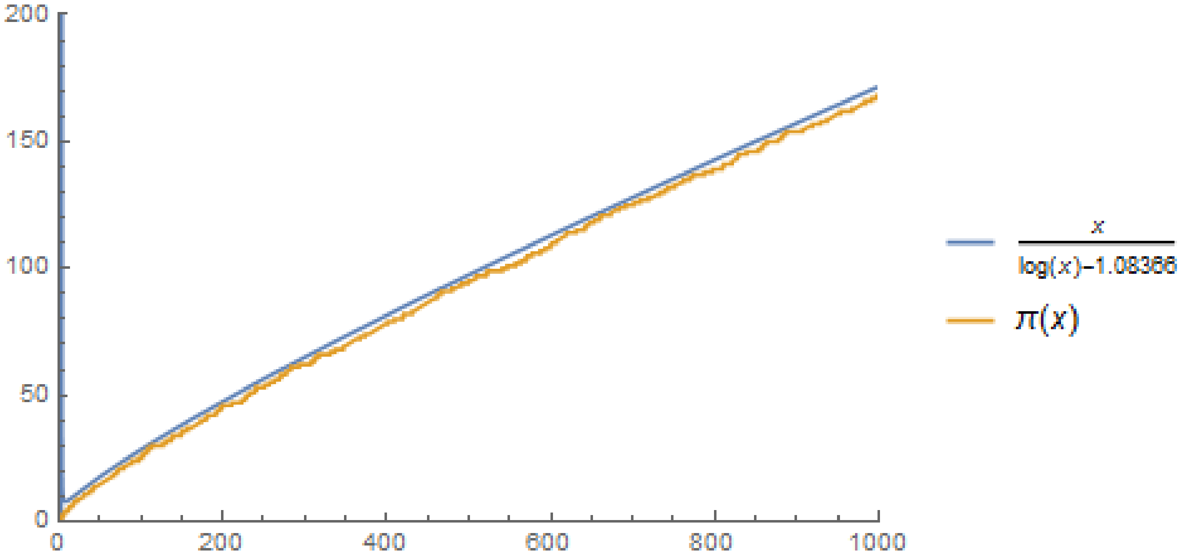


Figure 71. Comparison π(n) with Gauss approximation

```
Mathematica:

Plot[{x/(Log[x]-1.08366),PrimePi[x]},{x,1,1000},PlotRange-
>{{0,1000},{0,200}},PlotPoints->200,PlotLegends->"Expressions"]
```

A better approximation, also originating from C. F. Gauss:

$$\pi(x) = Li(x) + O(\sqrt{x} \cdot \ln(x))$$

$$\text{with: } Li(x) = \int_2^x \frac{dt}{\ln(t)} \text{ (logarithmic integral function)} \tag{129}$$

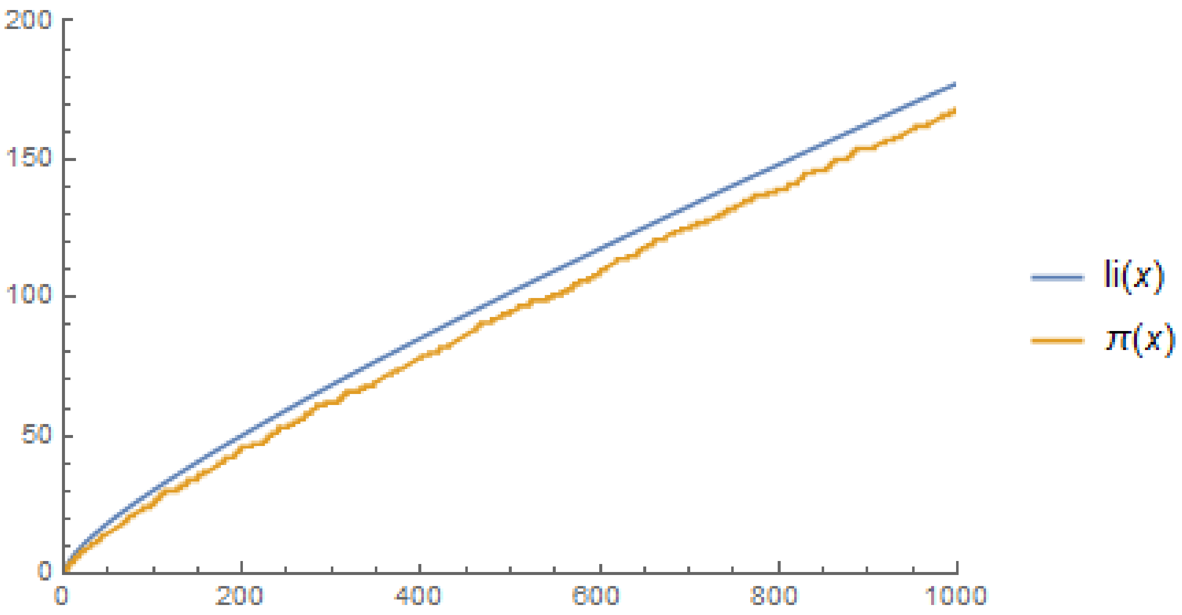


Figure 72. Comparison π(n) with logarithmic integral function, from 1 to 1000

Mathematica:

```
Plot[{LogIntegral[x],PrimePi[x]},{x,1,1000},PlotRange-
>{{0,1000},{0,200}},PlotPoints->200,PlotLegends->"Expressions"]
```

It looks as if $Li(x)$ is always greater than $\pi(x)$. For small x, this is right. However, for very large x, $Li(x)$ has been shown to be smaller than $\pi(x)$. In 1914, L. E. Littlewood proved that the difference $\pi(x) - Li(x)$ changes sign infinitely often. Since then it has been proven that the first change of sign cannot occur earlier than $1{,}39822 \cdot 10^{316}$ (Richard Hudson, 2000). However, the point of the first sign change cannot be smaller than 10^{14} (Kotnik, 2008).

This is a very amazing property of the prime counting function and the function $Li(x)$. It shows that very, very large numbers can have new, unexpected properties. In other words: unexpected phenomena can also occur in astronomically high number regions. Moreover, this shows that we cannot always trust the 'numerical evidence'!

An even better approximation is the Riemann function $R(x)$:

$$\boldsymbol{\pi(x) \approx R(x)} \tag{130}$$

The best asymptotic formula is:

$$\boldsymbol{\pi(x) \approx R(x) - \frac{1}{ln(x)} + arctan(\frac{\pi}{ln(x)})} \tag{131}$$

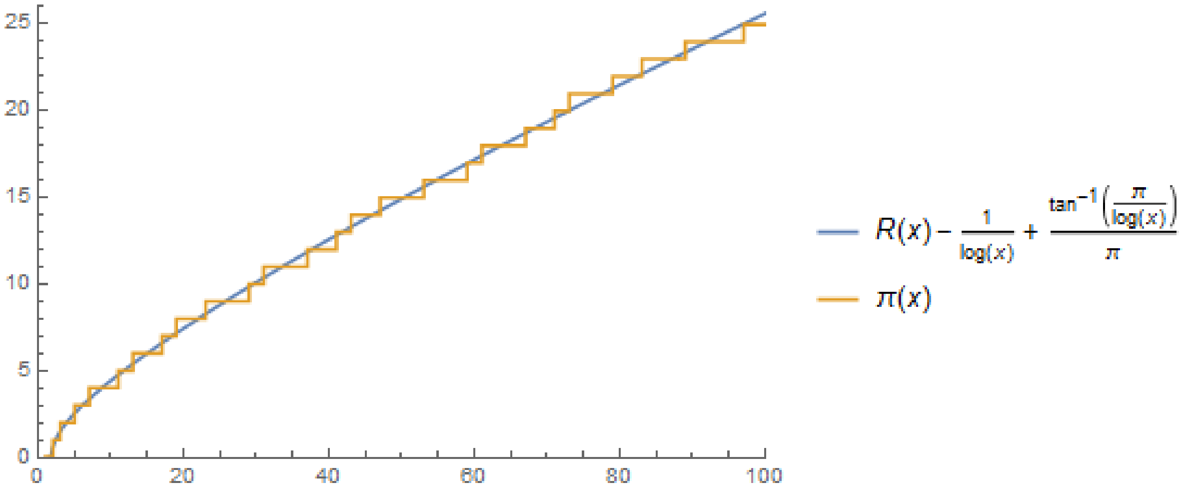


Figure 73. Comparison π(n) with Riemann function R(x), in the range from 0 to 100

```
Mathematica: Plot[{RiemannR[x]-
1/Log[x]+ArcTan[Pi/Log[x]]/Pi,PrimePi[x]},{x,1,100},PlotRange-
>{{0,100},{0,26}},PlotPoints->200,PlotLegends->"Expressions"]
```

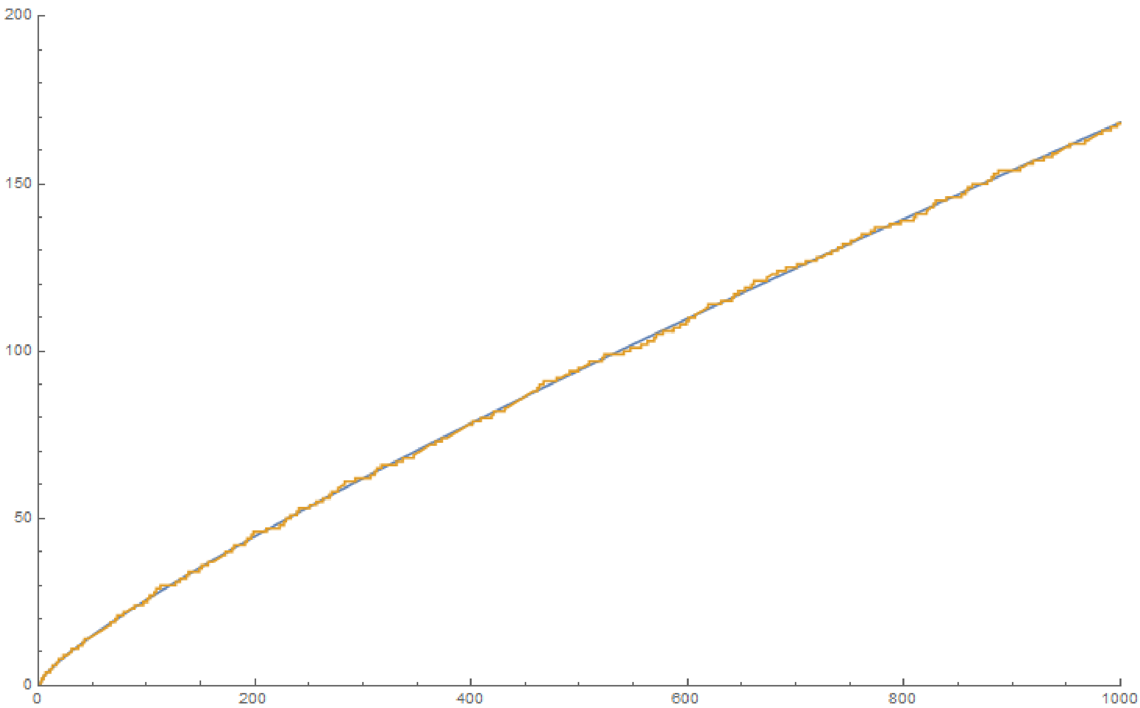


Figure 74. Comparison π(n) with Riemann function R(x), range from 0 to 1000

Mathematica:

```
Plot[{RiemannR[x]-
1/Log[x]+ArcTan[Pi/Log[x]]/Pi,PrimePi[x]},{x,1,1000},PlotRange-
>{{0,1000},{0,200}},PlotPoints->200,PlotLegends->"Expressions"]
```

And finally, here is the mysterious, exact formula found by Riemann:

$$\boldsymbol{\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})} \tag{132}$$

Using the Riemann function $R(x)$.

$$\boldsymbol{\pi_0(x) = R(x) - \sum_{\rho} R(x^{\rho}) - \frac{1}{ln(x)} + \frac{1}{\pi} arctan\left(\frac{\pi}{ln(x)}\right)} \tag{133}$$

The Riemann function is a very good approximation for $\pi(x)$, but Riemann's formula (132) is much more precise. It also takes into account the small local fluctuations of $\pi(x)$ and finally fits (if enough zero terms of the zeta function are evaluated) to the exact integer stairs function of the arithmetical calculation. Note: the summation over the zero terms of the zeta function must occur in ascending order of increasing values of $\mathrm{Im}(\rho)$, since the sum is only conditionally convergent. The calculation of $R(x^{\rho})$ requires the calculation of $\mathrm{Li}(x^{\rho})$ and is not quite as simple because the complex logarithm of x^{ρ} has to be calculated. This function, in turn, is not injective and is not defined unambiguously

The calculation of the principal value using the 'main branch' of the complex logarithm[46] would give incorrect results. In short, the problem lies in the fact that, for the complex logarithm, the equation $\ln(x^{\rho}) = \rho \ln(x)$ does not always obtain. However, the problem can be avoided by simply using $\mathrm{Ei}(\rho \ln(x))$ instead of $\mathrm{Li}(x^{\rho})$, where $\mathrm{Ei}(x)$ denotes the complex integral exponential function (Li(x) is the complex logarithmic integral function). $\mathrm{Ei}(x)$ is closely related to $\mathrm{Li}(x)$, since $\mathbf{Li}(\boldsymbol{x}) = \mathbf{Ei}(\mathbf{ln}\,\boldsymbol{x})$.

Note: the largest known values of $\pi(x)$ were obtained not by number theory but by the methods of analytical number theory.

The following graphs show how the analytical formula approximates to the exact stair function:

```
Mathematica program: please contact the author.
```

[46] The logarithm of the kth branch is defined as $w = \ln|z| + i\,(\arg z + 2k\pi), k \in \mathbb{Z}$. For $k = 0$ we have the main branch of the complex logarithm function.

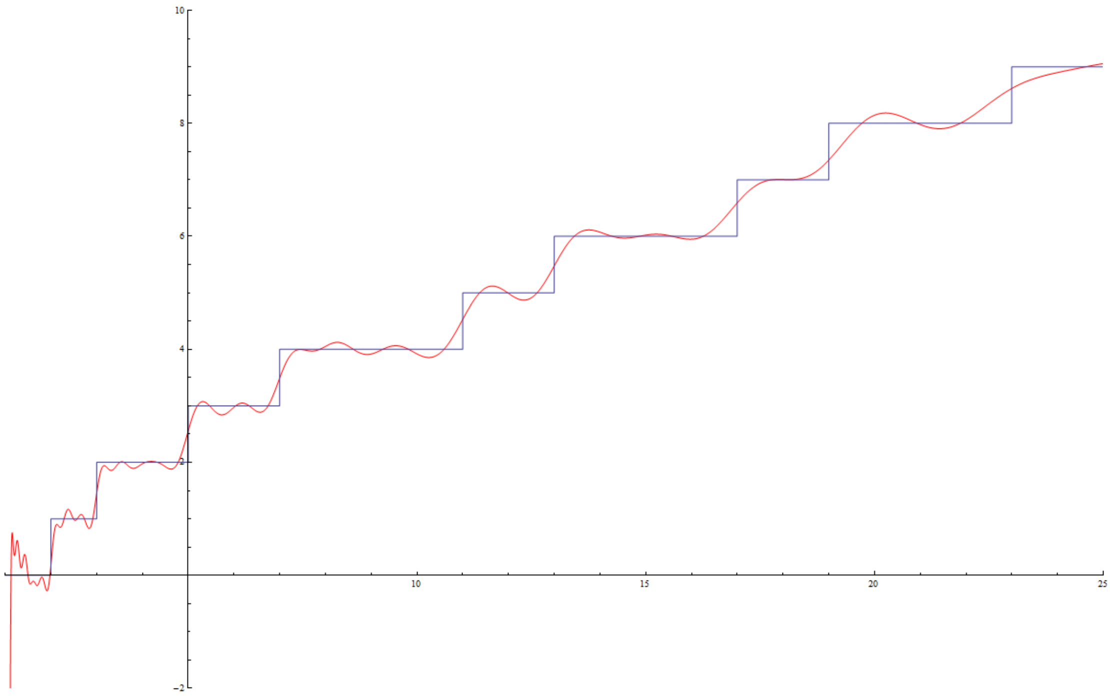


Figure 75. Riemann's exact formula for $\pi(x)$ (x = 1 to 25, summing over the first 10 zero pairs of the zeta function)

```
Mathematica-Program: please contact the author.
```

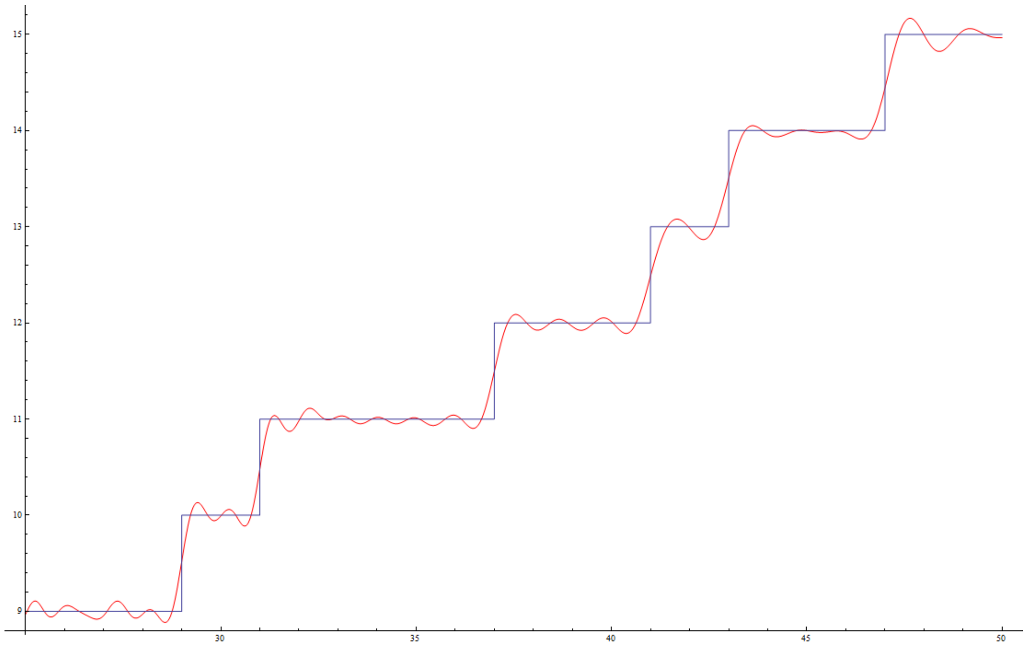


Figure 76. Riemann's exact formula for $\pi(x)$ (x = 25 to 50, summing over the first 100 zero pairs of the zeta function)

The number of composite numbers ('non primes') $\tilde{\pi}(n)$ up to an given limit n is simply

$$\boldsymbol{\tilde{\pi}(n) = n - \pi(n)}$$

```
Mathematica:
n-PrimePi[n]
```

More formulae for $\pi(x)$

$$\pi(x) \approx \mathrm{li}(x) - \frac{\sqrt{x}}{\ln x}\left(1 + 2\sum_{\gamma}\frac{\sin(\gamma \ln x)}{\gamma}\right), \text{where } \gamma = \mathrm{Im}(\rho) \tag{134}$$

Here, ρ are the complex zeros of the zeta function.

8.7 FORMULAE FOR CALCULATING THE *N*TH PRIME NUMBER

Unlike in Chapter 8.1, here we want to look for analytical, asymptotic solutions.

The calculation of the *n*th prime is difficult. No explicit, simple formula is known for this purpose.

The best asymptotic estimate currently known is (as of Dec. 2016):

$$p_n = n\left(\ln n + \ln\ln n - 1 + \frac{(\ln\ln n - 2)}{\ln n} - \frac{(\ln\ln n)^2 - 6\ln\ln n + 11)}{2(\ln n)^2}\right) \tag{135}$$

```
Mathematica:
Table[Prime[n],{n,1,100}]
prime[n_]:=Block[{logn=N[Log[n],15],loglogn=N[Log[Log[n]],15]}
,
n(logn+loglogn-1+(loglogn-2)/logn-(loglogn^2-
6loglogn+11)/(2logn^2))]
```

8.8 FORMULAE FOR CALCULATING THE *N*TH NON-PRIME (COMPOSITE NUMBER)

The *n*th non prime can be calculated using the following Mathematica program: (In this example: from n= 1 to 1000)

```
Mathematica:
composite[n_Integer]:=FixedPoint[n+PrimePi[#]&,n+PrimePi[n]]
ListLinePlot[Table[{k,composite[k]},{k,0,1000,10}],Filling->Axis]
```

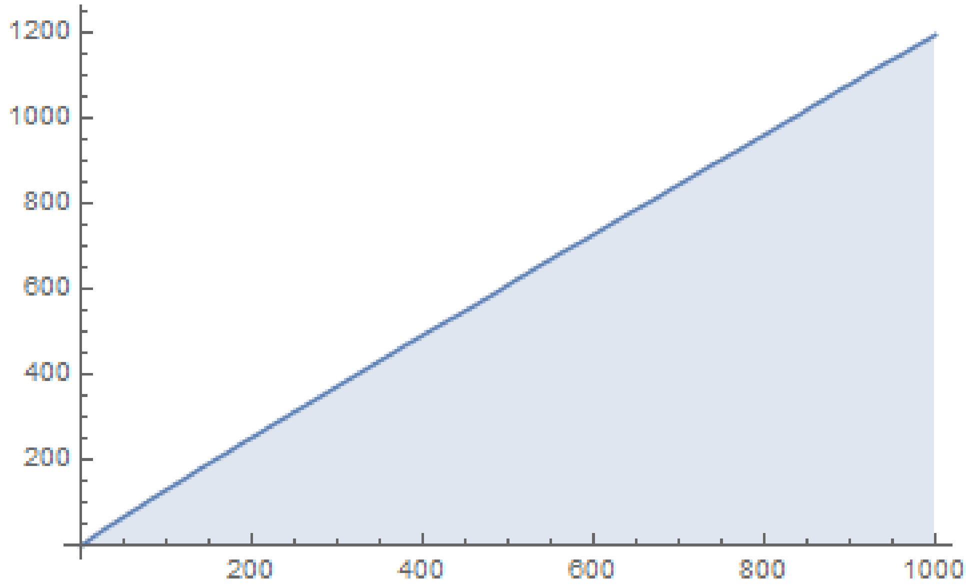


Figure 77. The *n*th composite number ('non prime')

An asymptotic approximation for the *n*th non prime c_n reads:

$$\boldsymbol{c_n = n\left(1 + \frac{1}{\ln n} + \frac{2}{\ln^2 n} + \frac{4}{\ln^3 n} + \frac{19}{2\ln^4 n} + \frac{181}{6\ln^5 n} + o\left(\frac{1}{\ln^5 n}\right)\right)} \tag{136}$$

9 NOW IT GETS INTERESTING: FOUR-DIMENSIONAL SPHERES AND PRIME NUMBERS

What have spheres – even four-dimensional spheres – to do with primes? We will pursue this question in this chapter. In principle, the question arises as to how many integer lattice points in the n-dimensional space have the same difference from the origin (i.e. lie on the 'surface' of an n-dimensional sphere). In mathematics the term 'n-sphere' is generally used for an n-dimensional sphere. So, for example, a 1-sphere is the circumference of a circle, a 2-sphere is the curved 2-dimensional surface of a sphere. A 3-sphere is the boundary of a 4-dimensional sphere, i.e. a three-dimensional space bent into the fourth dimension, which, for the sake of simplicity, we sometimes refer to as the 'surface' of the four-dimensional sphere. The term 'glome' is also used for this. In this chapter, we are looking for integer lattice points (of a Cartesian coordinate system) that 'sit' on n-spheres in two-, three-, or four-dimensional space.

The Mathematica software provides three powerful tools for solving this problem:

`FindInstance:`

finds all points that lie on an n-sphere, here e.g. on a 2-sphere with radius $\sqrt{n}$:

```
FindInstance[x^2+y^2+z^2==n,{x,y,z},Integers,numberOfInstances]
```

Since the solutions of `FindInstance` include many permutations and axis- and point-symmetrically mirrored solutions due to the symmetry properties (the degree of symmetry becomes higher as the number of dimensions increases), the following function is also interesting, because it just computes the "core" of the solutions – that is to say, without the 'mirrored' solutions from negative quadrants, octaves, etc., or which can be generated by permutations:

`PowersRepresentations:`

Finds all (actually different, integer and positive) solutions of the equation

$$x^2 + y^2 + z^2 = n$$

Example: PowersRepresentations`[n,3,2]`

And finally the function `SquaresR:`

this provides (only) the number of solutions of `FindInstance.`

`E.g.: SquaresR[3,n]` gives the number of solutions of
`FindInstance[x^2+y^2+z^2==n,{x,y,z},Integers,Infinity]]`

It will be shown that the spherical points on the n-spheres are not randomly distributed, but, on the contrary, form very beautiful structures that become more interesting the higher the dimensionality of the n-spheres. In the case of the 3-spheres ('surfaces' of four-dimensional spheres), a remarkable relation exists between the number of spherical points and the prime numbers. This connection is very simple and is anticipated here:

If the square $n = rad^2$ of the radius of a 4-dimensional sphere assumes the value of a prime number, then (and only then) the following relation applies:

$$rad^2 = n = \frac{r_4(n)}{8} - 1, \text{if } n \in \mathbb{P} \tag{137}$$

This relationship has long been known since the function $r_4(n)$ can be easily calculated from the sigma function $\sigma_1(n)$. However, there is no reference in the relevant literature to the beautiful connection with the 3-spheres ('surfaces') of four-dimensional spheres and primes.

Just a curiosity: the number 12 plays a special role in the sequence $r_4(n)$, since it is the only number for which

$$n = \frac{r_4(n)}{8}, \text{only if } n = 12 \tag{138}$$

Since it is difficult to imagine four-dimensional objects, it is always a good idea to start with the counterpart in one or two lower dimensions. Thus we begin with two-dimensional spheres (1-spheres) – 'circles', in common parlance.

9.1 THE SECOND DIMENSION: CIRCLES AND INTEGER LATTICE POINTS

We are looking for the integer lattice points of our Cartesian coordinate system that can lie along the circumference (we are not interested in any lattice points lying within the circle, but only in lattice points that lie exactly on the circumference). If we assume that the radius of the circle increases continuously, the circular line runs through the lattice points of our coordinate system in order, which lie exactly on the circle line. The number of these possible lattice points that are touched by the circumference, of course, depends strongly on the radius of the circle. Let us suppose that we increase the circle radius continuously, then the circumference will run through the grid points of our coordinate system. We are interested in the grid points that lie exactly on the circumference. Here are 6 examples for $r^2 = 8$ to 13:

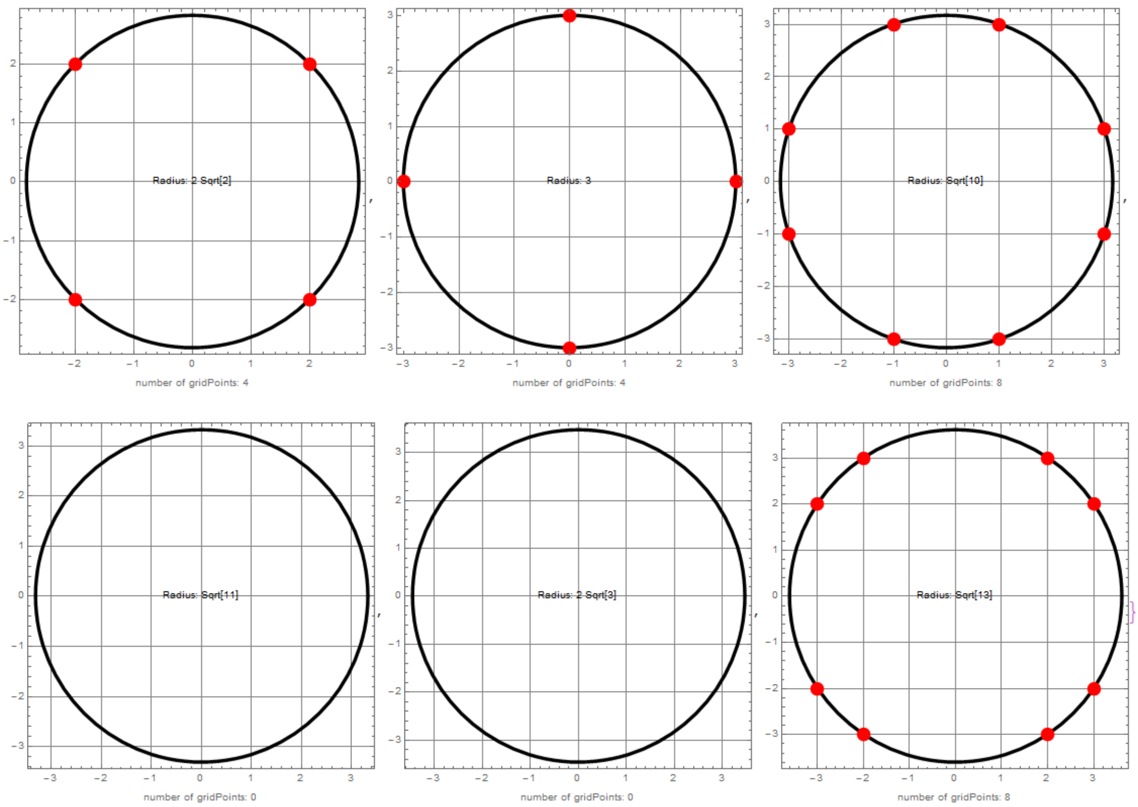


Figure 78. Lattice points on a 1-sphere, squared radius from 8 to 13

```
Mathematica program: please contact the author.
```

For some values with radius r, there are no integral solutions of the equation $x^2 + y^2 = r^2$, and therefore no corresponding lattice points either that are touched by the circle. The "crossing" of the circular line through the two-dimensional grid points can be viewed in an animation (as a video on the enclosed CD) or as a Mathematica animation (see Appendix: "Lattice points on n-spheres (n-dimensional spheres)").

The function which calculates the number of integer lattice points on a circumference in 2-dimensional space is called $r_2(n)$. Its function values are all divisible by 4. The first 100 values are:

```
{4,4,0,4,8,0,0,4,4,8,0,0,8,0,0,4,8,4,0,8,0,0,0,0,12,8,0,0,8,0,0,4,0,8,
0,4,8,0,0,8,8,0,0,0,8,0,0,0,4,12,0,8,8,0,0,0,0,8,0,0,8,0,0,4,16,0,0,8,
0,0,0,4,8,8,0,0,0,0,0,8,4,8,0,0,16,0,0,0,8,8,0,0,0,0,0,0,8,4,0,12}
Mathematica:
SquaresR[2,Range[100]]
```

Example: the 8 solutions $r^2 = 5$ read:

```
{{-2,-1},{-2,1},{-1,-2},{-1,2},{1,-2},{1,2},{2,-1},{2,1}}
Mathematica:
FindInstance[x^2+y^2==5,{x,y},Integers,8]
```

These solutions can be created from each other mutually by permutations or symmetrical mirroring. The number of genuinely different solutions is in this case 1:

```
{{1,2}}
Mathematica:
PowersRepresentations[5,2,2]
```

$r_2(n)$ is the number of lattice points in the 2-dimensional space lying on a circle with radius $\sqrt{n}$. We denote $r_2{}^*(n)$ as the number of different, positive lattice points, such that $0 \leq n_1 \leq n_2$ and $n_1^2 + n_2^2 = n$.
$r_2(n)$ has a value of 0 for many values of n. This means that not every natural number can be written as the sum of 2 squares. Here is a list of the first values of these 'non-representable' numbers:

```
{3,6,7,11,12,14,15,19,21,22,23,24,27,28,30,31,33,35,38,39,42,43,44,46,
47,48,51,54,55,56,57,59,60,62,63,66,67,69,70,71,75,76,77,78,79,83,84,
86,87, 88,91,92,93,94,95,96,99,…}

Mathematica:
Select[Range[199], Length[PowersRepresentations[ #, 2, 2]] == 0 &]
```

Here are two illustrations of $r_2(n)$:

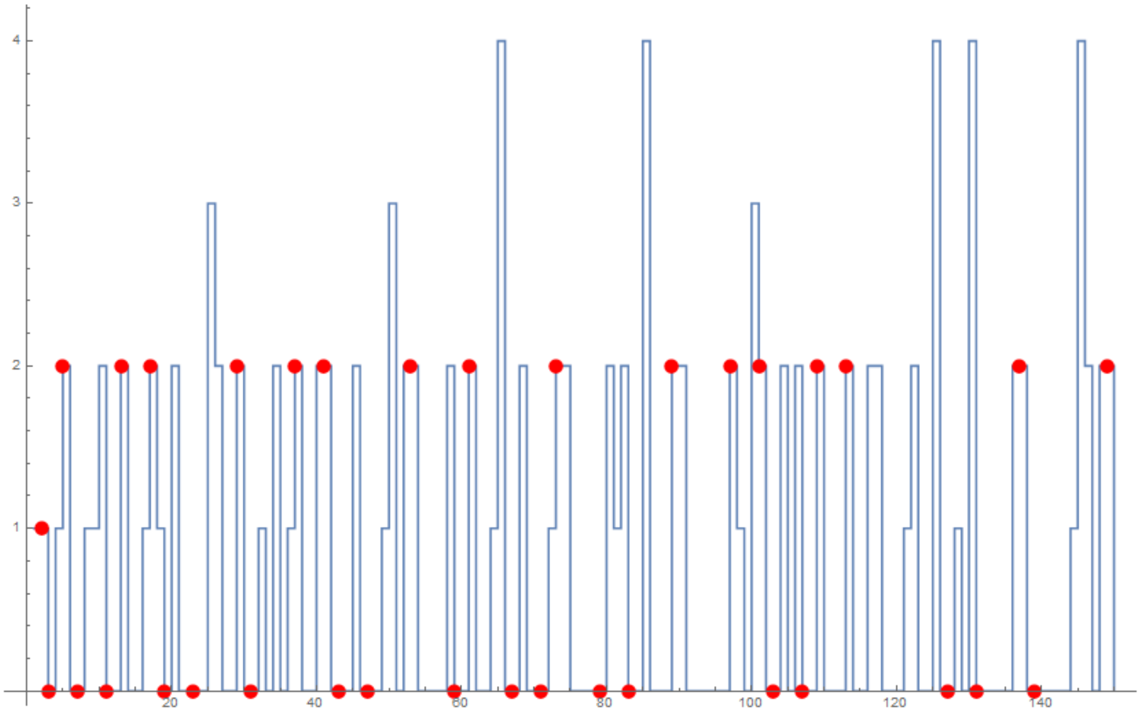


Figure 79. $r_2(n)$: number/4 of possible representations of n as a sum of 2 squares. No simple relationship to prime numbers can be observed.

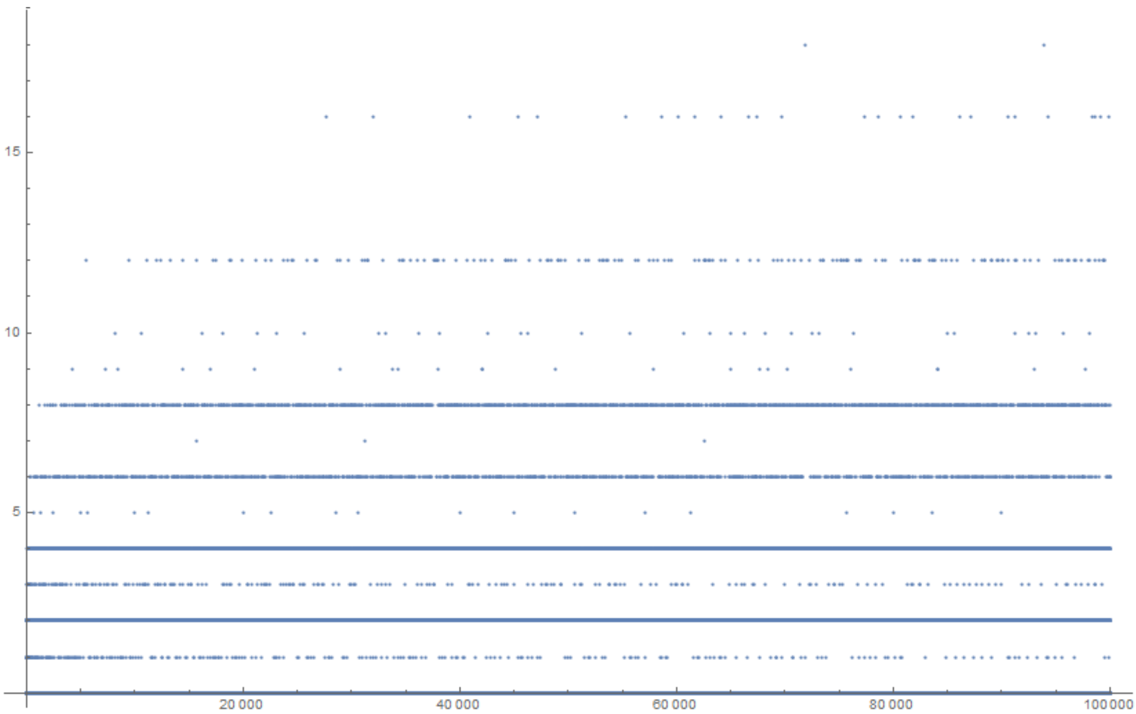


Figure 80. $r_2(n)$: number/4 of possible representations of n as a sum of 2 squares (up to n=100000)

And finally a few plots of $r_2{}^*(n)$ for different values of n:

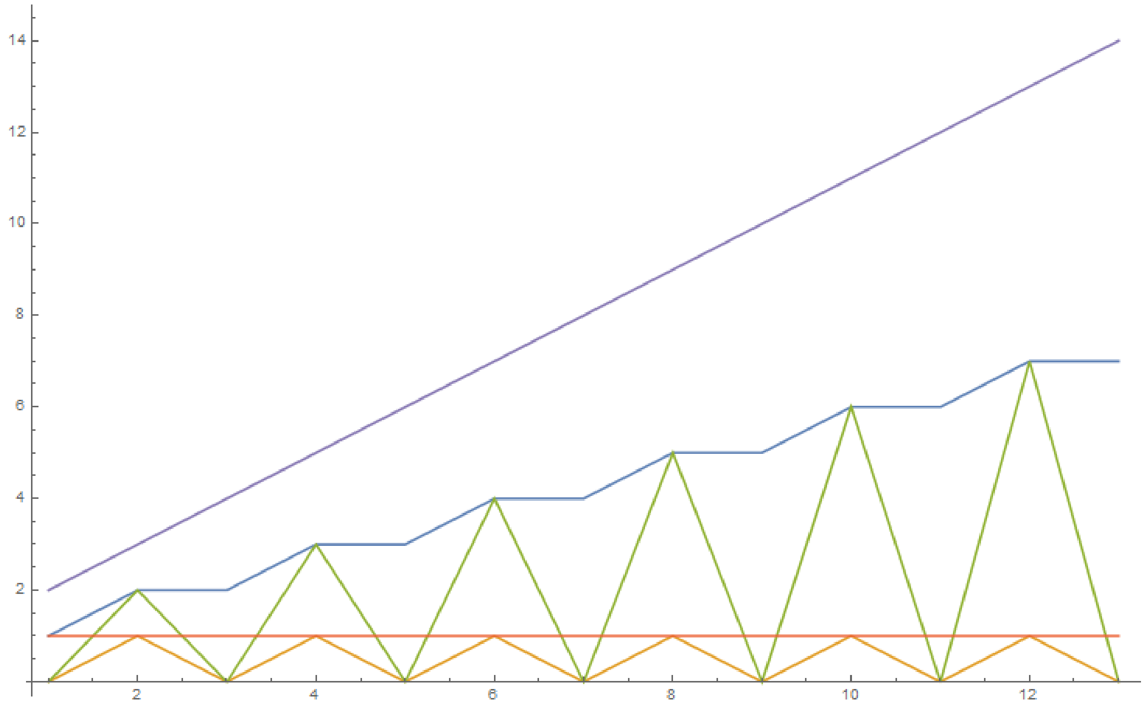


Figure 81. $r_2{}^*(\mathrm{b}^\mathrm{n})$: number of different representations of b^n as the sum of two squares

Mathematica program (Figure 79):
```
Show[ListLinePlot[Table[{n,SquaresR[2,n]/4},{n,1,150}],
InterpolationOrder->0],ListPlot[Table[{Prime[n],
SquaresR[2,Prime[n]]/4},{n,1,PrimePi[150]}],PlotStyle->Red]
]
```

Mathematica (Figure 80):
```
Show[ListPlot[Table[{n,SquaresR[2,n]/4},{n,1,100000}],PlotRange->Full]
]
```

Mathematica (Figure 81):
```
ListLinePlot[{Table[Length[PowersRepresentations[10^i,2,2]],{i,1,13}],
Table[Length[PowersRepresentations[14^i,2,2]],{i,1,13}],
Table[Length[PowersRepresentations[15^i,2,2]],{i,1,13}],
Table[Length[PowersRepresentations[16^i,2,2]],{i,1,13}],
Table[Length[PowersRepresentations[25^i,2,2]],{i,1,13}]
},PlotLegends->Automatic,PlotRange->All]
```

Note: integral solutions of $x^2 + y^2 = r^2$ (where r is also an integer) are also referred to as 'Pythagorean triplets'. These correspond to the lattice points on a circular line with an integer radius r.

9.1.1 FORMULAE AND PROPERTIES

We restrict ourselves to the function $r_2(n)$, which calculates the number of lattice points in two-dimensional space lying on a circle with radius $\sqrt{n}$. For the function $r_2{}^*(n)$, which

calculates the number of different, positive grid points, such that: $0 \leq n_1 \leq n_2$ and $n_1^2 + n_2^2 = n$, see note[47].
The generating function of $r_2(n)$ is the squared elliptic Jacobi $\vartheta_3(n)$ function:

$$\sum_{n=0}^{\infty} r_2(n)x^n = \vartheta_3^2(x) = 1 + 4x + 4x^2 + 4x^4 + 8x^5 + \cdots \tag{139}$$

Explicit formulae:

$$r_2(n) = 4[d_1 - d_3], d_k\text{: number of divisors of } n \text{ of the form } 4m + k \tag{140}$$

[47] http://mathworld.wolfram.com/SumofSquaresFunction.html

9.2 THIRD DIMENSION: SPHERES AND INTEGER LATTICE POINTS

Here, too, we are searching for lattice points in a Cartesian coordinate system that lie on the surface of a sphere. The number of these possible lattice points that are touched by the spherical surface also strongly depends on the radius of the sphere. Let us suppose that we continually enlarge the radius of the sphere, and that the spherical surface then passes through the integer grid points of our coordinate system. We are interested in the lattice points that lie exactly on the surface of the sphere. Here are some examples:

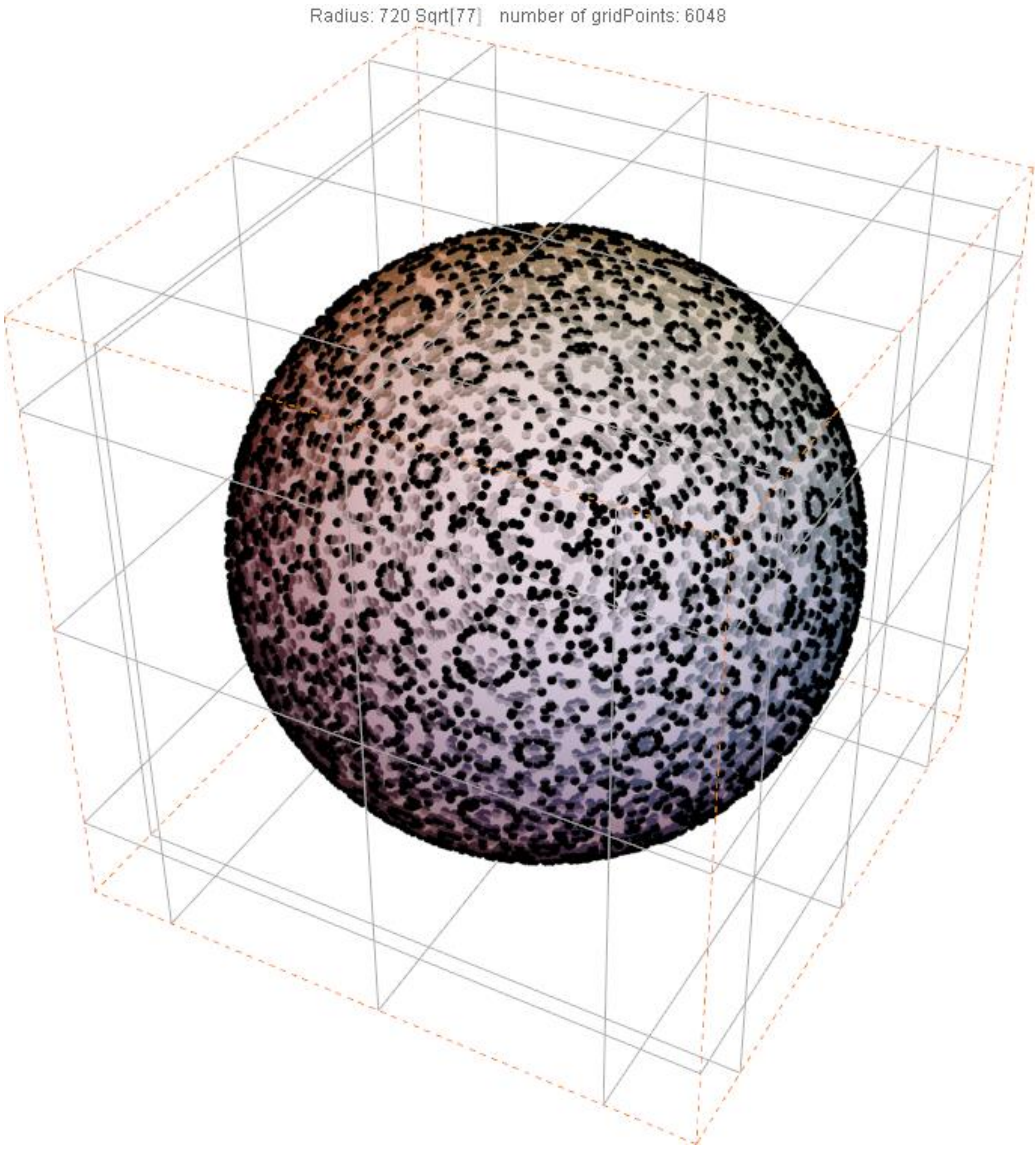


Figure 82. Integer lattice points of a sphere with squared radius 11!

`Mathematica program: please contact the author.`

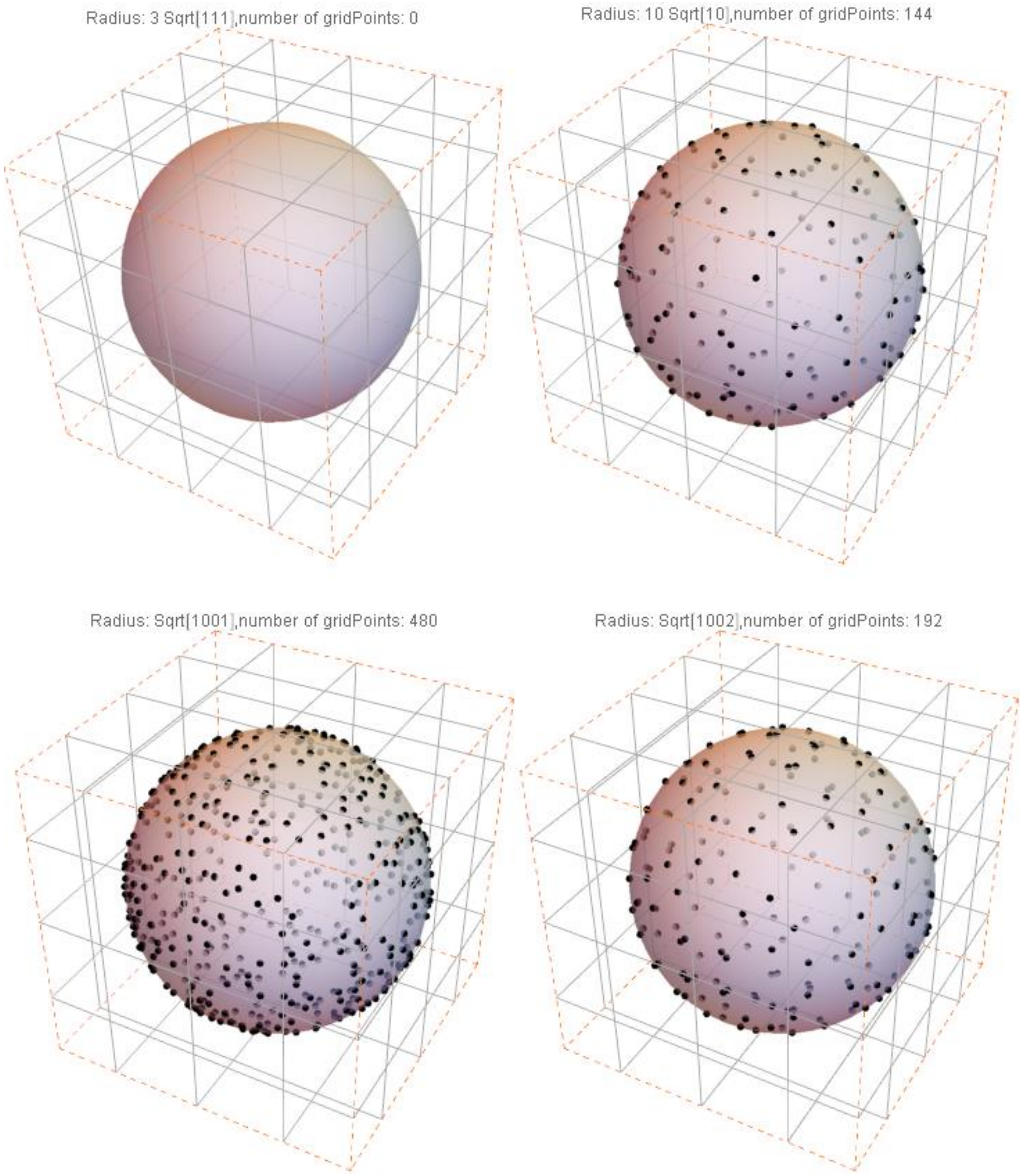


Figure 83. Integer lattice points of spheres (squared radius 999-102).

For some values with radius r there are no integer solutions of the equation $x^2 + y^2 + z^2 = r^2$ and therefore no corresponding lattice points either that are touched by the spherical surface. Some structures are only visible when the spheres are viewed from different viewing angles. This can be viewed in an animation (as a video on the enclosed CD) or as a Mathematica animation (see Appendix). The function that calculates the number of integer lattice points on the surface of a sphere in three-dimensional space is denoted in mathematical literature by $r_3(n)$. We define $r_3{}^*(n)$ as the number of different, positive lattice points, such that

$0 \leq n_1 \leq n_2 \leq n_3$ and $n_1^2 + n_2^2 + n_3^2 = n$.

Many (but not all) function values of $r_3(n)$ are divisible by 6. The first 100 values are:

```
{6,12,8,6,24,24,0,12,30,24,24,8,24,48,0,6,48,36,24,24,48,24,0,24,30,72
,32,0,72,48,0,12,48,48,48,30,24,72,0,24,96,48,24,24,72,48,0,8,54,84,48
,24,72,96,0,48,48,24,72,0,72,96,0,6,96,96,24,48,96,48,0,36,48,120,56,2
4,96,48,0,24,102,48,72,48,48,120,0,24,144,120,48,0,48,96,0,24,48,108,7
2,30}
Mathematica:
SquaresR[3,Range[100]]
```

Example: the 8 solutions of $r^2 = 3$ read:

```
{{-1,-1,-1},{-1,-1,1},{-1,1,-1},{-1,1,1},{1,-1,-1},{1,-1,1},{1,1,-1},{1,1,1}}
Mathematica:
FindInstance[x^2+y^2+z^2==3,{x,y,z},Integers,8]
```

These solutions can be created from each other mutually by permutations or symmetrical mirroring. The number of genuinely different solutions in this case is 1: `{{1,1,1}}`

```
Mathematica:
PowersRepresentations[3,3,2]
```

The first 100 values of $r_3{}^*(n)$ read:

```
{1,1,1,1,1,1,0,1,2,1,1,1,1,1,0,1,2,2,1,1,1,1,0,1,2,2,2,0,2,1,0,1,2,2,1
,2,1,2,0,1,3,1,1,1,2,1,0,1,2,3,2,1,2,3,0,1,2,1,2,0,2,2,0,1,3,3,1,2,2,1
,0,2,2,3,2,1,2,1,0,1,4,2,2,1,2,3,0,1,4,3,1,0,1,2,0,1,2,3,3,2}
Mathematica:
Table[Length[PowersRepresentations[i,3,2]],{i,1,100}]
```

or:

```
a[ n_] := If[ n < 0, 0, Sum[ Boole[ n == i^2 + j^2 + k^2], {i, 0,
Sqrt[n]}, {j, 0, i}, {k, 0, j}]];
```

As already mentioned above, $r_3(n)$ has the value 0 for some n. This means that not every natural number can be written as the sum of 3 squares. Here is a list of the first values of these 'non-representable' numbers:

```
{7,15,23,28,31,39,47,55,60,63,71,79,87,92,95,103,111,112,119,124,127,1
35,143,151,156,159,167,175,183,188,191,199}
Mathematica:
Select[Range[199], Length[PowersRepresentations[ #, 3, 2]] == 0 &]
```

Here are two plots of $r_3(n)$:

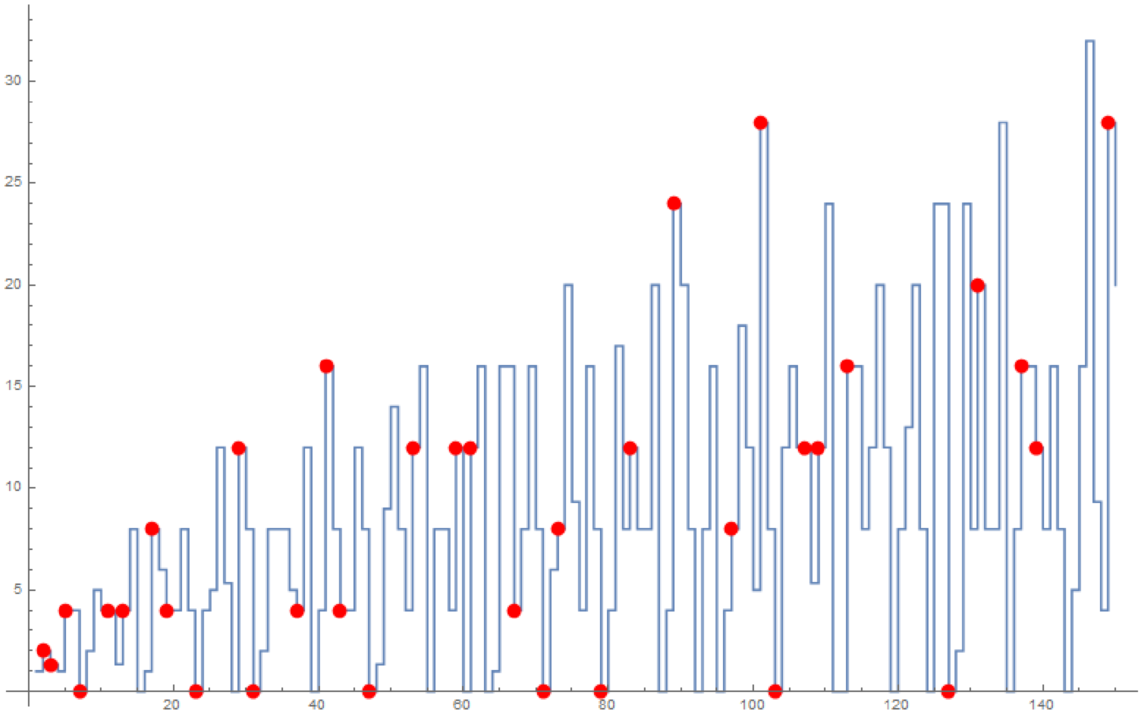


Figure 84. $r_3(n)$: number/6 of possible representations of n as the sum of 3 squares.

```
Mathematica:
Show[ListLinePlot[Table[{n,SquaresR[3,n]/6},{n,1,150}],InterpolationOrder->0],ListPlot[Table[{Prime[n],SquaresR[3,Prime[n]]/6},{n,1,PrimePi[150]}],PlotStyle->Red]]
```

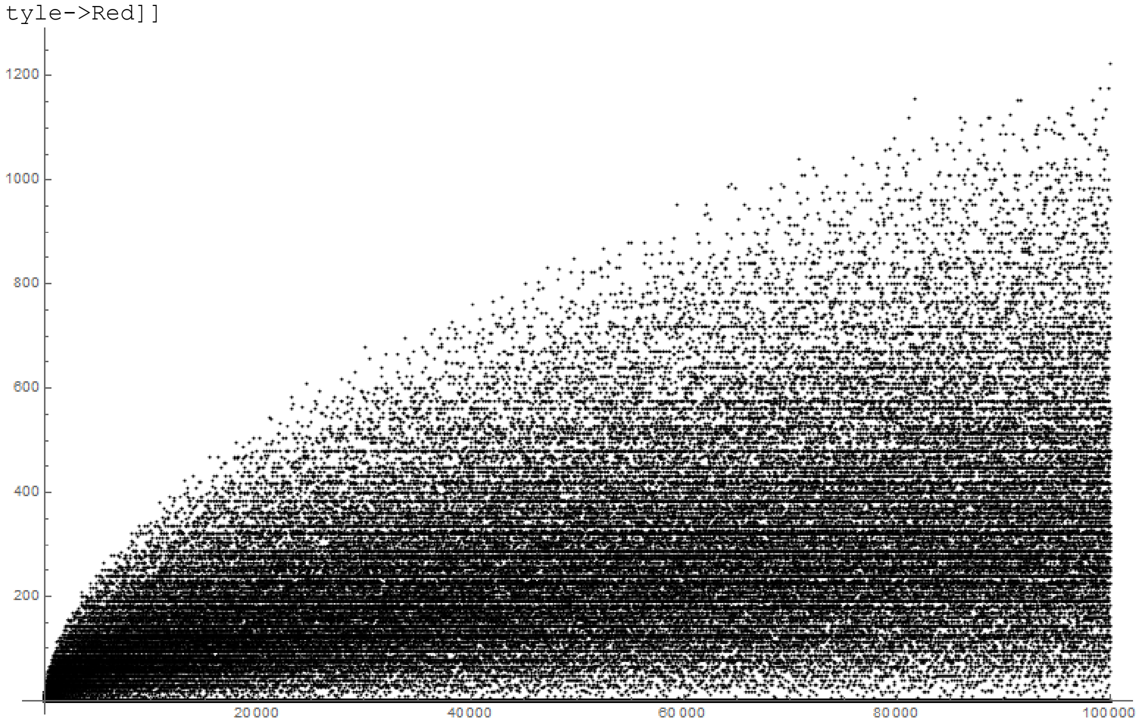


Figure 85. Number/6 of possible representations of n as the sum of 3 squares (up to 100000)

```
Mathematica:
ListPlot[Table[{n,SquaresR[3,n]/6},{n,1,100000}],PlotRange->Full,PlotStyle->Black]
```

Here are a few plots illustrating $r_3{}^*(n)$:

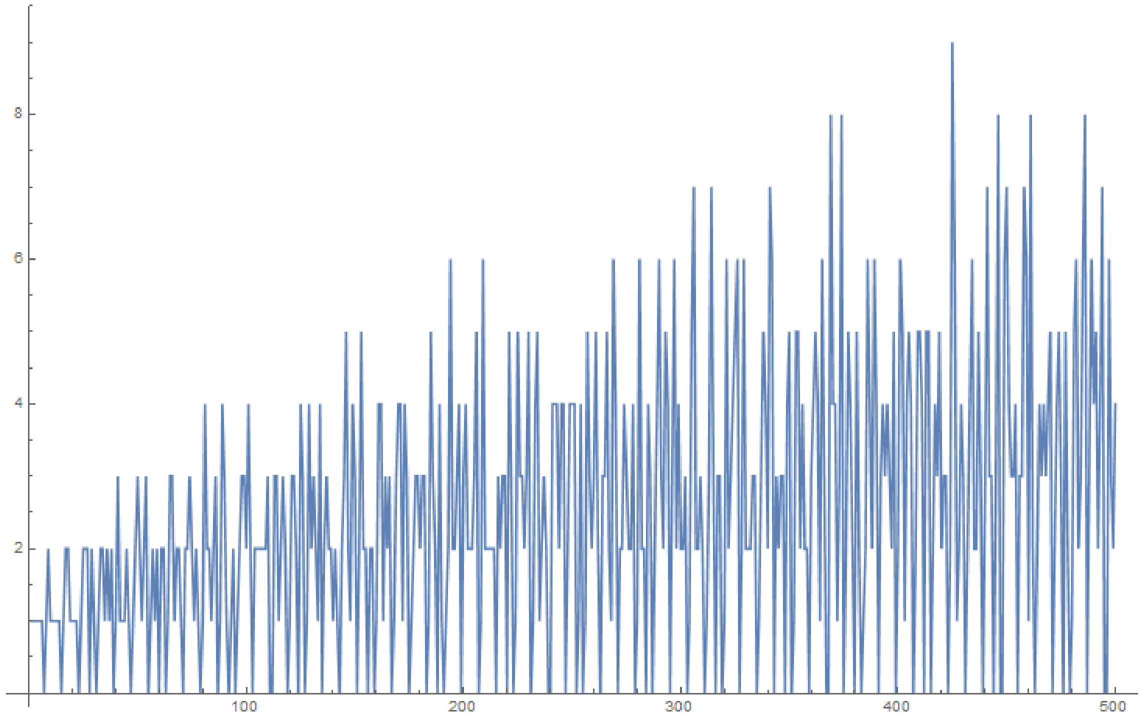


Figure 86. $r_3{}^*(n)$: Number of different representations of n as the sum of 3 squares, (n=1 to 500)

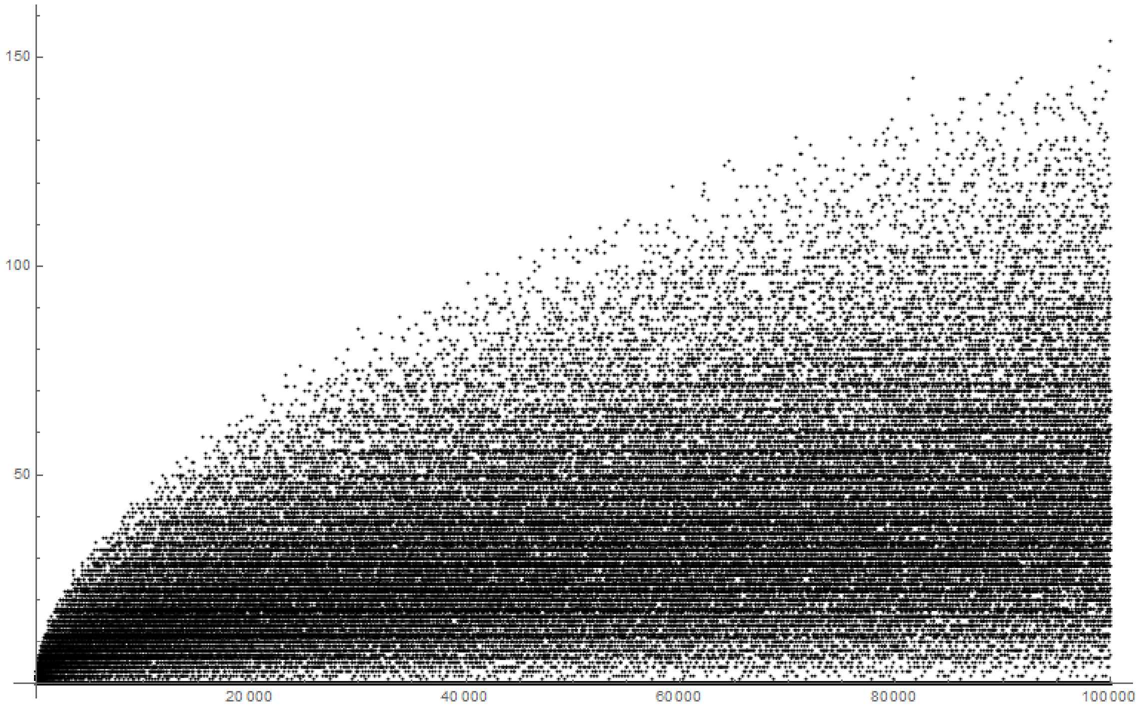


Figure 87. $r_3{}^*(n)$: Number of different representations of n as the sumof 3 squares, (n=1 up to 100000)

```
ListPlot[Table[{n,Length[PowersRepresentations[n,3,2]]},{n,1,100000}],
PlotRange->Full,PlotStyle->Black]
```

Since everything happens on an n-sphere (here a 2-sphere or spherical surface), spherical coordinates (r, φ, θ), obviously, must be used rather than Cartesian ones (x, y, z). The radius r of the sphere remains constant in our investigations, therefore only two degrees of freedom remain: the angles φ and θ.

This leads to the tempting idea of interpreting φ and θ as 2-dimensional Cartesian coordinates. All interesting patterns on the spherical surfaces can now be seen as two-dimensional representations:

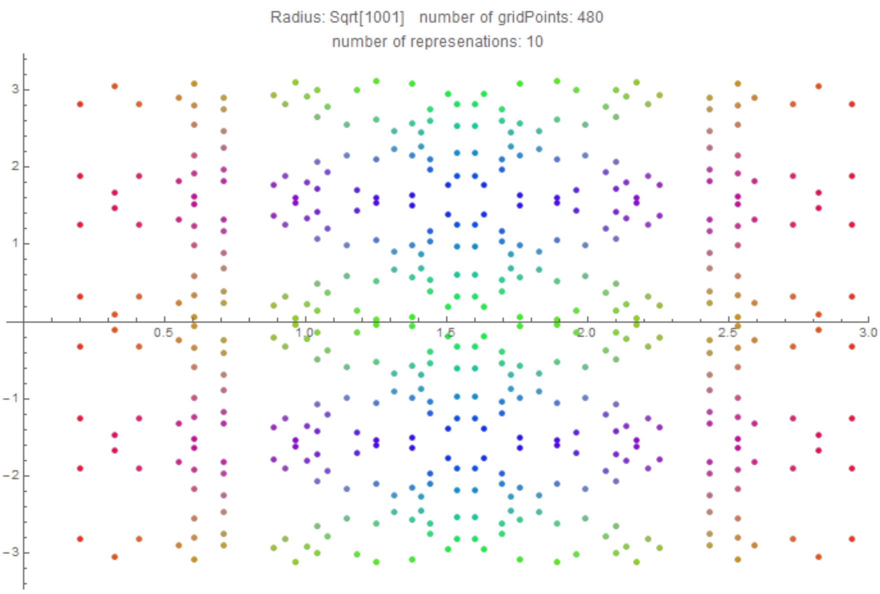


Figure 88. Lattice points of the surface of a sphere where $r^2 = 1001$, angles of the spherical coordinates interpreted as 2-dimensional Cartesian coordinates (same colour indicates identical points with respect to mirror operations)

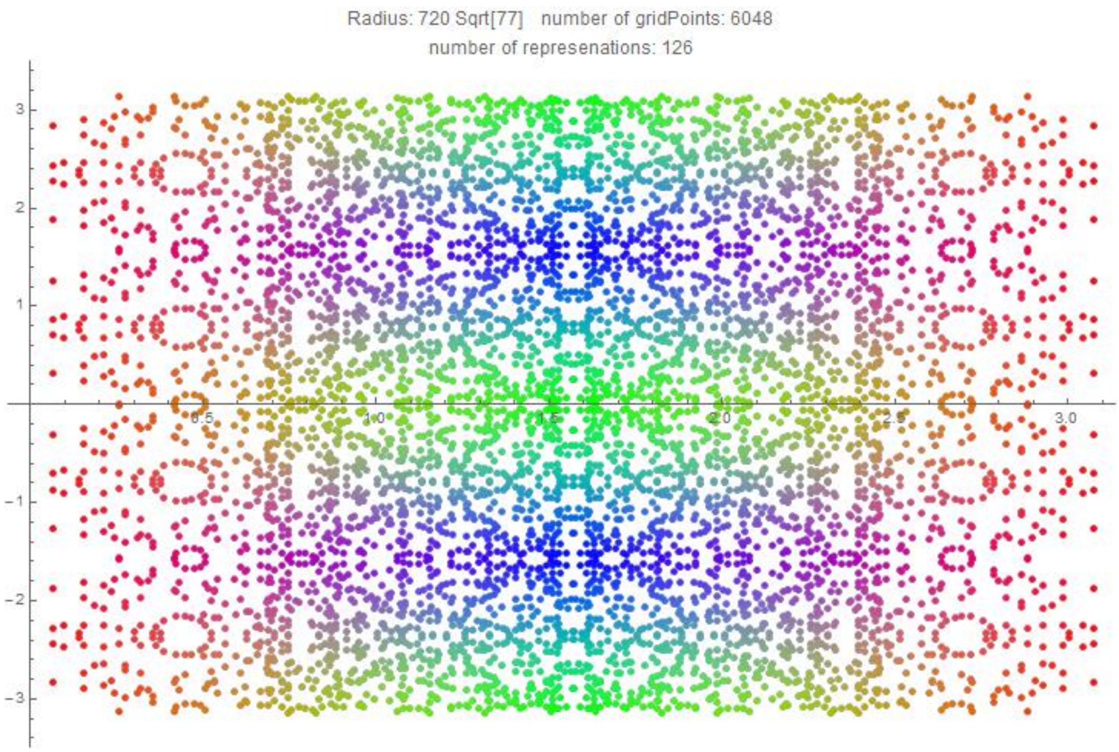


Figure 89. Same as above, however $r^2 = 11!$ (Mathematica programs in the Appendix)

9.2.1 FORMULAE AND PROPERTIES

We restrict ourselves to the function $r_3(n)$, which calculates the number of lattice points in 3-dimensional space lying on the surface of a sphere with radius $\sqrt{n}$. For the function $r_3{}^*(n)$, which calculates the number of different, positive grid points, such that:
$0 \leq n_1 \leq n_2 \leq n_3$ and $n_1^2 + n_2^2 + n_3^2 = n$, see e.g. note.[48]
The generating function of $r_3(n)$ is the elliptic Jacobi $\vartheta_3(n)$ function raised to a power of 3:

$$\sum_{n=0}^{\infty} r_3(n)x^n = \vartheta_3^3(x) = 1 + 6x + 12x^2 + 8x^3 + 6x^4 + 24x^5 + \cdots \tag{141}$$

Explicit formulae:

$$r_3(n) = \left\{\begin{matrix} 24h(-n), \text{if } n \equiv 3 \text{ (mod 8)} \\ 12h(-4n), \text{if } n \equiv 1{,}2{,}5{,}6 \text{ (mod 8)} \\ 0, \text{if } n \equiv 7 \text{ (mod 8)} \end{matrix}\right\} \tag{142}$$

(with $h(n)$ being the 'class number'[49] of n).

The 'Three-squares theorem' of C. F. Gauss is worth mentioning:

for each natural integer number $\boldsymbol{n}$ that can be represented as a sum of 3 squares ($\boldsymbol{x^2 + y^2 + z^2 = n, n, x, y, z} \in \mathbb{N}$), the following obtains:

$$\boldsymbol{n = 4^k m} \text{ where } \boldsymbol{4 \nmid m} \textbf{ and } \boldsymbol{m \not\equiv 7 \; mod \; 8}$$

[48] http://mathworld.wolfram.com/SumofSquaresFunction.html
[49] https://en.wikipedia.org/wiki/Class_numberl

9.3 FOURTH DIMENSION: HYPERSPHERES AND INTEGER LATTICE POINTS ON ‘GLOMES‘

In the same way as in three-dimensional space, we are searching for lattice points in a Cartesian coordinate system that lie on the ‘surface’ of a hypersphere. The number of these possible lattice points that are touched by the surface of the hypersphere depends heavily on the radius of the hypersphere. We will denote this ‘surface’ of the hypersphere in the following as "glome" and thus stick to the general language usage. Let us suppose that we increase the radius of the hypersphere continuously, then our glome will run through the lattice points of our four-dimensional coordinate system. Our interest now is directed to the lattice points that lie exactly on the glome. Here are a few examples:

(here, the author would have liked to have shown a few examples, but unfortunately it is not so easy to visualize four-dimensional objects. There is, however, a trick to doing so, see below).

Let us for the moment remain in the abstract, mathematical space.

In 3-dimensional space, some values r^2 always existed for which the equation $x^2 + y^2 + z^2 = r^2$ had no solutions (and therefore no corresponding lattice points). In the four-dimensional domain, this is no longer the case: for every integer r^2 (that is, every natural number), the equation $x^2 + y^2 + z^2 + t^2 = r^2$ has integer solutions! Each natural number can be expressed as the sum of four squares. This is the famous theorem of Lagrange from the year 1770.

The function that calculates the number of integer lattice points on a glome ('surface' of a four-dimensional hypersphere) is in mathematical literature denoted as $r_4(n)$. We denote the number of different positive lattice points such that $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$ and $n_1^2 + n_2^2 + n_3^2 + n_4^2 = n$

as $r_4{}^*(n)$

All function values of $r_4(n)$ are divisible by 8. The first 50 values read:

```
{8,24,32,24,48,96,64,24,104,144,96,96,112,192,192,24,144,312,160,144,2
56,288,192,96,248,336,320,192,240,576,256,24,384,432,384,312,304,480,4
48,144,336,768,352,288,624,576,384,96,456,744}
Mathematica:
SquaresR[4,Range[50]]
```

The 8 solutions for $r^2 = 3$ read:

```
{{1,-1,-1,0},{1,1,-1,0},{-1,0,-1,-1},{-1,-1,0,1},{1,1,0,1},{1,-
1,0,1},{-1,0,1,1},{0,1,-1,-1}}
Mathematica:
FindInstance[x^2+y^2+z^2+t^2==3,{x,y,z,t},Integers,8]
```

These solutions can be created from each other mutually by permutations or symmetrical mirroring. The number of genuinely different solutions in this case is 1:

```
{{0,1,1,1}}
```

```
Mathematica:
PowersRepresentations[3,4,2]
```

The first 50 values of $r_4^*(n)$ read:

```
{1,1,1,2,1,1,1,1,2,2,1,2,2,1,1,2,2,3,2,2,2,2,1,1,3,3,3,3,2,2,2,1,3,4,2
,4,3,3,2,2,3,4,3,2,4,2,2,2,4,5}
```

```
Mathematica:
Table[Length[PowersRepresentations[i,4,2]],{i,1,50}] (*or:*)
a[n_]:=If[n<0,0,Sum[Boole[n==i^2+j^2+k^2+l^2],{i,0,Sqrt[n]},{j,0,i},{k
,0,j},{l,0,k}]];
Table[a[n],{n,1,50}]
```

Here are two graphs of $r_4(n)$:

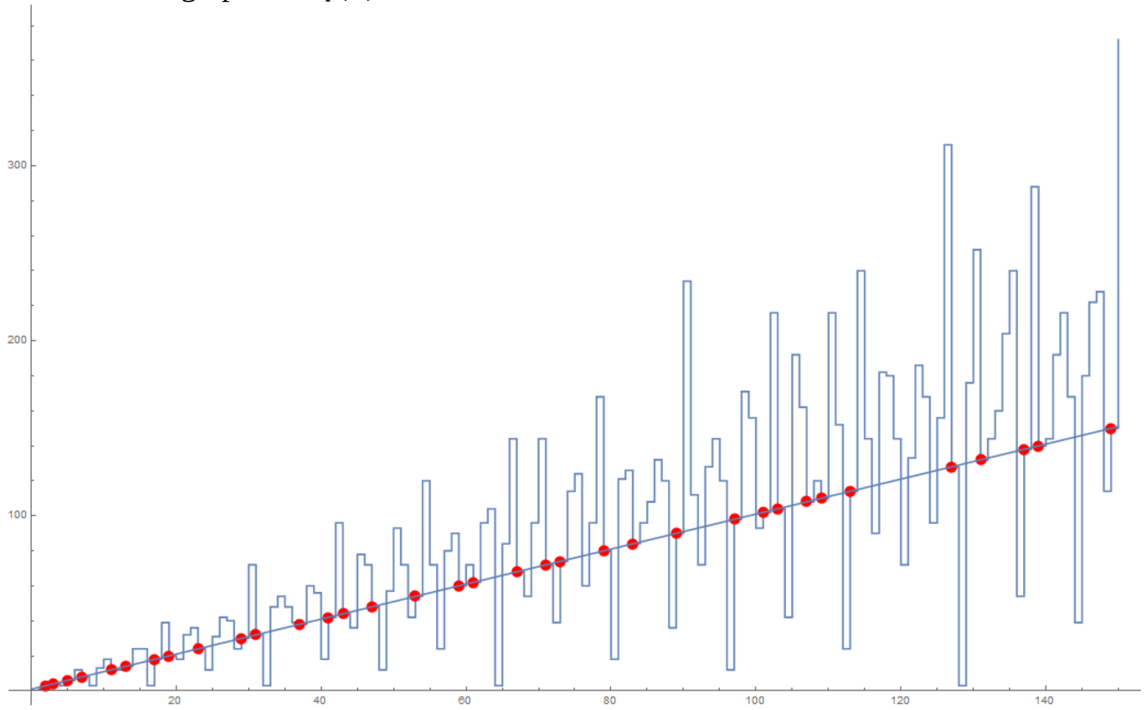


Figure 90. $r_4(n)$: Number/8 of possible representations of n as the sum of 4 squares. Values located at prime number positions are marked in red

```
Mathematica:
Show[ListLinePlot[Table[{n,SquaresR[4,n]/8},{n,1,150}],
InterpolationOrder->0],ListPlot[
Table[{Prime[n],SquaresR[4,Prime[n]]/8},{n,1,PrimePi[150]}],
PlotStyle->{Red,PointSize[0.01]}],Plot[x+1,{x,0,150}]]
```

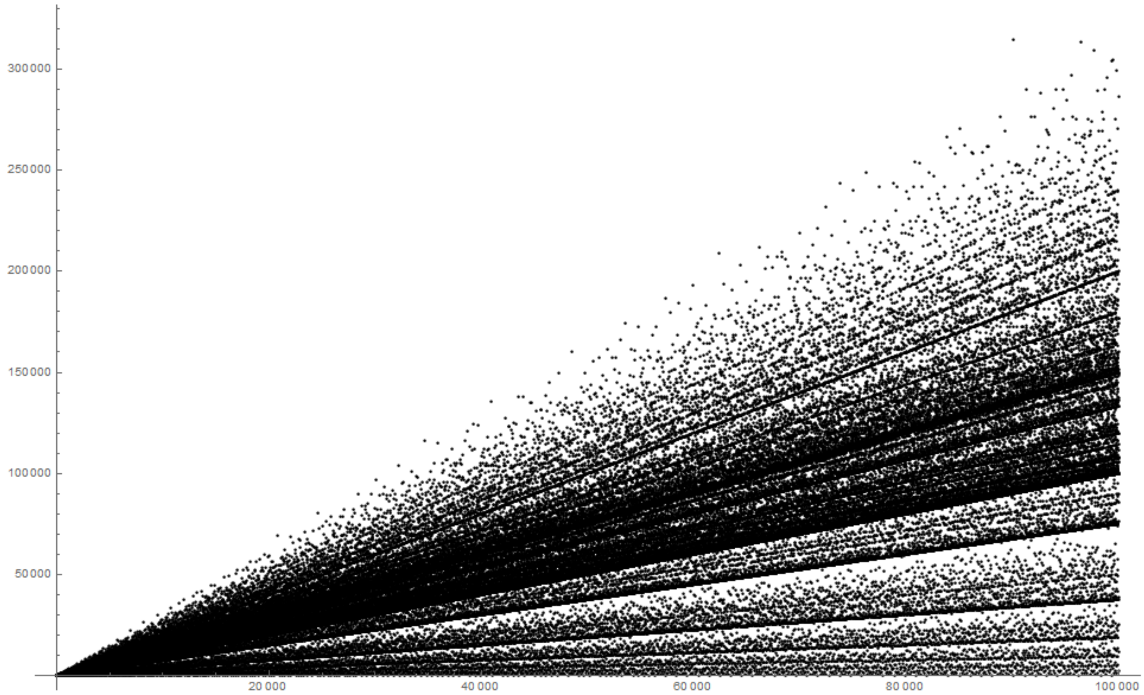


Figure 91. $r_4(n)$: number/8 of representations of n as the sum of 4 squares (up to 100000)

```
Mathematica:
ListPlot[ParallelTable[{n,SquaresR[4,n]/8},{n,1,100000}],
PlotRange->Full,PlotStyle->Black]
```

It can be clearly seen from Figure 90 that all values of $r_4(n)$ lie on a straight line when n is a prime number, see Formula (137). This phenomenon occurs only in the fourth dimension. Neither in the lower dimensions nor in higher dimensions can such a simple relationship between primes and the number of lattice points on n-spheres be observed.

Here are a few plots of $r_4^*(n)$:

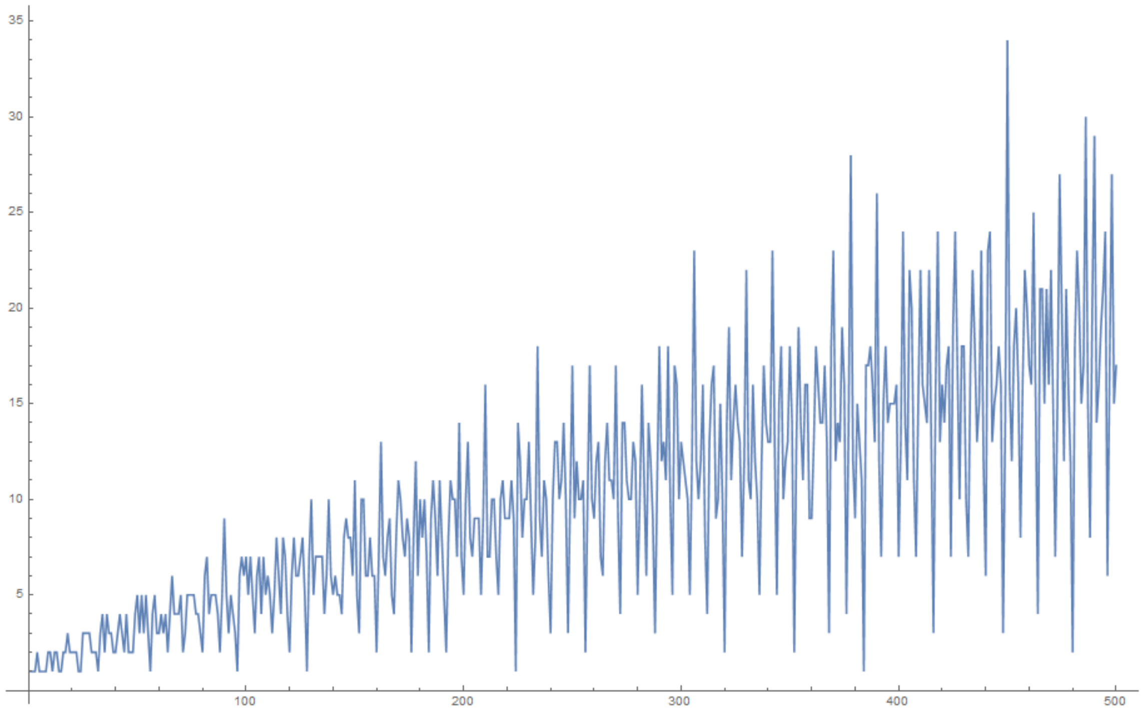
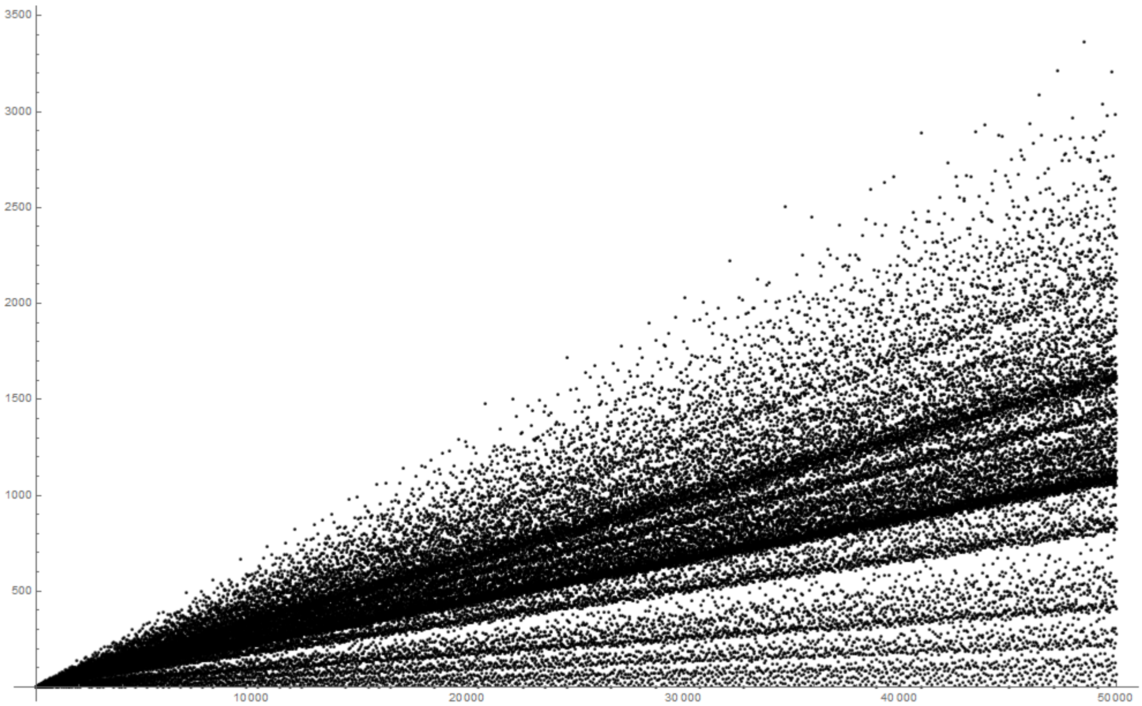


Figure 92. $r_4^*(n)$: number of different representations of n as the sum of four squares (n=1 up to 500)

Mathematica:
```
ListLinePlot[Table[Length[PowersRepresentations[i,4,2]],{i,1,500}],
PlotLegends->Automatic,PlotRange->All]
```



Mathematica:
```
ListPlot[ParallelTable[{n,Length[PowersRepresentations[n,4,2]]},{n,1,5
0000}],PlotRange->Full,PlotStyle->Black]
```

From the graphs for $r_4(n)$ and $r_4{}^*(n)$, we see that the asymptotic behavior of both functions is linear.
As in the last section for three-dimensional spheres, we use a trick to reduce the number of dimensions by one dimension by using hypersphere coordinates $(r, \varphi, \theta, \psi)$ instead of Cartesian coordinates(x, y, z, t) The radius r of the hypersphere remains constant and only three degrees of freedom remain: the angles φ, θ and θ. Φ, θ and ψ are interpreted as three-dimensional Cartesian coordinates. Thus all interesting patterns on the hyper-spherical surfaces can also be seen as a three-dimensional picture.
The colour representation was chosen such that the same (absolute) x, y, or z coordinates of a point represent the same R G B triple in the RGB colour space.

Example 1: r^2 =1001.
Example 2: r^2 =10007

Animations (views on the surface of the 4-dimensional spheres) can be found on the enclosed computer CD, or as a Mathematica program for the animations in the Appendix.

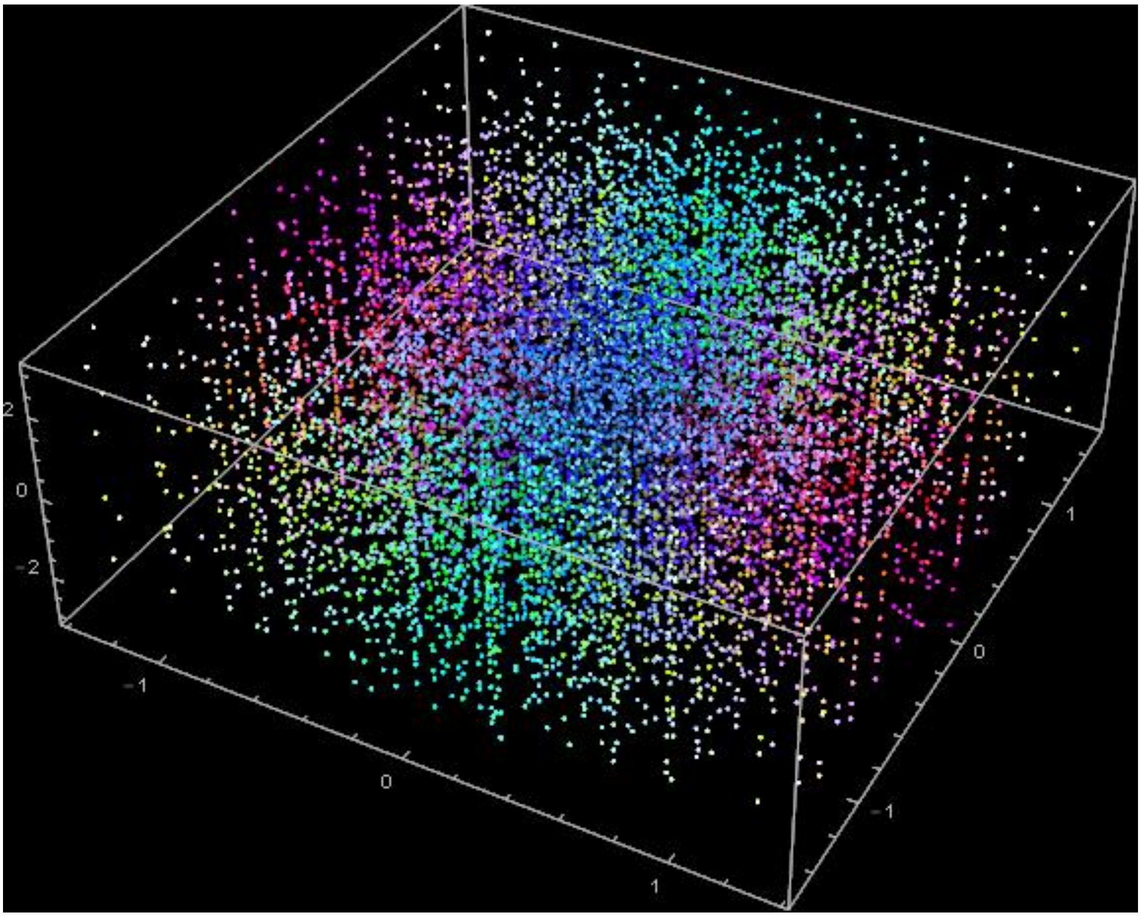


Figure 93. Example 1: lattice points on the 3-sphere of a 4 dim. sphere where r^2=1001

(Mathematica programs can be found in the Appendix).

The hidden structures appear only when viewed parallel to the coordinate axes:

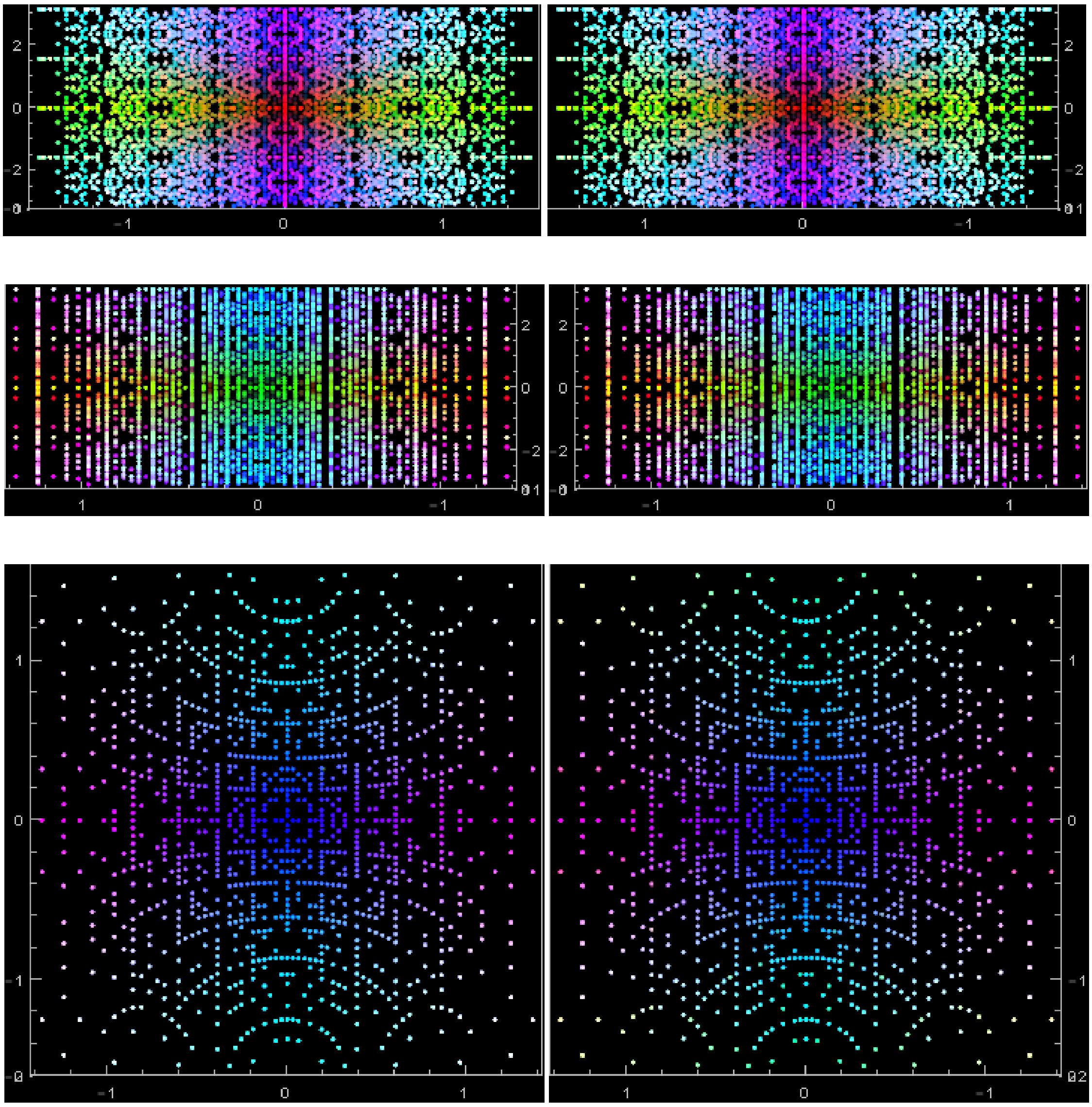


Figure 94. Six views of Figure 93: right/left, front/back, above/below

```
Mathematica:
grTable = {
Show[obj,ViewPoint->{Infinity,0,0},ImageSize->Medium],
Show[obj,ViewPoint->{-Infinity,0,0},ImageSize->Medium],
Show[obj,ViewPoint->{0,Infinity,0},ImageSize->Medium],
Show[obj,ViewPoint->{0,-Infinity,0},ImageSize->Medium],
Show[obj,ViewPoint->{0,0,Infinity},ImageSize->Medium],
Show[obj,ViewPoint->{0,0,-Infinity},ImageSize->Medium]
}
```

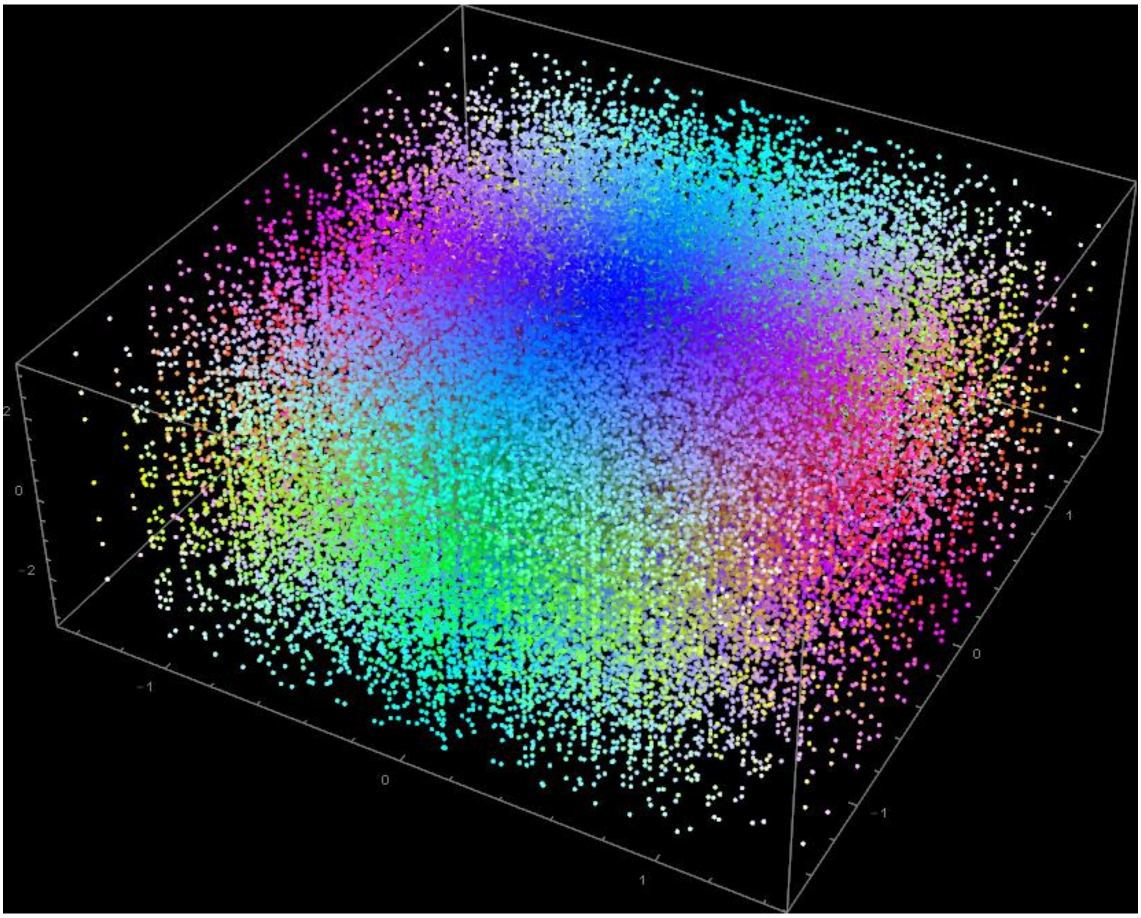


Figure 95. Example 2: lattice points on the 3-sphere of a 4 dim. sphere where r^2=10007

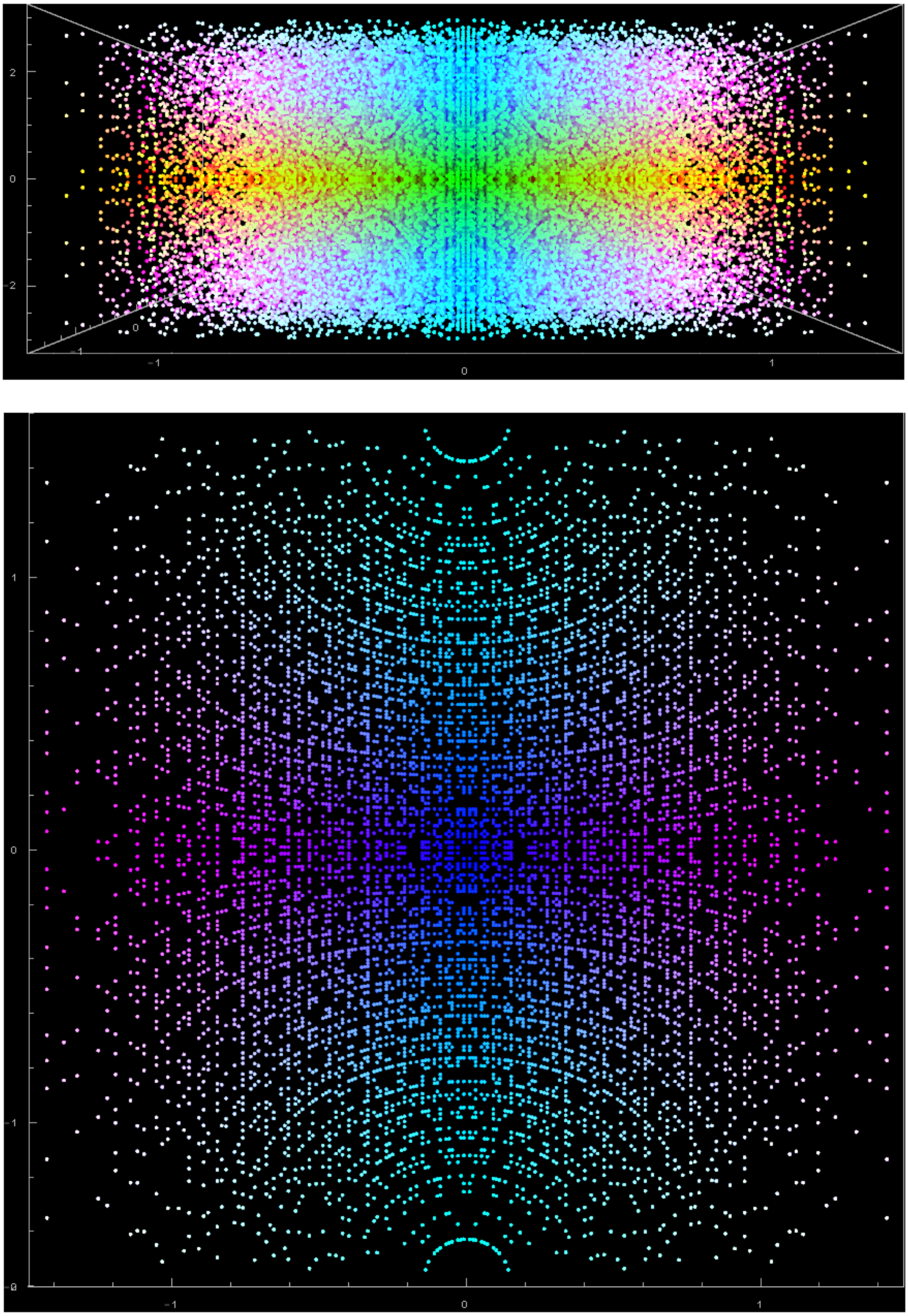


Figure 96. Two views of the illustration above

In the view of the author, the resulting images of the four-dimensional spherical surfaces are most attractive when the square of the spherical radius is a prime number. For them, the ratio between accumulations of points and empty spaces is the most balanced. This is also demonstrated by Figure 90. The density of the spherical points on 3-spheres is for primes always in the middle range.

9.3.1 FORMULAE AND PROPERTIES

As in the case of the lower dimensions, we restrict ourselves to the function $r_4(n)$, which calculates the number of lattice points in the 4-dimensional space that lie on a hypersphere surface (glome) with radius $\sqrt{n}$. For the function $r_4{}^*(n)$, which calculates the number of different positive grid points, such that: $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$ and $n_1^2 + n_2^2 + n_3^2 + n_4^2 = n$, please refer to other relevant sources.
The generating function of $r_4(n)$ is the elliptic Jacobi $\vartheta_3(n)$ function raised to the 4th power:

$$\sum_{n=0}^{\infty} r_4(n)x^n = \vartheta_3^4(x) = 1 + 8x + 24x^2 + 32x^3 + 24x^4 + 48x^5 + \cdots \tag{143}$$

Explicit formulae:

$$r_4(n) = \begin{cases} 8\sigma_1(n), \text{if } n \text{ odd} \\ 24\sigma^0(n), \text{if } n \text{ even, with } \sigma^0(\mathrm{n}) = \sum\limits_{2\nmid d, d|n} d \end{cases} \tag{144}$$

This can be written somewhat more easily:

$$r_4(n) = \begin{cases} 8\sigma_1(n), \text{if } 4 \nmid n \ (n \text{ not divisable by } 4) \\ 8\sigma_1(n) - 32\sigma_1\left(\frac{n}{4}\right), \text{otherwise } (n \text{ divisable by } 4) \end{cases} \tag{145}$$

or

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d \tag{146}$$

10 ABOUT OCRONS AND GOCRONS: SHADES OF GÖDEL

10.1 WHAT ARE OCRONS AND GOCRONS?

The acronym "OCRON" stands for "Operator Chain Representation Of Number". An OCRON is a representation method for natural numbers $n > 0$ that works procedurally (similar to a small computer program written in a programming language with very simple instructions) and operators that can be executed sequentially from left to right. For the processing of the operator sequence (= OCRON), we use the so-called "Polish notation", which works with a stack: numbers and basic symbols appearing in the list are simply 'pushed' onto the stack. Operators process the lowest two stack entries getting a single value and let the stacked entries above slip down one position. The stack can become arbitrarily large in the course of processing an OCRON, but in the end only one entry should remain: the value of the OCRON. Thus, any number can be converted into an OCRON. An OCRON, in contrast to a normal sum representation with number systems, describes not only the value of the number but also the procedure by means of which the number is generated.

Of course, the converse does not apply: not every string consisting of symbols from the symbol stock is a number. The logicians speak of 'well-formed' and 'non-well-formed' character strings. For most types of OCRONs, there are more non-well-formed OCRONs than well-formed OCRONs. Below, however, we will present methods that make possible the interpretation non-well-formed OCRONs and the assignment of numbers to them.

There are also OCRON systems (see 'prime OCRONs') which, by their very nature, always lead to well-formed operator sequences. These most interesting systems represent a bijective mapping from the natural numbers onto a set of symbols that is unambiguously reversible. We anticipate here (what is explained in detail below) that a GOCRON is a 'Gödelized' OCRON, freely following the method of the brilliant Austrian mathematician Kurt Gödel, who invented this method (Gödel assigned mathematical assertions, theorems, or formulae to natural numbers), we will assign a numerical value to each OCRON chain. This process is called 'Gödelization'. It describes a change in the 'level of meaning': from a procedural meaning to an arithmetical meaning. In contrast to Gödel, who used 'Gödelization' only hypothetically and theoretically (for the proof of his 'theorem of incompleteness'), we will here work quite concretely with 'Gödelized' numbers.

First, we will repeat the simple number representations and show that they can also be interpreted as OCRONs.

10.1.1 REPRESENTATION BY SUMS IN NUMERAL SYSTEMS

First, a brief overview of number representations is provided here. The usual methods that are suitable for processing natural numbers in a computer are:

the **sum representation** in numeral systems with a suitable base: the base is typically 10 (decimal system), 2 (binary system), 16 (hexadecimal system), or 8 (octal system). Let b be the base, $z_i < b$ the 'digits', and N the highest occurring power to the base b of the numeral representation. Then every natural number $n \geq 0$ can be written as:

$$n = \sum_{i=0}^{N} z_i\, b^i, \text{where } N = \lfloor (ln\,(n))/ln\ b \rfloor \tag{147}$$

Both the digits z_i and the exponents i are represented in the same sum representation with the same base b, so that we have a total representation with $b+3$ symbols (namely the b numeral symbols as well as the three operator symbols $'+', '*'$ and $'\wedge'$ (addition, multiplication and exponentiation). This is a mixed representation, since all three operators occur. Generally, the digits, base and exponents will also be represented in the same system of numbers. We can, however, get a 'pure' representation, consisting only of the operators $'+'$ and $'\wedge'$, by adding the terms $z_i b^i$ as $(b^i + b^i + b^i + \ldots)$ and discarding terms with '0'. This leads to a sum representation that uses only the two operators $'+'$ and $'\wedge'$. Here, we give the base its own symbol b. This has the advantage that the symbol $'0'$ no longer appears in the reduced representation, in which only the individual digits and the operators actually occur.

The minimum number of different symbols for the sum representation with the operators $'+', '*'$ and $'\wedge'$ is five (binary system), the maximum number $b+3$ (in the b system).

The minimum number of different symbols for the sum representation with the operators $'+'$and $'\wedge'$ is four (binary system), the maximum number $b+2$ (in the b system).

Example: the number **$12800000 = 110000110101000000000000_{binary}$**

(using the decimal system, with operators $'+', '*'$ and $'\wedge'$):

$$8 * 10^5 + 2 * 10^6 + 10^7$$

or in operator-notation (stack method[50], b=10):

8b5^*2b6^*+b7^+

(decimal system, with operators $'+'$ and $'\wedge'$):

$$10^5 + 10^5 + 10^5 + 10^5 + 10^5 + 10^5 + 10^5 + 10^5 + 10^6 + 10^6 + 10^7$$

or in operator notation (stack method, b=10):

b5^b5^+b5^+b5^+b5^+b5^+b5^+b5^+b6^+b6^+b7^+

[50] Stack method: inverse Polish notation, 'b' and '1' will be pushed on the stack '+' and '^' evaluate and the two lowest stack values by applying the actual operator, write the result in to the lowest stack register and decrement the stack by 1.

(binary system, with operators $'+','*'$ and $'\wedge'$):

$$1*2^{12}+1*2^{14}+1*2^{16}+1*2^{17}+1*2^{22}+1*2^{23}\ (decimal)$$
$$=1*10^{1*10^{11}+1*10^{10}}+1*10^{1*10^{11}+1*10^{10}+1*10^{1}}+1*10^{1*10^{100}}+1*10^{1*10^{100}+1}+1*10^{1*10^{100}+1*10^{10}+1*10^{1}}+1*10^{1*10^{100}+1*10^{10}+1*10^{1}+1}(binary)$$

Obviously the multiplication by 1 is redundant, so we finally get:

(binary system, with operators $'+'$ and $'\wedge'$):

$$\mathbf{10^{10^{10+1}+10^{10}}+10^{10^{10+1}+10^{10}+10}+10^{10^{10^{10}}}+10^{10^{10^{10}}+1}+10^{10^{10^{10}}+10^{10}+10}+10^{10^{10^{10}}+10^{10}+10+1}}$$

or using the operator notation (stack method, b=10):

bbb1+^bb^+^bbb1+^bb^+b+^+bbbb^^^+bbbb^^1+^+bbbb^^bb^+b+^+bbbb^^bb^+b+1+^+

The method of reducing the description of a number to a small number of symbols (e.g. 1,2, $'+','*'$ and $'\wedge'$) was described back in 1944 by the British mathematician Reuben Louis Goodstein[51] when he was studying the 'Goodstein sequence'[52] (named after him). This sequence has interesting properties, since its terms reach unimaginably large values and (according to the theorem of Goodstein) returns to the value 0 after a finite number of steps. Some mathematicians argue that this theorem belongs to Gödel's category of unprovable statements: true, but not provable!

Note that in this example the 0 is no longer present, so that in the case of the binary system we have a pure sum representation of a number with only two operators (, +' and '^') and the symbols 1 and b, thus only 4 symbols. This sum representation is, of course, ambiguous: because of the commutativity of the operators, '+' and '*', the order can be changed at many positions in the sequence. Let us summarize for the sake of completeness, what is in any case trivial:

The sum representation within a numeral system with a base *b* and its powers can be written as a sequence of operators and symbols. The fewer the symbols used, the longer the sequence and the smaller the base of the numeral system. Representations with two ('+' and '^') and three (, +', '*' and '^') operators are possible.

[51] R.L. Goodstein(1945), „Function Theory in an Axiom-Free Equation Calculus". *Proceedings of the London Mathematical Society*

[52] https://de.wikipedia.org/wiki/Goodstein-Folge

10.1.2 PRODUCT REPRESENTATION USING PRIME FACTORS

The **product representation** works with prime factor decomposition. Every natural number $n > 1$ can be written as the product of prime factors p_{n_i}, which occur raised to the power e_i. Let N be the number of different occurring prime factors.

$$\boldsymbol{n = \prod_{i=1}^{N} p_{n_i}{}^{e_i}\,, \text{where } N = \omega(n)} \tag{148}$$

Note: $\omega(n)$ behaves asymptotically $\approx \ln\ln n$ and can be calculated:

$$\omega(n) = \ln\ln n + B_1 + \sum_{k=1}^{\infty}\left(-1 + \sum_{j=0}^{k-1}\frac{\gamma_j}{j!}\right)\frac{(k-1)!}{(\ln n)^k} \tag{149}$$

In which B_1 is the Mertens constant 0.2614972128 and γ_j are the Stieltjes constants. B'_1 can also be used to calculate the variance $\mathrm{var}\big(\omega(n)\big)$:

$$\mathrm{var}\big(\omega(n)\big) = \ln\ln n + B'_1 + \sum_{k=1}^{\infty}\frac{c_k}{(\ln n)^k}$$

$$\begin{aligned} B'_1 = B_1 - t - \frac{\pi^2}{6} &= 1.83568427, t = P(2) \\ &= \sum_{k=1}^{\infty}\frac{1}{{p_k}^2}\ (\text{Prime zeta function}) = 0.452247 \end{aligned} \tag{150}$$

Here $c_1 = 1.0879488865$, and $c_2 = 3.3231293098$

In Mathematica $\omega(n)$ and $\Omega(\mathrm{n})$ are implemented as arithmetical functions `PrimeNu[n]` and `PrimeOmega[n]`.

For the representation of the $p_{n_i}$and e_i, we can again choose: generally, the p_{n_i} and e_i are represented in the summation representation of a number system to a base b. Thus we have a mixed number representation: e.g. p_{n_i} and e_i in the decimal system as a summation representation, but the total number n as a product representation. However, we can also achieve here a 'pure' representation (in which we mean by 'pure': in such a way that the representation contains only ' $*$ ' and '^' operators, but not the ' $+$ ' operator). This leads us again to the idea of the 'OCRONs'. Suppose we restrict ourselves to the first N prime numbers. The next step is to convert the n_i (not the p_{n_i}!) and e_i into the product representation. Here, however, arises the phenomenon of recurrence, since the product representation of n_i or e_i can again contain p_{n_i} and e'_i, which in turn can be written in a normal sum representation or as product representation. The recursive process of the transformation from sum representations to product presentations can be continued until only the first N prime numbers still occur. Then we have a pure product representation of a number in which only the first N prime numbers occur (also in the powers of the prime numbers).

Example: the number **12800000**

• Using product representation with the first 3 prime numbers ($p_1 = 2, p_2 = 3, p_3 = 5$), operators $'*'$ and $'\wedge'$:

$$\mathbf{12800000 = 2^{12} * 5^5 = 2^{2^2 * 3} * 5^5 = p_1{}^{p_1{}^{p_1} * p_2} * p_3{}^{p_3}}$$

or in operator notation (stack method):

222^3*^55^*

• Using product representation with the first 2 prime numbers ($p_1 = 2, p_2 = 3$), operators $'*'$ and $'\wedge'$:

$$\mathbf{12800000 = p_1{}^{p_1{}^{p_1} * p_2} * p_3{}^{p_3} = p_1{}^{p_1{}^{p_1} * p_2} * p_{p_2}{}^{p_{p_2}}}$$

or in operator notation (stack method):

222^3*^(p_3)(p_3)^*

This idea of further reducing the number of primes required for representation leads us to OCRONs with a prime operator in the next chapter. The representation by means of indices, e.g. p_{p_3} is confusing and unclear, therefore we introduce a so-called prime operator P, which simply yields the nth prime number when applied to n.

10.2 OCRONS WITH PRIME OPERATORS

We continue to implement the idea of the product representation and replace all the values occurring in the bases and exponents recursively by smaller, simpler prime factor decompositions resulting in indices of (indices of … etc.) prime numbers, until we arrive at the last basic prime number $p_1 = 2$, which cannot be further reduced. This last, 'irreducible' prime number 2 is called the $'2'$-operator. Let us continue the last example in the last chapter:

$$\mathbf{12800000 = p_1{}^{p_1{}^{p_1} * p_2} * p_3{}^{p_3} = p_1{}^{p_1{}^{p_1} * p_2} * p_{p_2}{}^{p_{p_2}} = p_1{}^{p_1{}^{p_1} * p_{p_1}} * p_{p_{p_1}}{}^{p_{p_{p_1}}} =}$$

$$\mathbf{2^{2^2 * p_2} * p_{p_2}{}^{p_{p_2}}}$$

or in operator notation (stack method, operators: $2, P, *$ and ^):

$$\mathbf{12800000 = 222\wedge 2P * \wedge 2PP * 2PP\wedge *}$$

The operator notation is much easier. Note that the $'2'$ operator does nothing else but 'push' the $'2'$ on the stack; the P-operator simply calculates the xth prime number (with x being the actual stack value). The $'*'$ and $'\wedge'$ operators work as usual and process the two lowest stack entries, write the result to the lowest stack cell, and let all stack records above slip down one position.

Here is a simple example using the number 1763: 1763 is the product of the prime numbers 41 and 43. We use the P-operator in slightly different notation: $P(n)$ yields the nth prime number. Instead of $1763 = 41 * 43$ we write:
$1763 = P(13) * P(14)$. Well, we know that 13 is the 6th prime number and $14 = 2 * 7 = 2 * P(4)$.
Thus we can write: $P(13) = P(P(6))$ and $P(14) = P(2 * P(4))$ etc. …
(possible exponents are decomposed in the same way as the bases…). Therefore:

$$1763 = 41 * 43 = P(13) * P(14) = P(P(6)) * P(2 * P(4))$$
$$= P(P(2 * P(2))) * P((2 * P(2\hat{}2)) =$$

$P(P(2 * P(2))) * P((2 * P((2)\hat{}2))$, or using operator notation with inverse Polish notation:

$$\mathbf{1763 = 22P * PP222\hat{}P * P *}$$

In their 'simplicity', these operator sequences bear a certain similarity to the programming language 'Brain-Fuck'.[53]
Among the OCRON sequences, there are 'well-formed' and 'non-well-formed' sequences. The well-formed parts can be processed without any problem. The non-well-formed, for example **^^ p * 222**, have no meaning (at present).

Note: OCRONs of types 3 to 5 (with * and ^ operators) can be redundant and yet still well-formed. The redundancy occurs because there is a certain ambiguity in arithmetic representations. OCRONs that cannot be shortened, we call 'minimal' OCRONs. Here is an example:

Redundant (arithmetically): $2 * 2 * 5 * 5 * 2$. Redundant (OCRON:) 22*52^*2*
Minimal (arithmetically): $2^3 * 5^2$. Minimal (OCRON:) 23^52^*

OCRONs are not unique. They can have different elements in a different order, but still give the same value. This property is called "degeneration". This comes from the commutativity of the calculations performed. OCRONs can easily be multiplied by simply hooking the OCRON chains together and appending a $'*'$ operator:

Example $\mathbf{5 * 7 = 35}$ $\mathbf{2PP \cdot 22\hat{}P = 2PP22\hat{}P *}$
Example $\mathbf{6 * 12 = 72}$ $\mathbf{2P2 * \cdot\ 2P22\hat{} * = 2P2 * 2P22\hat{} ** = 2P2\hat{}22P\hat{} *}$

Note: the transforming (or simplifying) of the redundant OCRON "$2P2 * 2P22\hat{} **$" into the minimal OCRON "$2P2\hat{} 22P\hat{} *$" by typographical means, however, is difficult and still an unsolved problem. More about this in Chapter **10.3.**

10.2.1 OCRONS WITH THE PRIME "P" AND "*" OPERATORS

The simplest OCRON obtained from the prime factor decomposition of a number contains three operators: $2, P, *$. As discussed in the last chapter, a recurring decomposition of the occurring bases and exponents yields an OCRON consisting of three symbols. We call it the OCRON type $'3'$. Each well-formed sequence begins with a $'2'$ and ends with $'*'$ or $'P'$ (i.e. one can immediately see whether a type 3 OCRON is a prime number or a composite number). Here is an example: the first 50 natural numbers in OCRON type 3 representation:

[53] https://en.wikipedia.org/wiki/Brainfuck

Table 13. Numbers 2 to 50 in OCRON type 3 format

n	OCRON type 3	n	OCRON type 3
1	-	26	2P2*P2*
2	2	27	2P2P*2P*
3	2P	28	22*P2*2*
4	22*	29	2PP2*P
5	2PP	30	2PP2P*2*
6	2P2*	31	2PPPP
7	22*P	32	22*2*2*2*
8	22*2*	33	2PPP2P*
9	2P2P*	34	22*PP2*
10	2PP2*	35	22*P2PP*
11	2PPP	36	2P2P*2*2*
12	2P2*2*	37	2P2*2*P
13	2P2*P	38	22*2*P2*
14	22*P2*	39	2P2*P2P*
15	2PP2P*	40	2PP2*2*2*
16	22*2*2*	41	2P2*PP
17	22*PP	42	22*P2P*2*
18	2P2P*2*	43	22*P2*P
19	22*2*P	44	2PPP2*2*
20	2PP2*2*	45	2PP2P*2P*
21	22*P2P*	46	2P2P*P2*
22	2PPP2*	47	2PP2P*P
23	2P2P*P	48	2P2*2*2*2*
24	2P2*2*2*	49	22*P22*P*
25	2PP2PP*	50	2PP2PP*2*

OCRONs of type 3 do not have a power operator and are therefore not as interesting. For high powers, OCRONs of type 3 become unwieldy. Just think of large composite numbers or powers of 2, such as $2^{57885161}$, whose OCRON representation would then have a length of millions of characters!

10.2.1.1 DEGENERATION OF TYPE 3 OCRONS

By ‘degeneration’ we understand the fact that there are generally several OCRON representations for a unique number n. The converse does not apply, of course. To an OCRON there is only one unique number n. This degeneration increases very fast with n, as the following graphic shows:

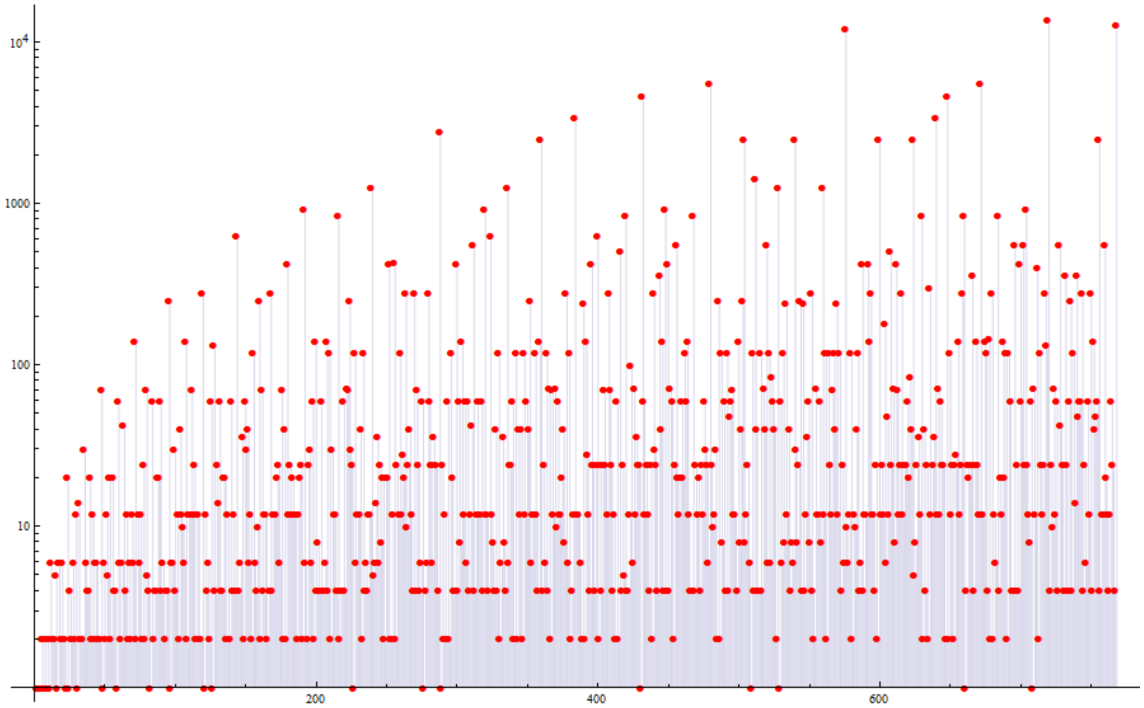


Figure 97. Degeneration of well-formed OCRON3s up to n=768 (logarithmic plot)

```
Mathematica:
data =
Import["primes/data/ocron3_wellformed_Degeneration_OK_upto_768.txt","C
SV"]
ListLogPlot[data,PlotStyle->Red,AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic,PlotRange->All]
```

10.2.2 OCRONS WITH THE PRIME “P”, “*” AND “^” OPERATORS

We want to pay most attention to this type of OCRON. We call this OCRON a ‘type 4 OCRON’, since it contains the 4 operators: $2, P, *, \wedge$. For the type 4 OCRONs, we have in addition a power operator. **It reflects the prime factor decomposition of a number.** Each well-formed sequence begins with a $'2'$ and ends with $' * ', '\wedge'$, or $'P'$ (i.e. one can immediately see if an OCRON is a prime number, a composite number, or a power number). Here is an example: the first 50 natural numbers in OCRON type 4 format:

Table 14. The numbers 2 to 49 in OCRON type 4 format

n	OCRON type 4	n	OCRON type 4
2	2	26	222P*P*
3	2P	27	2P2P^
4	22^	28	22^22^P*
5	2PP	29	22PP*P
6	22P*	30	22P*2PP*
7	22^P	31	2PPPP
8	22P^	32	22PP^
9	2P2^	33	2P2PPP*
10	22PP*	34	222^PP*
11	2PPP	35	2PP22^P*
12	22^2P*	36	22^2P2^*
13	22P*P	37	22^2P*P
14	222^P*	38	222P^P*
15	2P2PP*	39	2P22P*P*
16	222^^	40	22P^2PP*
17	22^PP	41	22P*PP
18	22P2^*	42	22P*22^P*
19	22P^P	43	222^P*P
20	22^2PP*	44	22^2PPP*
21	2P22^P*	45	2P2^2PP*
22	22PPP*	46	22P2^P*
23	2P2^P	47	2P2PP*P
24	22P^2P*	48	222^^2P*
25	2PP2^	49	22^P2^

OCRONs of type 4 provide a compact representation of very large values. By way of example, here is a table of the first Mersenne numbers:

Table 15. Mersenne numbers, as well as the exponents in OCRON type 4 representation

n	Mersenne prime exponent p	Mersenne number $M_p = 2^P$-1	OCRON4(p)
1	**2**	**3**	**2**
2	**3**	**7**	**2P**
3	**5**	**31**	**2PP**
4	**7**	**127**	**22^P**
5	11	2047	2PPP
6	**13**	**8191**	**22P*P**
7	**17**	**131071**	**22^PP**
8	**19**	**524287**	**22P^P**
9	23	8388607	2P2^P
10	29	536870911	22PP*P
11	**31**	**2147483647**	**2PPPP**
12	37	137438953471	22^2P*P
13	41	2199023255551	22P*PP
14	43	8796093022207	222^P*P
15	47	140737488355327	2P2PP*P
16	53	9007199254740991	222^^P

17	59	576460752303423487	22^PPP
18	61	2305843009213693951	22P2^*P
19	67	147573952589676412927	22P^PP
20	71	2361183241434822606847	22^2PP*P
21	73	9444732965739290427391	2P22^P*P
22	79	604462909807314587353087	22PPP*P
23	83	9671406556917033397649407	2P2^PP
24	89	618970019642690137449562111	22P^2P*P

Table 16. Mersenne numbers in OCRON type 4 format (prime numbers in red)

n	p	Mersenne number $M_p = 2^P - 1$	OCRON4(M_p)
1	2	3	2P
2	3	7	22^P
3	5	31	2PPPP
4	7	127	2PPPPP
5	11	2047	2P2^P22P^2P*P*
6	13	8191	22^2PP2PPP*P*P
7	17	131071	22P^2P*22P2^*P*PP
8	19	524287	22PP*22^2P2P^*PP*P
9	23	8388607	2P2PP*P2PP2^P2P22P*P*P*P*
10	29	536870911	2P22^PP*P2PP22^2P*P*P*22^22PPP*P*P*
11	31	2147483647	2PP2P*22P^*PP2P2*P2PP*2P2^*P*2PP*P
12	37	137438953471	2P222^^*P2P2^2*P22^P2*P*22^P2^*2PP2P^*2*P*
13	41	2199023255551	2P2^2*P2P2*P*2*P22^PP2PPP*2P*P2PP2*P*2P2*P*2P*2*P*
14	43	8796093022207	2P2^PPP2PP2*PPP2*P*2PPP2^2*P2PP*P2*P*
15	47	140737488355327	22^P2PP*2*PP22^PP2P2^*22^*P*2PPPP2PP*2P*P2PPP*P2*P*
16	53	9007199254740991	2PP2*P2PP*PP22P^P2P*PP22^*P*2PP2*P2PP*22P^*P2P2^P*2P*2*P*
17	59	576460752303423487	22^P2*P22^*P222^^*P2P22^P22P^*PP2P2P^P*2P2*PP*2P2*P*2P2^*2*P**
18	61	2305843009213693951	2PP2P2^2*P*222^P*22PPP*2P2*P2P*P*2P2^2*PP22^P*2P2^*P*P*P*P
19	67	147573952589676412927	22^22^2P*2PP*22P*P*22P2P^*P*P*P2PP2P^222^P*PP*22P^2PP2^*22^P*22P^PP*P*P*
20	71	2361183241434822606847	2P22P^PP*222P*P*P*P22P*P2P2PP22^P*PPP*P*P*2P2^22^P*2P22P*P*P*2PPP22^PP*P*P*
21	73	9444732965739290427391	2PP22^PP*P22PP^22P^PP*22PPP*P*P*2P2^22PPP*P*2P2P2^^2P22P^PP*P*P*P*
22	79	604462909807314587353087	22P*2PP*22P*P*P2P2PPPP*222^P*P*2PPP22^2P*P*P*P*22^2P*P222^2PP*P*P*22P*2P2^P*222^P*P*P*PP*
23	83	9671406556917033397649407	2P22P*P*P2P222^P*P*222P2^*P*P*22P^2P2P^*P*22^2PP2P^*P*222^P*22^P2^P*22P^222P*P*PP*P*P*P*
24	89	618970019642690137449562111	??????

Table 17. Wagstaff prime exponents in OCRON type 4 format (resulting primes in red)

n	Wagstaff prime exponent p	Wagstaff number $\frac{M_p=2^P+1}{3}$	OCRON4(p)
1	2	5/3	2
2	3	3	2P
3	5	11	2PP
4	7	43	22^P
5	11	683	2PPP
6	13	2731	22P*P
7	17	43691	22^PP
8	19	174763	22P^P
9	23	2796203	2P2^P
10	29	178956971	22PP*P
11	31	715827883	2PPPP
12	37	45812984491	22^2P*P
13	41	733007751851	22P*PP
14	43	2932031007403	222^P*P
15	47	46912496118443	2P2PP*P
16	53	3002399751580331	222^^P

17	59	192153584101141163	22^PPP
18	61	768614336404564651	22P2^*P
19	67	49191317529892137643	22P^PP
20	71	787061080478274202283	22^2PP*P
21	73	3148244321913096809131	2P22^P*P
22	79	201487636602438195784363	22PPP*P

Table 18. Wagstaff numbers in OCRON type 4 format (prime numbers in red)

n	Wagstaff prime exp. p	Wagstaff number $\frac{W_p=2^P+1}{3}$	OCRON4$\left(\frac{2^P+1}{3}\right)$
1	2	5/3	-
2	3	3	2P
3	5	11	2PPP
4	7	43	222^P*P
5	11	683	22^2PPPP*P
6	13	2731	2P22^P*22P^P*P
7	17	43691	22P^22P^22P*P*P*P
8	19	174763	222^22P2P^*P*P*P
9	23	2796203	2PP22^2P2^*P*2P22P^P*P*P
10	29	178956971	22^PPP222^^2P22^^*22P*P2^*P*
11	31	715827883	22^PPP22^22^P*2PPP*22^PP*PP*P
12	37	45812984491	2PPP2*P2PP*PP2PPP*2P*2*P2PPP2PP2^*P*
13	41	733007751851	222^^P2P*P2PPP2*P*2P2*P*22^P*2PP*2P*22^*P2P2^PP*
14	43	2932031007403	22^P2P^P2PP2P*P*P2P2*P22^*P2*P*2PPP*2*P
15	47	46912496118443	2PP2P2^*2*P2PP2*P*P2PPP2P*P2*P*2P2^2*P*2P*P2P2^2*PP*
16	53	3002399751580331	2PP2P^2*P2P22^*P2P*P*2P*P22^PP2P*P22^P*2P*22^**P22^P22^*P*
17	59	192153584101141163	22^P2^2PP2^*22P^*P22^PP*2P*PP2P2*PPP2PPP*2*P*2PPP2P*P2P*P*
18	61	768614336404564651	22^222^PP*PP*222^PP*222^P*222^2P*22^2P*P*P*PP*P*P*PP
19	67	49191317529892137643	22PP*22^22P2P^P*P*P*P222^22P^PP*2P22P^P*222^P*PPP*P*P*PP*
20	71	787061080478274202283	22^2P*22^PP*2P2^P2P2^PP*P*P222^P*22PP*P*2PP22^PP*P*222^^22P22P*^*P*P*P*
21	73	3148244321913096809131	2P22^P*22P*P*P22^2P*PP22P*222PP*2PPP*P*PP*P*2P2PP*22P2^P*2PP2PPP*P*P*P*P*
22	79	201487636602438195784363	??????

The lengths of the OCRONs in Mersenne numbers grow approximately proportional to p:

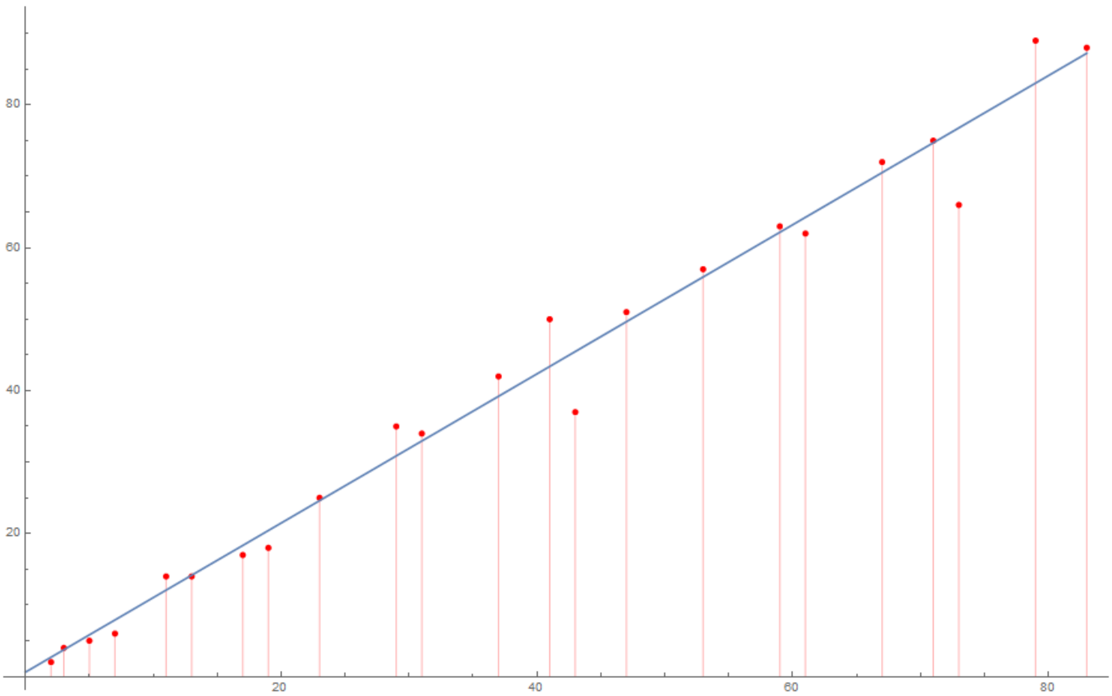


Figure 98. Lengths of type 4 OCRONs of Mersenne numbers up to M_{83}

```
Mathematica:
data={{2,2},{3,4},{5,5},{7,6},{11,14},{13,14},{17,17},{19,18},{23,25},
{29,35},{31,34},{37,42},{41,50},{43,37},{47,51},{53,57},{59,63},{61,62
},{67,72},{71,75},{73,66},{79,89},{83,88}}
line = Fit[data, {1,x},x]
Show[ListPlot[data,PlotStyle->Red, AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic],Plot[line,{x,0,83}]]
```

If we extrapolate for high Mersenne primes, we expect OCRON lengths of some million characters (about three times as long as in decimal representation, but much shorter than in binary representation).

The lengths of the OCRONs in Wagstaff numbers also grow approximately proportional to p:

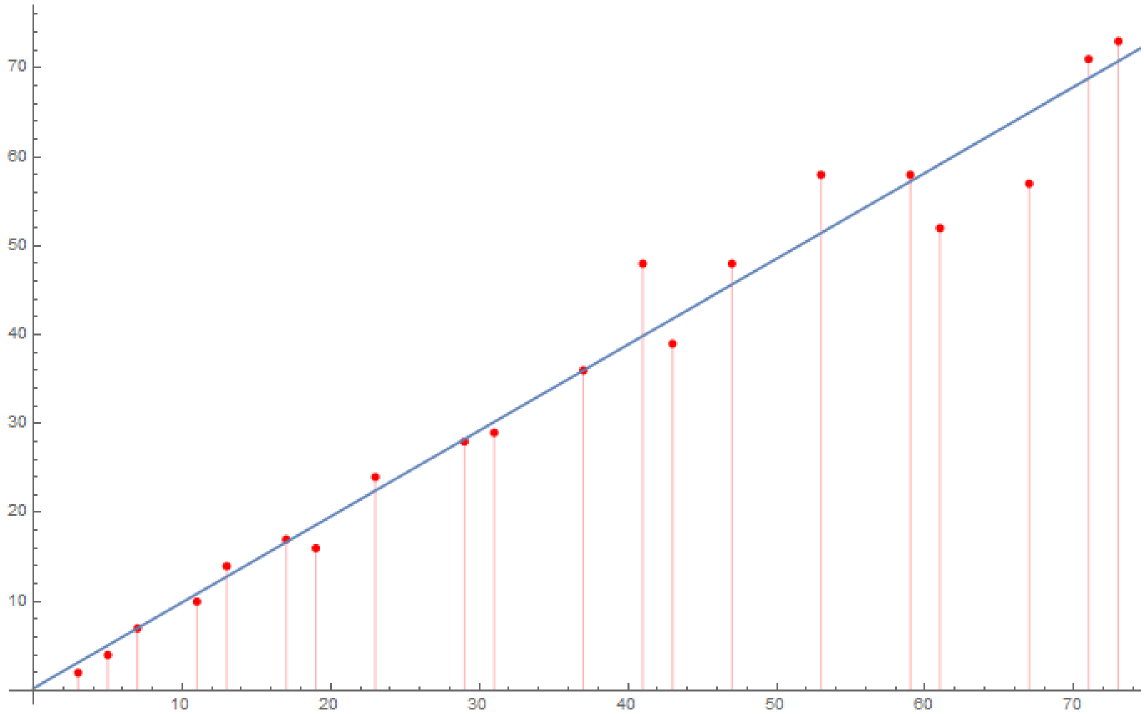


Figure 99. Lengths of type 4 OCRONs of Wagstaff numbers up to W_{73}

```
Mathematica:
{{3,2},{5,4},{7,7},{11,10},{13,14},{17,17},{19,16},{23,24},{29,28},{31
,29},{37,36},{41,48},{43,39},{47,48},{53,58},{59,58},{61,52},{67,57},{
71,71},{73,73}}
line = Fit[data, {1,x},x]
Show[ListPlot[data,PlotStyle->Red, AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic],Plot[line,{x,0,73}]]
```

If we extrapolate for high Wagstaff numbers to similarly high ranges as the largest known Mersenne primes, we also expect OCRON lengths of some million characters.

Note that the OCRONs for Mersenne prime numbers incremented by 1 (2^p) are only 2 characters longer than the prime exponent p itself! Here is an example:

The Mersenne prime number $M_{48} = 2^{57885161} - 1$ has a decimal representation of 17425170 digits. Its representation as a type 4 OCRON has an estimated length of approximately 60 million characters.

The prime exponent 57885161 has the OCRON type 4 representation 2PP2*P2PP*2P*P22^P*2P22^^*2*P having a length of 29 characters! $2^{57885161}$ has the OCRON representation **2**2PP2*P2PP*2P*P22^P*2P22^^*2*P^ with a length of 31 characters!

The following consideration is even more amazing: let us assume that the number $2^{57885161} + 1$ $(= 3 *$ possible Wagstaff candidate) has a similar complexity (with estimated 60 million characters OCRON length) as its 'Mersenne neighbor' $2^{57885161} - 1$.

Now one can simply write down the (unimaginably gigantic) number $2^{2^{57885161}+1}$ in OCRON type 4 representation because of the multiplicative property of the OCRONs:

$$2^{2^{57885161}+1} = 2^{2^{57885161}} * 2 = \mathbf{22}2\mathrm{PP}2 * \mathrm{P}2\mathrm{PP} * 2\mathrm{P} * \mathrm{P}22\text{^}\mathrm{P} * 2\mathrm{P}22\text{^^} * 2 * \mathrm{P}\text{^} *$$

Having an OCRON length of 33 characters! This could mean that a great portion of redundancy is present in the OCRON type 4 representation of $2^{57885161} + 1$ (HAVING presumably a length of millions of characters). There could therefore be an unknown algorithm that eliminates this redundancy!

If the hypothesis above about similar large complexities is true, then the question arises, why $2^{2^{57885161}+1}$ has less complexity by a factor of 2 million than its seemingly simpler exponent $2^{57885161} + 1$! In order that this idea can also be applied to M_{48} and its power of 2 ($2^{M_{48}}$), it would have to be decomposed just as easily in $2^{2^{57885161}} * 2^{-1}$. This would, in turn, indicate an extension of the OCRON concept to negative integers and will be the subject of further studies.

Curiosities:

The sequential operator representation used in the OCRONs is 'without alternative'. Here is an example of what the OCRON of the 17th Mersenne prime number M_{59} looks like when the expression is displayed 'conventionally'. We have up to five levels of nested indices (both in the base and in the mantissa). This number is now practically unreadable:

$$M_{59} = p_{p_2 p_{p_2 p_{p_{p_2}}}} p_{p_{p_2 p_{{p_2}^2} p_{2 p_{2 ({p_2}^2)^2 (p_{p_2})^2}}}} p_{2 p_{p_{p_2}} p_{p_{p_2 p_2}}}$$

10.2.2.1 PROPERTIES AND EXTENSION OF TYPE 4 OCRONS: EOCRONS

Let us return to the well-formed and non-well-formed OCRONs.

Not all OCRONs that can be represented by the symbols $' * ', '\wedge', '2'\ and\ 'P'$ are 'well-formed' and give meaning so that they can be processed (e.g. the sequence $\wedge\wedge \boldsymbol{p} * \mathbf{222}$ is not a well-formed OCRON).

There is now a possibility of assigning these degenerate OCRONs in a reasonable, meaningful way also an indication and thus a numerical value. For the interpretation of an OCRON, the symbols of the OCRON sequence must be 'pushed' onto the stack or processed according to the rules of the 'Polish' notation.

- We 'prime' the stack with infinitely many virtual $'2'$ symbols so that in the case of 'unexpected', '$*$' or '^' symbols ('unexpected' here means that at the time of the processing of an operator symbol, the stack has less than 2 values) the operator can still applied.
- If there are still more than one stack entries at the end of the processing of an OCRON, then we append virtual $'*'$ symbols, as many as needed (i.e. multiplications with a virtual 2s) from left, until the stack has only one entry (the final result).

Using these rules, non-well-formed OCRONs can be transformed into normal OCRONs. Any OCRON (even if non-well-formed) is thereby given an unambiguous value. Conversely, normal OCRONs can be shortened by discarding the leading $'2'$, which is always present, and removing all '*'s at the end, which would have eventually reduced the stack to a single value.
This has the advantage that, at the end of the OCRON processing, as many stack entries remain as factors are present, unless our number is a power number (number which can be written as a power). The number of prime factors can also be 'extracted' from the OCRONs without having to go to the meaning level of 'numbers'. We can remain on the formal typographic OCRON level of meaning without explicit decoding.
This means in practice that we simply remove the last $' * '$s (if the end of the OCRON consists only of successive $' * 's$, so the stack remains unchanged). These 'erased' multiplications can be added again afterwards (see above, 'virtual' $' * '$s), so that again a well-formed OCRON arises.

Let us denote these non-well-formed OCRONs together with the set of well-formed OCRONs 'EOCRONs' (= Enhanced OCRONs). Either types may be converted into the other. In order to make a well-formed OCRON from a non-well-formed EOCRON, it must always be enlarged (to the left or to the right).

We distinguish three types of OCRONs: (well-formed OCRONs), standardized EOCRONs, any EOCRONs.

Properties or transformation rules of (well-formed) OCRON4s

- **An OCRON4 consists of an arbitrarily long string of OCRON4 symbols (2, P, ^, *) which, when interpreted, yield a value.**

- **OCRON4s may be concatenated (that is, linked together, and finally appended by a $'*'$). This concatenation is associative and commutative and corresponds to a multiplication on the number significance level.**

- **Each OCRON4 (= well-formed) begins with a '2' and ends with a '*', '^' or 'P'. The only OCRON4 that can end with a '2' is '2' itself.**

- **Except for the concatenation rule, there are at present no further important rules known (for example, addition rule, or transformation rules).**

- **The number 1 has no representation by type 4 OCRONs**

- **The length of a type 4 OCRON typically increases proportional to the value of the corresponding number.**

- **The difference between the maximum length and the minimum length of OCRONs resulting in a set of degenerate OCRONs (i.e. having the same numerical value) can be arbitrarily large.**

Properties or transformation rules of (any) EOCRON4s

- **An EOCRON4 consists of an arbitrarily long string of OCRON4 symbols (2, P, ^, *) that can be arranged arbitrarily.**

- **For a non-well-formed EOCRON4 to be interpreted, it must be converted into a well-formed OCRON4, resulting in an enlargement.**

- **There is an empty EOCRON4: by converting to an OCRON4 this is the '2'.**

- **There are EOCRONs, which result in the same numerical value as the standard EOCRONs when interpreted, but are shorter.**

Properties or transformation rules of (standardized) EOCRON4s

- **A standardized EOCRON4 consists of an arbitrarily long string of OCRON4 symbols (2, P, ^, *).**

- **For a standardized EOCRON4 to be interpreted, it must be converted into a well-formed OCRON4 (possibly by inserting '2' symbols at the beginning and adding '*' symbols at the end). This results in an enlargement.**

- **There is an empty standardized EOCRON4: by converting to an OCRON4, this is the '2'.**

- **The number of prime factors of an EOCRON4 is simply the number of stack entries after interpretation of all standardized EOCRON symbols (with insertion of the leading, ‘2’ before conversion into a well-formed OCRON).**

- **Each standardized EOCRON4 (= well-formed) ends with a '^' or 'P'. It can never end with a '*' or '2'.**
- **Standardized EOCRONs can be concatenated (corresponding to a multiplication), but are then no longer standardized.**
- **The difference between the maximum length and the minimum length of EOCRONs which result in a set of degenerate EOCRONs (i.e. having the same numerical value) can become arbitrarily large.**

The last property suggests that any transformation rules between degenerate OCRONs or EOCRONs are not trivial at all!

Here is a type 4 EOCRON table (generated with the software: "kmatrix[54], red: ‘minimal EOCRONs blue background: well formed):

Table 19. The first 100 type 4 EOCRONs (in ascending order)

type 4 EOCRON	Value (n)	type 4EOCRON	Value (n)
*	4	^*2	16
P	3	^*^	256
2	4	^P*	14
^	4	^PP	17
P*	6	^P2	14
PP	5	^P^	128
P2	6	^2*	8
P^	8	^2P	12
2*	4	^22	16
2P	6	^2^	16
22	8	^^*	32
2^	4	^^P	53
^*	8	^^2	32
^P	7	^^^	65536
^2	8	P***	24
^^	16	P**P	37
P**	12	P**2	24
P*P	13	P**^	4096
P*2	12	P*P*	26
P*^	64	P*PP	41
PP*	10	P*P2	26
PPP	11	P*P^	8192
PP2	10	P*2*	12
PP^	32	P*2P	18
P2*	6	P*22	24
P2P	9	P*2^	36
P22	12	P*^*	128
P2^	9	P*^P	311
P^*	16	P*^2	128
P^P	19	P*^^	18446744073709551616
P^2	16	PP**	20
P^^	256	PP*P	29
2**	8	PP*2	20
2*P	7	PP*^	1024

54 Kmatrix: http://kmatrix.eu

2*2	8	PPP*	22
2*^	16	PPPP	31
2P*	6	PPP2	22
2PP	10	PPP^	2048
2P2	12	PP2*	10
2P^	8	PP2P	15
22*	8	PP22	20
22P	12	PP2^	25
222	16	PP^*	64
22^	8	PP^P	131
2^*	8	PP^2	64
2^P	7	PP^^	4294967296
2^2	8	P2**	12
2^^	16	P2*P	13
^**	16	P2*2	12
^*P	19	P2*^	64

10.2.2.2 DEGENERATION OF TYPE 4 OCRONS

Degeneration was defined in 10.2.1.1. The degree of degeneration depends strongly on the composition of the number, i.e. how many prime factors it contains. 'Highly' composite numbers have a high OCRON degeneration, whereas primes often have a small degeneration. Some primes have a degeneration value of 1 (and thus no degeneration). Here is a small table of degeneration values of the first 100 type 4 OCRONs:

Table 20. Degeneration values of the first 100 type 4 OCRONs

n	Degener.	n	Degener.	n	Degener.	n	Degener.
1	0	26	4	51	4	76	40
2	1	27	5	52	16	77	4
3	1	28	16	53	18	78	24
4	2	29	2	54	34	79	2
5	1	30	12	55	2	80	156
6	2	31	1	56	68	81	18
7	2	32	57	57	10	82	4
8	5	33	2	58	4	83	2
9	2	34	4	59	2	84	144
10	2	35	4	60	72	85	4
11	1	36	46	61	8	86	8
12	8	37	8	62	2	87	4
13	2	38	10	63	16	88	34
14	4	39	4	64	220	89	34
15	2	40	34	65	4	90	72
16	18	41	2	66	12	91	8
17	2	42	24	67	5	92	16
18	8	43	4	68	16	93	2
19	5	44	8	69	4	94	4
20	8	45	8	70	24	95	10
21	4	46	4	71	8	96	714
22	2	47	2	72	244	97	2
23	2	48	156	73	4	98	28

24	34	49	6	74	16	99	8
25	2	50	8	75	8	100	46

A table of type 4 OCRONs of the first 25 natural numbers, including degenerate OCRONs as well as other tables about OCRONs, can be found in the Appendix.

The degeneracy grows very fast with n (albeit not as fast as with type 3 OCRONs), as the following graphic shows:

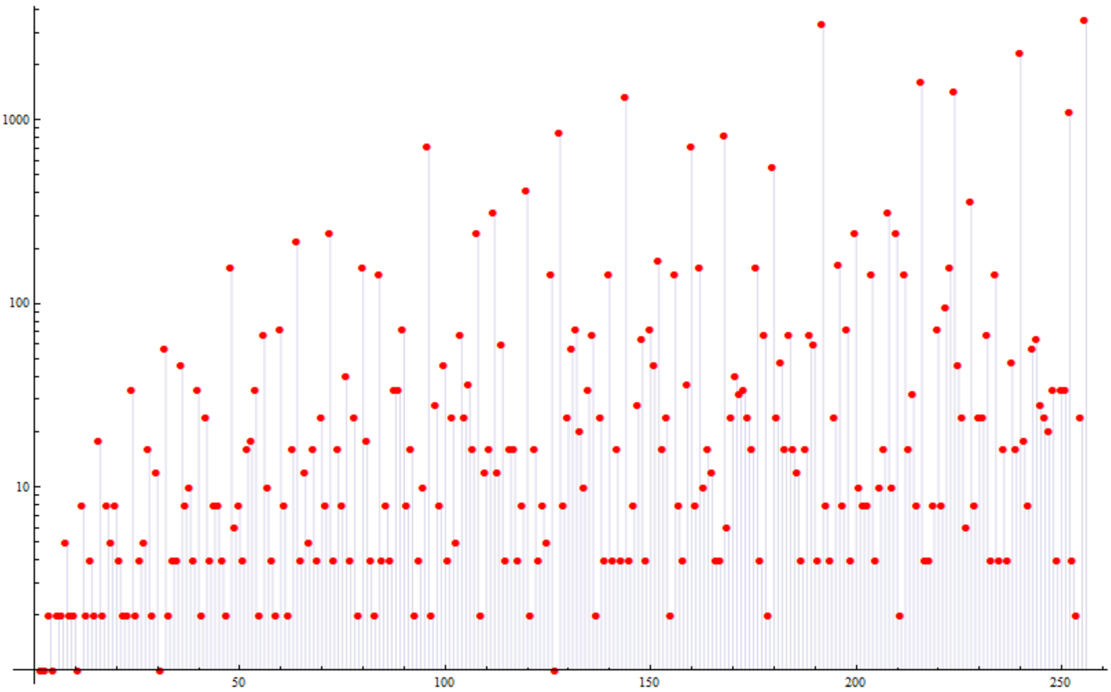


Figure 100. Degeneration of well formed OCRON4s up to n=256 (logarithmic plot)

```
Mathematica:
data =
Import["primes/data/ocron4_wellformed_Degeneration_OK_upto_256.txt","C
SV"]
ListLogPlot[data,PlotStyle->Red,AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic,PlotRange->All]
```

10.2.2.3 STANDARDIZATION OF TYPE 4 OCRONS AND EOCRONS

Because of the high degree of degeneration of these OCRON types, we want to pick out from the many possible (E)OCRON representations the so-called ‘standard type’, which corresponds to the following OCRON rules:

- The standardized form should correspond to the prime factor decomposition (that is, each prime number may only occur once for a decomposition together with its exponent).

- Whenever products appear, the rule of ascending sorting (first the small factors, then the large factors) applies.
- ‘ ^′ has a higher priority than the ′ ∗ ′ operator; that is, whenever possible, we take the ‘ ^’ operator instead of the ′ ∗ ′ operator (for example, ‘ 22 ^′ instead of ‘ 22 ∗’).
- The standardization should result in a reduced, minimal form (as a minimum EOCRON), in which the prime factor assignment can simply be read off from the stack values.

Before OCRONs are converted to EOCRONs, they should be converted into the standardized form.

Note: the transformation of any type 4 OCRON into standardized type 4 OCRONs only at the symbol level (without evaluation as a number) is a difficult and unsolved problem!

10.2.2.4 THE GÖDELIZATION OF TYPE 4 OCRONS

By 'Gödelization' we mean a change in the level of meaning from a procedural point of view (each OCRON represents a small computer program by means of which its value can be calculated) into a static ‘value-defined’ interpretation. We assign a positive, integer value n to each OCRON (which initially consists only of a chain of formal symbols). This approach was originally invented by the mathematician Kurt Gödel, who succeeded in proving his famous ‘incompleteness theorem’ with this method.

This assignment is arbitrary and can be done in various ways. However, it is common to all methods that the so-called ‘Gödel code’ is assigned to each OCRON symbol (e.g. the ‘^’ symbol gets the value of 3). The entire OCRON chain then results (when interpreted by arithmetical processing of the individual OCRON symbols) in a total numerical value. Here, we also have complete freedom with respect to the choice of a suitable algorithm which combines the individual Gödel codes of the OCRON symbols into a total value. We want to limit this freedom of choice by demanding that the resulting Gödel values should become as small as possible, so that we can examine their possible arithmetic laws as easily as possible. In the Gödelization used by Gödel himself, astronomically high values arise that are useless for further arithmetic investigation.

More information can be found in Hofstadter's book 'Gödel-Escher-Bach' (Hofstadter, 1991/1985)[55]. The change of the level of meaning from formal symbols ('typographic') into the world of numbers is amazing (we quote Hofstadter from his famous book):

“Stepping out of one purely typographical system into another isomorphic typographical system is not a very exciting thing to do; whereas stepping out of the typographical domain into an isomorphic part of number theory has some kind of unexplored potential. It is as if somebody has known musical scores all his life, -but purely visually- and then, all of a sudden, someone introduced him to the mapping between sounds a musical scores. What a rich, new world! Then again, it is as if somebody has been familiar with string

[55] p. 271: The Boomerang: Gödel-Numbering TNT

figures all his life, but purely as string figures devoid of meaning - and then, all of a sudden, someone introduced him the mapping between stories and strings. What a revelation! The discovery of Gödel numbering has been likened to the discovery, by Descartes, of the isomorphism between curves in a plane and equations in two variables; incredibly simple, once you see it - and opening onto a vast new world."

Here are a few conceivable possibilities of 'Gödelizations':

The simple prime number Gödelization[56]

For this purpose, for each OCRON $\boldsymbol{o}$ of length l, we need the first l prime numbers, e.g. to 'Gödelize' the string '22^P2*P' (of length 7), we need the first 7 prime numbers $\boldsymbol{P_n} =$ 2,3,5,7,11,13,17, as well as the Gödel codes $\boldsymbol{gc}(\mathbf{symbol})$ for the OCRON symbols (e.g. 1 for ' $*$ ', 2 for $'2'$, 3 for $'\mathrm{P}'$and 4 for $'\text{^}'$).

The Gödel codes must have integer values > 0. The value 0 is not allowed. The total value is then obtained by multiplying the factors $\boldsymbol{P_n}^{\boldsymbol{gc}(\mathbf{symbol})}$ by one another (where n runs up to the OCRON length $l-1$):

Example: the OCRON '22^P2*P' (corresponding to a value of 43) has

- in the first position, the value $\boldsymbol{P_1}^{\boldsymbol{gc}(\mathbf{2})} = \mathbf{2^2} = \mathbf{4}$
- in the second position, the value $\boldsymbol{P_2}^{\boldsymbol{gc}(\mathbf{2})} = \mathbf{3^2} = \mathbf{9}$
- in the third position, the value$\boldsymbol{P_3}^{\boldsymbol{gc}(\wedge)} = \mathbf{5^4} = \mathbf{625}$
- in the fourth position, the value $\boldsymbol{P_4}^{\boldsymbol{gc}(\boldsymbol{P})} = \mathbf{7^3} = \mathbf{343}$
- in the fifth position, the value $\boldsymbol{P_5}^{\boldsymbol{gc}(\mathbf{2})} = \mathbf{11^2} = \mathbf{121}$
- in the sixth position, the value $\boldsymbol{P_6}^{\boldsymbol{gc}(*)} = \mathbf{13^1} = \mathbf{13}$
- in the seventh position, the value $\boldsymbol{P_7}^{\boldsymbol{gc}(\boldsymbol{P})} = \mathbf{17^3} = \mathbf{4913}$

The total value $g(o)$ (Gödel number of 22^P2*P) results in:

$$\boldsymbol{g}(\boldsymbol{o}) = \boldsymbol{g}(\mathbf{22\text{^}P2*P}) = \mathbf{4} * \mathbf{9} * \mathbf{625} * \mathbf{343} * \mathbf{121} * \mathbf{13} * \mathbf{4913} = \mathbf{59641989907500}$$

or generally (with l=length of the OCRON)

$$g(o) = \prod_{n=1}^{l} P_n^{\,gc(\mathrm{OCRON}[n-1])} \tag{151}$$

From the prime factor decomposition of 59641989907500, it is possible to reconstruct the OCRON '22 ^ P2 * P'', and finally the original number 43.

[56] https://en.wikipedia.org/wiki/Gödel_numbering

The advantage of this method is the small number of 'degrees of freedom' (here 4 for the choice of the Gödel codes of the OCRONs) and the independence in the representation of any possible numeral system (for example, decimal system or binary system). The disadvantage is clear: we get unwieldy large numbers for the Gödel numbers, which are also difficult to decode.

The main drawback, however, is that there are a lot of numbers that do not correspond to any Gödel number (and therefore cannot be converted into an OCRON), namely all numbers having a prime factor decomposition that is not in the complete order of the first numbers of n prime numbers, or numbers whose prime factor decomposition contain a prime power that is greater than all the occurring Gödel codes of our OCRONs. For example $\mathbf{32} = \mathbf{2^5}$ would not be a valid Gödel code.

The simple prime number Gödelization represents an injective mapping of the set of OCRONs onto the set of positive natural numbers $\mathbb{N}_+$.

The differential prime number Gödelization

In this method, we need more than the first l prime numbers, where l is the OCRON length, as well as the fixed Gödel codes 1, 2, 3 and 4 for the symbols '*', '2', 'P' (note that the assignment is arbitrary, so that we have here also $4! = 24$ possible code assignments). With which algorithm is it now possible to construct from an arbitrarily long OCRON chain (composed of the symbols, '*', '2', 'P', '^' with the corresponding Gödel codes (e.g. 1, 2, 3, 4)) a unique Gödel numbering?

By treating a Gödel code as an offset of indices in the prime number table. For the above example, we therefore get:

The OCRON '22^P2*P' has (using a slightly different Gödel code assignment $* -> 0, P -> 1, 2 -> 2, ^\wedge -> 3$):

- in the first position, the value $P_{0+gc(2)} = P_2 = 3$
- in the second position, the value $P_{2+gc(2)} = P_4 = 7$
- in the third position, the value $P_{4+gc(^\wedge)} = P_7 = 17$
- in the fourth position, the value $P_{7+gc(P)} = P_8 = 19$
- in the fifth position, the value $P_{8+gc(2)} = P_{10} = 29$
- in the sixth position, the value $P_{10+gc(*)} = P_{10} = 29$
- in the seventh position, the value $P_{10+gc(P)} = P_{11} = 31$

The total value $g(o)$ (Gödel number of 22^P2*P) finally results in:

$$\boldsymbol{g(o) = g(\text{22^P2*P}) = 3 * 7 * 17 * 19 * 29^2 * 31 = 176839593}$$

This looks quite a bit better, but this method still has the disadvantages described in the last method.

Gödelization by using numeral systems

Here, we simply replace the symbols of the OCRON chains with the respective Gödel codes, and receive e.g. from ‘22^P2*P’ the Gödel number, 2231201 which is best represented in the base 4 system: 2231201_4. This coding is easy to perform in both directions (OCRON->Gödel number and Gödel number->OCRON) and has a great advantage: it is bijective, unambiguous in both directions, i.e. there is a definite EOCRON for any given Gödel number. We denote here explicitly EOCRON, since the conversion often results in non-well-formed OCRONs (= EOCRONs), which can, however, easily be transformed into well-formed ones, by the method described in 10.2.2.1. **Using this Gödelization method, we now have a tool to transform arbitrary numbers with the help of this ‘Gödel transformation’ into a Gödel number (in which somehow the construction principle of this number is hidden). A transformation that leads us into another world of numbers, which involves a change in the fundamental meaning of the numbers!**

Here are a few tables to give an idea of the abstract descriptions (with o(n) = OCRON(n) and g(n) = g(OCRON(n))=Gödel number:

Properties of EGOCRONs

Note that the ‘EGOCRONs’ are almost always larger than the numbers from which they originate and are always odd.

Table 21. Type 4 EOCRONs (standard representation) and Gödel numbers from 2 to 100. Gödel codes:(‘ *’=0,’P’=1,’2’=2,’^’=3)

N	o(n)	g(n)	N	o(n)	g(n)
2	(Leer)	(-)	51	P22^PP	1717
3	P	1	52	2^22P*P	11921
4	2^	11	53	22^^P	701
5	PP	5	54	2P2P^	615
6	2P	9	55	PP2PPP	1429
7	2^P	45	56	2P^22^P	10157
8	2P^	39	57	P22P^P	1693
9	P2^	27	58	22PP*P	2641
10	2PP	37	59	2^PPP	725
11	PPP	21	60	2^2P2PP	11877
12	2^2P	185	61	2P2^*P	2481
13	2P*P	145	62	2PPPP	597
14	22^P	173	63	P2^22^P	7085
15	P2PP	101	64	22P*^	659
16	22^^	175	65	PP22P*P	5777
17	2^PP	181	66	2P2PPP	2453
18	2P2^	155	67	2P^PP	629
19	2P^P	157	68	2^22^PP	11957
20	2^2PP	741	69	P2P2^P	1645
21	P22^P	429	70	2PP22^P	9645
22	2PPP	149	71	2^2PP*P	11857
23	P2^P	109	72	2P^2P2^	10139
24	2P^2P	633	73	P22^P*P	6865
25	PP2^	91	74	22^2P*P	11153
26	22P*P	657	75	P2PP2^	1627
27	P2P^	103	76	2^22P^P	11933
28	2^22^P	2989	77	2^P2PPP	11669

29	2PP*P	593	78	2P22P*P	9873
30	2P2PP	613	79	2PPP*P	2385
31	PPPP	85	80	22^^2PP	11237
32	2PP^	151	81	P22^^	431
33	P2PPP	405	82	22P*PP	2629
34	22^PP	693	83	P2^PP	437
35	PP22^P	1453	84	2^2P22^P	47533
36	2^2P2^	2971	85	PP22^PP	5813
37	2^2P*P	2961	86	222^P*P	10961
38	22P^P	669	87	P22PP*P	6737
39	P22P*P	1681	88	2P^2PPP	10133
40	2P^2PP	2533	89	2P^2P*P	10129
41	2P*PP	581	90	2P2^2PP	9957
42	2P22^P	2477	91	2^P22P*P	46737
43	22^P*P	2769	92	2^2P2^P	11885
44	2^2PPP	2965	93	P2PPPP	1621
45	P2^2PP	1765	94	2P2PP*P	9809
46	2P2^P	621	95	PP22P^P	5789
47	P2PP*P	1617	96	2PP^2P	2425
48	22^^2P	2809	97	PP2^P	365
49	2^P2^	731	98	22^P2^	2779
50	2PP2^	603	99	P2^2PPP	7061

Properties of inverse EGOCRONs

Note that the inverse numbers of the Gödel numbers (inverse EGOCRONs) are almost always smaller than the original Gödel numbers from which they originate.

Table 22. Type 4 EOCRONs (inverse Gödelization from g=0 to 99)

Gödel number g	EOCRON for g	n	Gödel number g	EOCRON for g	n
0	*	4	50	^*2	16
1	P	3	51	^*^	256
2	2	4	52	^P*	14
3	^	4	53	^PP	17
4	P*	6	54	^P2	14
5	PP	5	55	^P^	128
6	P2	6	56	^2*	8
7	P^	8	57	^2P	12
8	2*	4	58	^22	16
9	2P	6	59	^2^	16
10	22	8	60	^^*	32
11	2^	4	61	^^P	53
12	^*	8	62	^^2	32
13	^P	7	63	^^^	65536
14	^2	8	64	P***	24
15	^^	16	65	P**P	37
16	P**	12	66	P**2	24
17	P*P	13	67	P**^	4096
18	P*2	12	68	P*P*	26
19	P*^	64	69	P*PP	41
20	PP*	10	70	P*P2	26
21	PPP	11	71	P*P^	8192
22	PP2	10	72	P*2*	12
23	PP^	32	73	P*2P	18

24	P2*	6	74	P*22	24
25	P2P	9	75	P*2^	36
26	P22	12	76	P*^*	128
27	P2^	9	77	P*^P	311
28	P^*	16	78	P*^2	128
29	P^P	19	79	P*^^	18446744073709551616
30	P^2	16	80	PP**	20
31	P^^	256	81	PP*P	29
32	2**	8	82	PP*2	20
33	2*P	7	83	PP*^	1024
34	2*2	8	84	PPP*	22
35	2*^	16	85	PPPP	31
36	2P*	6	86	PPP2	22
37	2PP	10	87	PPP^	2048
38	2P2	12	88	PP2*	10
39	2P^	8	89	PP2P	15
40	22*	8	90	PP22	20
41	22P	12	91	PP2^	25
42	222	16	92	PP^*	64
43	22^	8	93	PP^P	131
44	2^*	8	94	PP^2	64
45	2^P	7	95	PP^^	4294967296
46	2^2	8	96	P2**	12
47	2^^	16	97	P2*P	13
48	^**	16	98	P2*2	12
49	^*P	19	99	P2*^	64

One may wonder what the frequencies of the ‘*’, ’P’, ’2’ and ‘^’ symbols are. A statistical evaluation of the EOCRONs of the type 4 for the natural numbers from 3 to 10000 yields the following frequencies:

Total number of symbols: 123952
of which P symbols: 52664 (42, 487 %)
of which 2 symbols: 42794 (34, 525 %)
of which ^ symbols: 16711 (13, 482 %)
of which * symbols: 11783 (9, 506 %)

The following illustration shows a logarithmic representation of the Gödel numbers for the first 5,000 natural numbers using the Gödel codes ‘*’ = 0, ‘P’ = 1, ‘2’ = 2, and ‘^’ = 3. In the Gödelization, the base 4 numeral system was used. One can clearly see a striated structure.

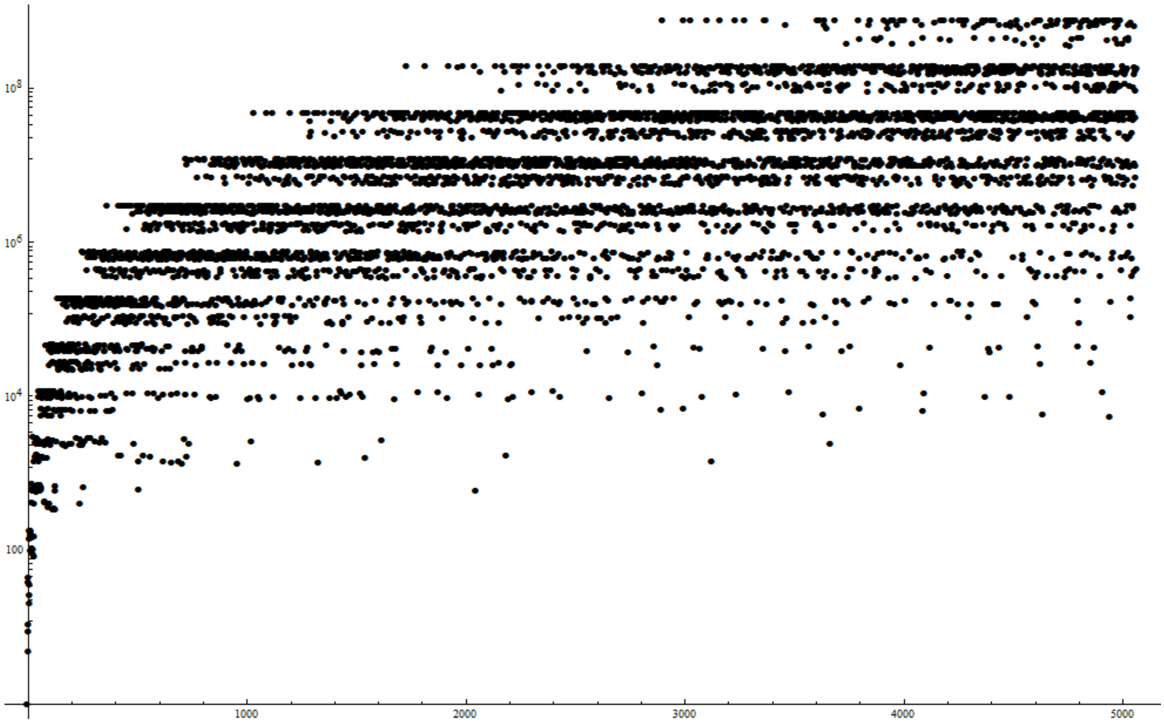


Figure 101. Logarithmic representation of the Gödel numbers for the first 5000 natural numbers by using the Gödel codes ' *'=0, ' P'=1, ' 2'=2, and also ' ^'=3

```
Mathematica:
data=Import["primes/data/EGOCRONsTyp8_3.txt",{"Data",All,{1}}];
ListLogPlot[{data,{All}{1}},PlotStyle->Black,PlotMarkers-
>Automatic,AxesLabel->Automatic,PlotRange->All,ImageSize->Large]
```

And here in the region from 1 to 200:

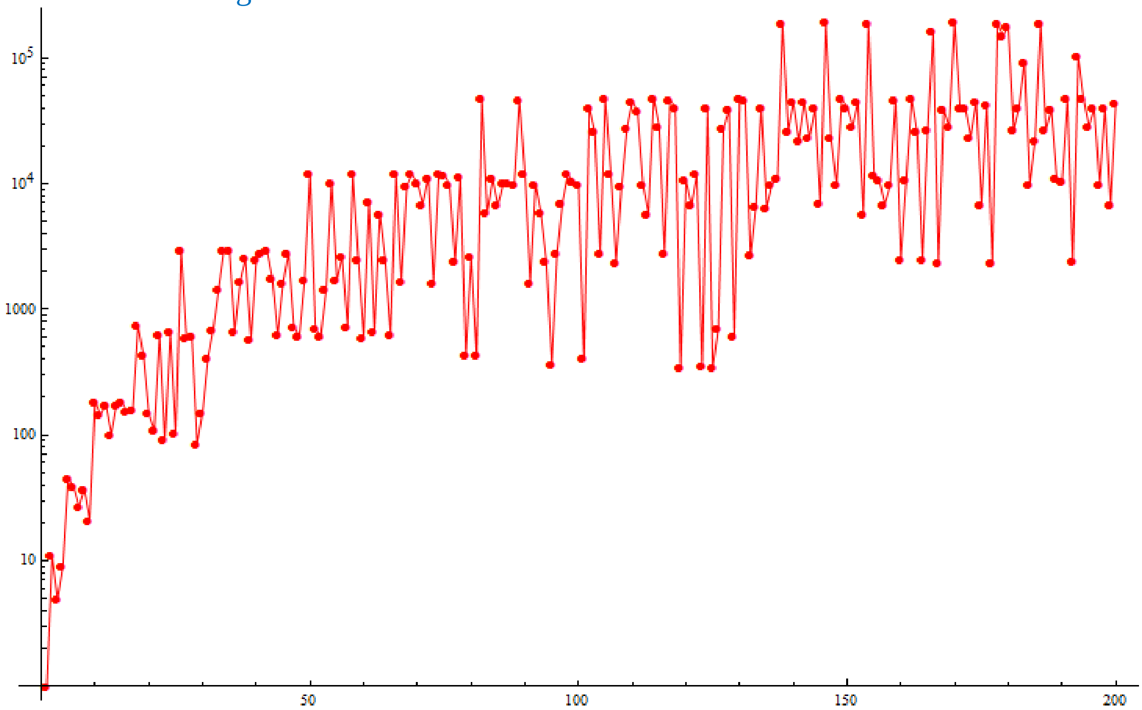


Figure 102. Logarithmic representation of the Gödel numbers for the first 200 natural numbers by using the Gödel codes ' *'=0, ' P'=1, ' 2'=2, and also ' ^'=3

10.2.3 OCRONS WITH THE PRIME “P”, “*”, “^” AND “Q” OPERATORS

The Q operator replaces multiple consecutive ‘P’ operators by the sequence <n>Q, where n is represented in the corresponding OCRON coding and contains the number of successive P's. All other operators are identical to the OCRON type 4. We call this type OCRON type 5.

10.2.3.1 DEGENERATION OF TYPE 5 OCRONS

Degeneration was defined in 10.2.1.1. The degeneration increases very quickly with n as shown in the following graph:

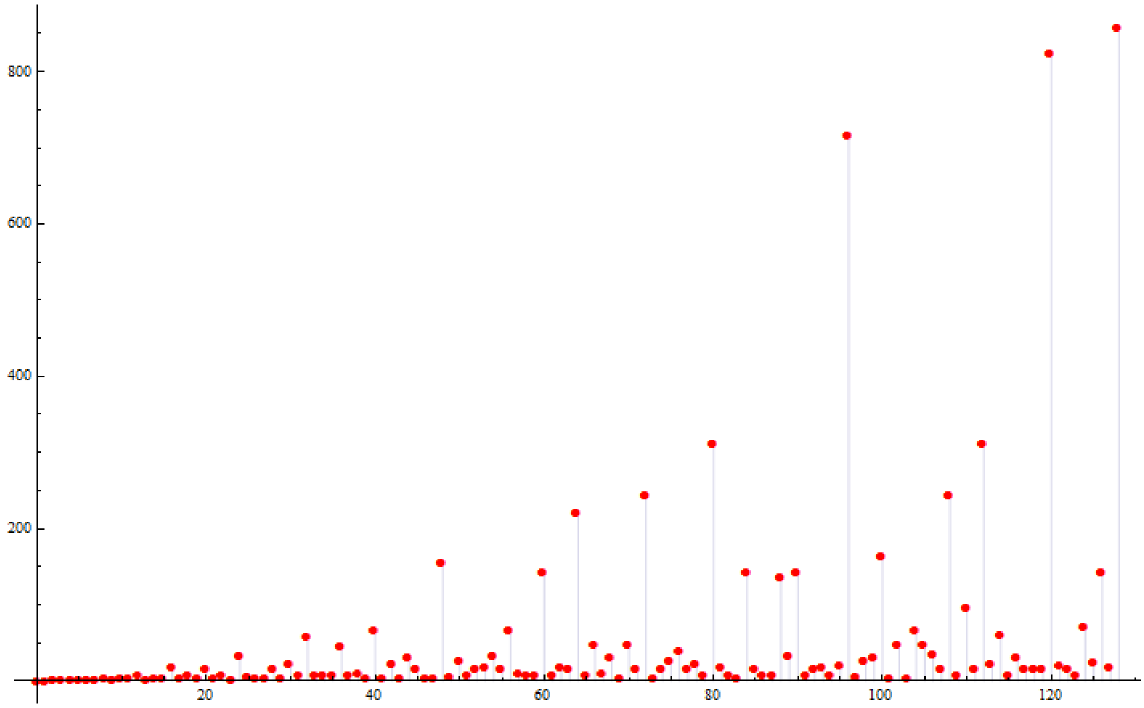


Figure 103. Degeneration of well-formed OCRONs of type 5 up to n=128

```
Mathematica:
data =
Import["primes/data/ocron5_wellformed_Degeneration_OK_upto_128.txt","C
SV"]
ListPlot[data,PlotStyle->Red,AxesLabel->Automatic,Filling-
>Axis,PlotMarkers->Automatic,PlotRange->All]
```

10.2.4 OCRONS WITH PRIME AND NON-PRIME OPERATORS

This type of OCRON has only two operators: the prime operator ‘P’, and the non prime operator, which for the sake of simplicity we may also refer to by ‘$*$’ (not to be confused with the multiplication operator used by the OCRON types 3 to 5. The interpretation of

the operators is the same as for the type 3, type 4 and type 5 operators: let n be the current numeric value, which is always in the lowest stack drawer. The 'P' operator calculates the nth prime number and thus overwrites the stack value. The '$*$ 'operator calculates the nth non prime number (=composite number) and thus also overwrites the lowest stack value.
In this way, any number $n \geq 1$ can be written as a sequence of '$*$' and 'P' operators. Note that there is no multiplication operator or power operator any more! The new set of OCRONs is given the type 6. For the calculation of the OCRONs, it is only important that the stack be 'pre'-occupied by the value 1, so that the value 1 will be the result for the '*' OCRON (first not prime number) and 'P' has the value 2 (first prime number). The zero has no correspondence in the '$P\,*$' OCRON coding.
Prime OCRONs have a number of very interesting and remarkable properties:

1) The '$P\,*$' representation is unique (bijective), i.e. for each number there is a unique '$P\,*$' representation and vice versa! Using '$P\,*$' OCRON representation, the set of natural numbers can be rearranged, and in a unique way.
2) A direct consequence is that there is no more degeneration and the corresponding Gödel numbers remain manageably small.

To illustrate here the first 100 '$P\,*$' OCRONs, together with their Gödel numbers, based on the Gödel codes $* = \ 0$ and $P \ = \ 1$; for the Gödel number GN (g) we also use the term 'GOCRON' (= Gödelized OCRON).

Table 23. Prime OCRONs (P and * operator) with Gödel numbers (GCodes P=1, *=0)

N	G=OCRON6(N)	GN(g)	N	G=OCRON6(N)	GN(g)
0	(-)	(-)	50	PPP*****	224
1	*	0	51	PP******	192
2	P	1	52	P*P*P*	42
3	PP	3	53	PP***P	49
4	P*	2	54	P*PP***	88
5	PPP	7	55	PPPP****	240
6	PP*	6	56	P**P***	72
7	P*P	5	57	PP*PP*	54
8	P**	4	58	PP**P**	100
9	PPP*	14	59	P*PPP	23
10	PP**	12	60	P***P*	34
11	PPPP	15	61	PPPP*P	61
12	P*P*	10	62	P*P*****	160
13	PP*P	13	63	PPPPP**	124
14	P***	8	64	PP*P****	208
15	PPP**	28	65	PPP**P*	114
16	PP***	24	66	P*******	128
17	P*PP	11	67	P**PP	19
18	PPPP*	30	68	PPP*P***	232
19	P**P	9	69	PPP******	448
20	P*P**	20	70	PP*******	384
21	PP*P*	26	71	P*P**P	41
22	P****	16	72	P*P*P**	84
23	PPP*P	29	73	PP*P*P	53
24	PPP***	56	74	PP***P*	98
25	PP****	48	75	P*PP****	176
26	P*PP*	22	76	PPPP*****	480
27	PPPP**	60	77	P**P****	144

28	P**P*	18	78	PP*PP**	108
29	PP**P	25	79	P****P	33
30	P*P***	40	80	PP**P***	200
31	PPPPP	31	81	P*PPP*	46
32	PP*P**	52	82	P***P**	68
33	P*****	32	83	PPP*PP	59
34	PPP*P*	58	84	PPPP*P*	122
35	PPP****	112	85	P*P******	320
36	PP*****	96	86	PPPPP***	248
37	P*P*P	21	87	PP*P*****	416
38	P*PP**	44	88	PPP**P**	228
39	PPPP***	120	89	PPP***P	113
40	P**P**	36	90	P********	256
41	PP*PP	27	91	P**PP*	38
42	PP**P*	50	92	PPP*P****	464
43	P***P	17	93	PPP******	896
44	P*P****	80	94	PP********	768
45	PPPPP*	62	95	P*P**P*	82
46	PP*P***	104	96	P*P*P***	168
47	PPP**P	57	97	PP****P	97
48	P******	64	98	PP*P*P*	106
49	PPP*P**	116	99	PP***P**	196

```
Mathematica (calculation n->GOCRON(n):
Please contact the author.
```

Note that in the binary representation of the Gödel numbers GN(g), prime numbers always end with a '1' digit and composite numbers with a '0' digit! Or in the decimal notation: prime numbers always have an odd Gödel number, composite numbers always have an even Gödel number! The resulting sequence of Gödel numbers is not quite unknown; it can be found on the Internet at https://OEIS.org (A071574 and A237739)[57].

Here is a logarithmic plot of the prime GOCRONs:

57 https://oeis.org/A071574

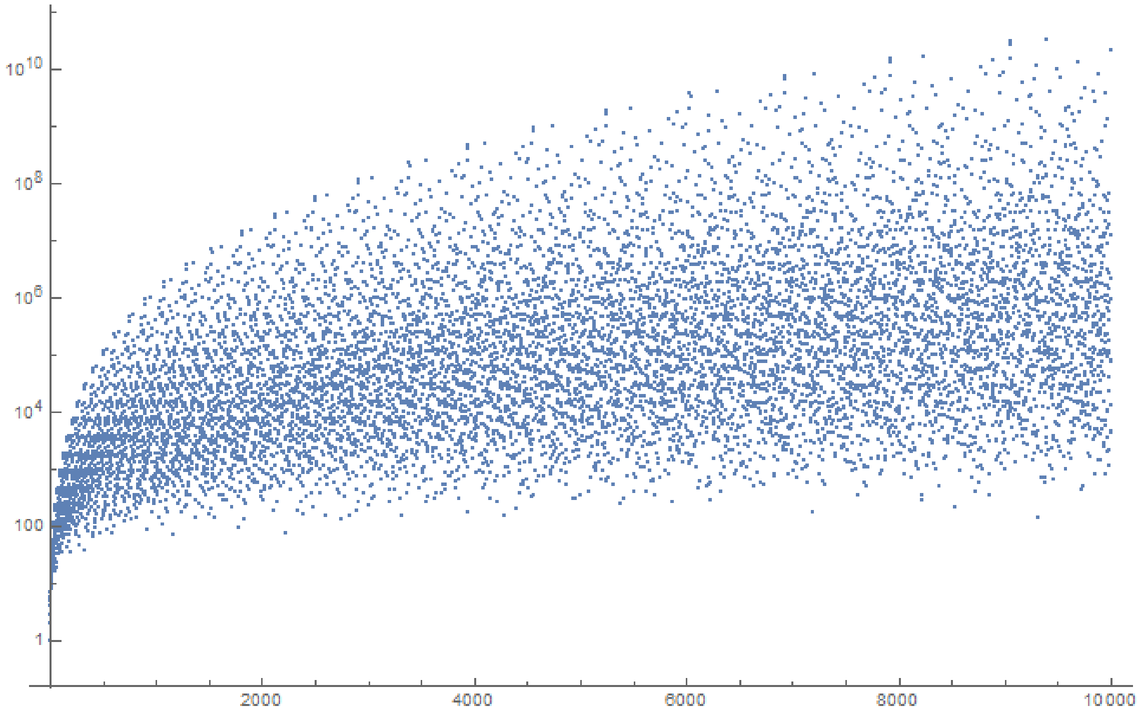


Figure 104. Prime GOCRONs of type 6 (n->GOCRON[n]) from 1 to 10000

The following table constitutes the inverse of Table 23.

Table 24. Prime GOCRONs, OCRONs and the corresponding inverse numbers from 0 to 99

GOCRON	OCRON	N	GOCRON	OCRON	N
0	*	1	50	PP**P*	42
1	P	2	51	PP**PP	109
2	P*	4	52	PP*P**	32
3	PP	3	53	PP*P*P	73
4	P**	8	54	PP*PP*	57
5	P*P	7	55	PP*PPP	179
6	PP*	6	56	PPP***	24
7	PPP	5	57	PPP**P	47
8	P***	14	58	PPP*P*	34
9	P**P	19	59	PPP*PP	83
10	P*P*	12	60	PPPP**	27
11	P*PP	17	61	PPPP*P	61
12	PP**	10	62	PPPPP*	45
13	PP*P	13	63	PPPPPP	127
14	PPP*	9	64	P******	48
15	PPPP	11	65	P*****P	137
16	P****	22	66	P****P*	106
17	P***P	43	67	P****PP	401
18	P**P*	28	68	P***P**	82
19	P**PP	67	69	P***P*P	281
20	P*P**	20	70	P***PP*	244
21	P*P*P	37	71	P***PPP	1153
22	P*PP*	26	72	P**P***	56
23	P*PPP	59	73	P**P**P	173
24	PP***	16	74	P**P*P*	141
25	PP**P	29	75	P**P*PP	587
26	PP*P*	21	76	P**PP**	121
27	PP*PP	41	77	P**PP*P	467
28	PPP**	15	78	P**PPP*	411
29	PPP*P	23	79	P**PPPP	2221
30	PPPP*	18	80	P*P****	44
31	PPPPP	31	81	P*P***P	113
32	P*****	33	82	P*P**P*	95
33	P****P	79	83	P*P**PP	353
34	P***P*	60	84	P*P*P**	72
35	P***PP	191	85	P*P*P*P	239
36	P**P**	40	86	P*P*PP*	203
37	P**P*P	107	87	P*P*PPP	919
38	P**PP*	91	88	P*PP***	54
39	P**PPP	331	89	P*PP**P	163
40	P*P***	30	90	P*PP*P*	133
41	P*P**P	71	91	P*PP*PP	547
42	P*P*P*	52	92	P*PPP**	110
43	P*P*PP	157	93	P*PPP*P	419
44	P*PP**	38	94	P*PPPP*	345
45	P*PP*P	101	95	P*PPPPP	1787
46	P*PPP*	81	96	PP*****	36
47	P*PPPP	277	97	PP****P	97
48	PP****	25	98	PP***P*	74
49	PP***P	53	99	PP***PP	241

```
Mathematica program (calculation GOCRON->n (inverse): please contact
the author.
```

Here is a logarithmic plot of the inverse prime GOCRONs (of type 6):

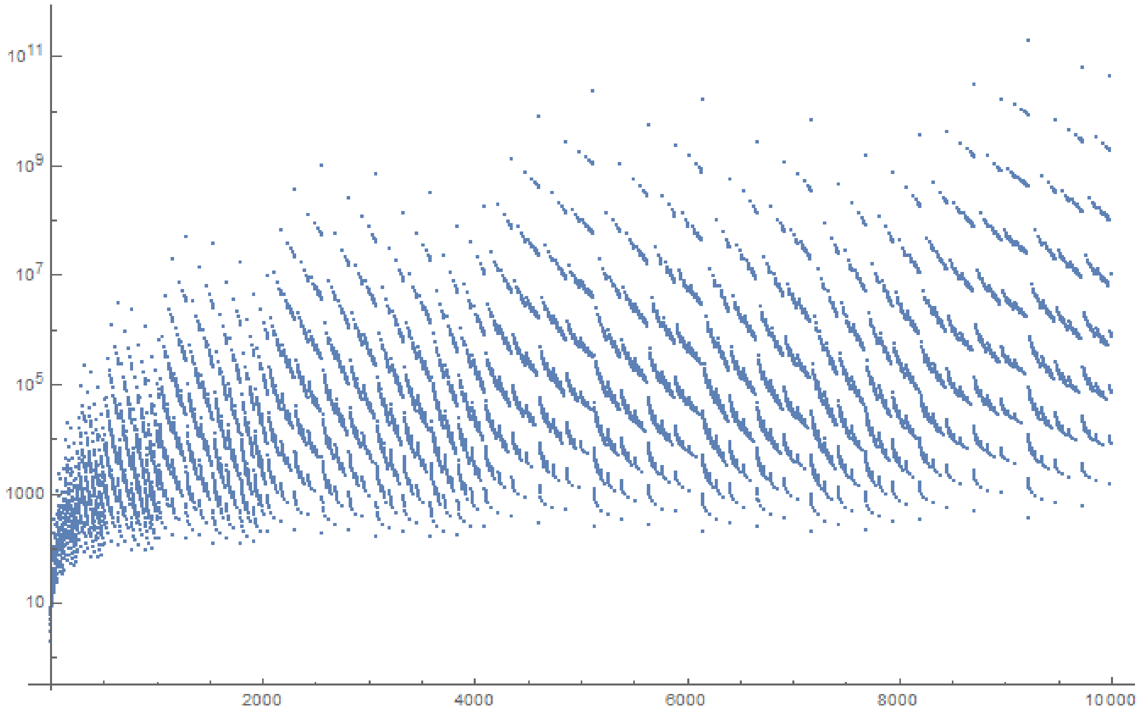


Figure 105. Inverse prime GOCRONs of type 6 (GOCRON->n) from 1 to 10000

10.3 THE WORLD OF OCRON BEINGS AND MATHEMATICAL DYNAMITE

Note: in order to understand this chapter properly, the reader should at least be a bit familiar with the type 4 OCRONs (see Chapter 10.2.2).
We want to make an excursion into the world of OCRON beings. This is a thought experiment, since we do not know for certain whether this world exists together with its inhabitants, which we will call 'OCRONians'. Thought experiments are a successful tool in science to make complex relations clearer. It is known that Einstein had the decisive idea for his general theory of relativity by using thought experiments. One of these thought experiments was that he imagined himself to be in a huge falling elevator that is large enough to hold a whole physics laboratory with all possible measuring instruments, and which would not have any contact with the environment outside. He compared this situation with a closed room (also having all possible measuring instruments and devices and also without contact to the outside world) that is moving with constant velocity through space far away from any planets or other space objects (today in the space age this is no longer hard to imagine). A physicist who is in the first or in the second room can perform all the measurements and experiments he wants. All types of measuring instruments are available in the two rooms. However, the physicist cannot determine, solely by means of measurements performed within the respective room, whether it is situated in a falling elevator (it may also be said to be within reach of a gravitational field), or in a space ship moving with constant velocity far away in space.
A similar thought experiment would be to put the first closed room, together with the measuring instruments simply on the surface of the earth, to place the other room on the top of a rocket that has turned its engines on, and which moves with a constant acceleration through the universe. Again, a physicist cannot determine solely by measurements performed within the respective rooms in which situation he is.
The logical conclusion was that the two respective situations do not only appear to be identical, but are actually identical. A gangway can be constructed between the respective situations using appropriate mathematical tools. In the case of the general theory of relativity, it was the idea of a 'curved spacetime' that finally produced the equivalence of the two situations.

Our thought experiment leads us into another world, a world that is so completely different from ours that we can hardly imagine it. The cosmologists often speak of other worlds. There is the concept of a multiverse, which includes many or even infinitely many universes of a certain kind. The universe, in which we want to go now originated in a multiverse, which has a name: in his book 'Our Mathematical Universe', Max Tegmark[58] calls it the ‘level IV' multiverse. Tegmark speaks of different parallel universes that form a four-stage hierarchy, each multiverse being a single element among many of the other ones existing one level higher. According to his theory, level I and II universes emerged physically after the Big Bang in the so-called ‘inflationary phase’. In level I, however, each universe has the same physical natural laws and natural constants, the same mathematics, however different initial conditions. In level II each universe has the same natural laws and the same mathematics but different natural constants and different particles. This type of multiverse can also have higher spatial dimension. The level III multiverse corresponds to the level II multiverse, but it consists of infinitely many individual

[58] Max Tegmark: *Our mathematical universe*, Ullstein Buchverlage GmbH, Berlin

universes that continuously split up by generating new universes, always when anybody perceives something (or someone else). It takes account of the quantum mechanical nature of our world.
By "perceiving" is meant the most general form of perception, including the following situations: 'looking' or 'viewing', a physicist who measures a physical quantity, but also any completely abstract interaction between a complex quantum mechanical system and another.
At an atomic and subatomic scale, all physical processes are calculated by quantum mechanics and by a wave function that describes the process spatial and temporal. It only has the disadvantage that all potentially measurable quantities exist as an infinite superposition of all possible discrete states. At least, as long as no measurement is made – that is, as long as nobody 'looks' at the system, since in the case of a measurement the quantum mechanical system has to decide for one of these infinitely many possible states. The physicists call this the collapse of the wave function following the 'Copenhagen interpretation' and are still not happy about it. One conceivable alternative that avoids this collapse of the wave function is the 'many worlds' hypothesis of quantum mechanics, which states that our entire universe splits into several universes, depending on which process is considered (caused by a measurement).
This leads to the level III multiverse. From a philosophical point of view it can be said that in such a multiverse everything occurs (all possible events happen, in any of the infinite many plane III universes), which can occur at any time.
In the level VI multiverse, the restriction of the uniform equations of physics also fails. Each universe contains its own set of mathematical structures. Many of these type VI universes will be uninteresting, but many will be complex and powerful enough to create their own worlds within this level VI universe.

Why do we digress so far? In order to show that the world in which we want to go is so unimaginably different to our own world, since it is a level VI world (after Max Tegmark) in which there is a completely different mathematics to that of our world.

Now we finally come to our OCRON beings. Let us suppose that the world of mathematics in this world is versatile and powerful enough to create a world with a similar complexity and diversity to ours. There exists also life in this world: the inhabitants are the OCRONians mentioned above.
They live there in communities, are intelligent, and they also pursue science, including mathematics, out of curiosity. They can also construct machines, computers and other devices for which they need mathematics as a tool. Their own mathematics is fundamentally different from our mathematics.
The OCRONians cannot add, they can only multiply and raise to the power (and in some mysterious way also calculate logarithms to the base 2). They also do not calculate using numbers to which a unique value can be assigned, but they only calculate using OCRONs: the four different formal symbols '$*$', '2', '^', and 'P'. They can not say how large a number is, the terms 'size' or value of a number (in our sense) do not exist in their world.
The concept of addition is alien to them, not just alien, for it simply does not exist in their world. They can multiply huge numbers effortlessly (in their world of course OCRONs) from childhood onward. Not even prime factors are a problem for them: they look at a number and can see in a fraction of a second whether it is a prime

number, or of which prime factors it is composed.
Their computers also work without addition, since numbers are not stored in them as sum totals, but only as OCRON representations (which ultimately result in a product of prime factors). For programming and the unavoidable calculations that must occur when constructing machines, however, they must also be able to compare numbers and can determine if two numbers have the same ‘size’.
And they may also have to carry out operations of the kind that we call ‘addition’ in our world (which actually can be done in their world, but only in a very roundabout way). Although the term ‘value’ of a number (of an OCRON) does not exist in their world, they also have an ability to determine without a concept of ‘size’ or ‘value’ which is the larger of two numbers, or whether they are of the same size. For this purpose they consult the ‘MATHOracle’. (MATHOracle consultation, see below).
They can also ask the MATHOracle for a second operation, which they use in their computations: the ORACLELog operation (in our world, this is called the logarithm to base 2).
The OCRONians have to perform 'additions' (this term is known only to us) becau

se of their physical equations, but they do not know that we call this process in our world 'addition' and that we have a much simpler method of doing so. They use a rather complicated method for this: to add two OCRONs o_1 and o_2, they write (we call the result o_3):

o_3=ORACLELog[$2o_1$^$2o_2$^ $*$].

In our mathematical language this reads as follows: $o_3 = \mathrm{Log}_2(2^{o_1} \cdot 2^{o_2})$.

The OCRONians call the oracle logarithm the 'ORACLELog' symbol. 'ORACLELog' returns either an OCRON or nothing. Together with the function 'ORACLEValue' (which can only supply one of the three values 'smaller', 'greater' or 'equal', these are the two mathematical operations for which the OCRONians can query the MATHOracle).

The methods 'ORACLEValue' and 'ORACLELog' are not really understood by the OCRONians, but this method is intuitively familiar to all OCRONians. They can interrogate the 'MATHOracle' and it will always give them the right answer to these questions within a fraction of a second. Each OCRONian has access to the MATHOracle from any location and at any time in a mysterious way. The OCRONian computers also enjoy such access. The mathematicians among the OCRONians now state that there can exist quite different OCRONs, which provide the same result in the MATHOracle consultation with 'ORACLEValue'.
By probing (‘trial and error’) and ORACLEValue consultations, they discover all possible OCRONs that give the same value.
The cleverest mathematicians among the OCRONians have therefore wrestled for many years with the question of whether there is a method based on an algorithm, instead of the random 'try-out' in the search for 'equivalent' OCRONs, with which 'equivalent' OCRONs could be transformed into each other (and thereby, for example, simplified). This would have made the work of the OCRONian engineers much easier, since they would have found a quick method for the conversion of OCRONs

instead of random testing, and they would not need to consult the MATHOracle (at least for their complicated 'addition') with 'ORACLEValue'.
Clever OCRONian mathematicians also found that the consultation with ORACLELog would also be invalid, provided that they had an algorithm that could produce all the equivalent OCRONs by reforming.
For the discovery of such an algorithm, a prize was offered in the OCRONian world.

Here is an example: multiplication of $8 * 4$ looks in our world like: $8 * 4 = 32$.

In the world of the OCRONians: 22P^ times 22^ gives 22P^22^ $*$ (OCRONians multiply by simply concatenation of OCRONs and appending a $'*'$).
The OCRONians are now able to determine by random testing and ORACLEValue interrogations that, for example, 22P ^ 22 ^ * has the same value as 22PP ^.
However, they have no algorithm that produces the equivalent OCRON 22PP ^.

In our world, mathematicians and logicians speak of a (typographic) formal system. OCRONians have to work hard with symbols to solve simple things like additions by randomly ‘rolling dice’. For them the access into the 'higher' logical world in which addition exists is denied!
Poor OCRONians! How does the story continue? Will the smartest OCRONians succeed in finding such an algorithm?
Let us think of two possible (fictional) scenarios of the story.

Scenario 1: an OCRONian mathematician finds an algorithm for transforming equivalent OCRONs into each other. He is celebrated and receives the OCRONian ‘Fields Medal’. The MATHOracle has no longer to be consulted for the 'additions'. Indeed, the solution is complicated (for the calculation, solutions have to be found with the aid of complicated recurrence rules and rules that in turn invented new rules), but it is still a method to bypass the MATHOracle consultation. In addition, the old method associated with the 'ORACLEValue' consultation always involves boring random probing, with the result that the complicated method with recursive rules for long OCRONs works unbeatably better than the random method.

Scenario 2: a clever OCRONian named ‘Gocroedel’ finds a proof that the axiomatic system of OCRONian mathematics is simply too 'weak' and too lacking in 'power' to solve the problem of the transformation. He claims that the statement: ‘22P^22^*’ is equivalent to ‘22PP^’ is indeed true but that it cannot be proved with OCRONian mathematics. This implies that no such algorithm can be found. For this, Gocroedel also receives the OCRONian ‘Fields Medal’, but the OCRONians cannot really look forward to it.

The attentive reader has probably realized what ‘explosive’ is hidden in scenario 1:

If Scenario 1 were true, then we could learn from the OCRONians (above referred to as 'poor'): we could adopt their 'transformation algorithm' and would have a quick method for the factorization of numbers: we would simply have to separate the number to be factored into a sum of two numbers whose

prime factorization we know (more precisely, whose OCRON representations we know). Then we prepend a '2' symbol at the beginning of the respective OCRONs, append an '^' symbol at the end of both OCRONs, concatenate both the new OCRONs (OCRON multiplication) and append a '∗' symbol at the end. If the second summand is 1, then the thing is even simpler: the second OCRON is simply the symbol '2'. Finally, we transform the concatenated OCRON into an equivalent OCRON with the help of the mysterious algorithm, so that at the end of the OCRON a '^' symbol is placed, then easily get the logarithm to the base 2 (by discarding the leading '2' and the last symbol '^'), and *voilà!*
We have a product representation (which is implicit in every OCRON) of the number to be factored! If the resulting OCRON does not have a '∗' or '^' symbol at the end, but a '*P*' symbol, then the number to be factored is a prime number. We would have solved the factorizing problem on a pure typographical level by applying typographic transformation rules.

Here a few examples:

We examine the number $\mathbf{37}$. Additive composition: $\mathbf{37 = 36 + 1}$:
In OCRON notation:
$36 = 22\text{^}2P2\text{^}*$ -> (brackets inserted for the sake of clarity)
$(\mathbf{222\text{^}2P2\text{^}*\text{^}})(\mathbf{2})*$->(MATHOracle consultation) $\mathbf{222\text{^}2P*P\text{^}}$
(logarithm: discard the 2 and ^)-> $\mathbf{22\text{^}2P*P}$, prime number!
Thus, we have shown that 37 is a prime number, only by dealing with OCRONs.

We examine the number $\mathbf{143}$. Composition: $\mathbf{143 = 71 + 72}$:
In OCRON notation:
$71 = 22\text{^}2PP*P, 72 = 22P\text{^}2P2\text{^}*$ ->
$(\mathbf{222\text{^}2PP*P\text{^}})(\mathbf{2\ 22P\text{^}2P2\text{^}*\text{^}})*$->(MATHOracle consultation)
$\mathbf{22PPP22P*P*\text{^}}$
(logarithm: discard the 2 and ^)-> $2PPP22P*P* = (2PPP)(22P*P)*$
Result: factors $\mathbf{2PPP}$ $(= \mathbf{11})$ and $\mathbf{22P*P}$ $(= \mathbf{13})$
Thus we have factorized 143 by means of OCRON manipulation into the factors 11 and 13.

Note: the transformation without a MATHOracle consultation is also difficult in our world, because we have to calculate explicitly the value of
$\mathbf{222\text{^}2PP*P\text{^}2\ 22P\text{^}2P2\text{^}*\text{^}*}$:
$11150372599265311570767859136324180752990208$, and then reconvert this value back to an OCRON (which is then transformed).

These examples show that by means of type 4 OCRONs we can factorize numbers by pure, formal typographic manipulation of symbols, provided that we have access to the MATHOracle!

There remains only the 'small' problem, how we can circumvent the MATHOracle consultation and find the fabulous algorithm!
Before we begin searching for this algorithm, we should first determine whether or not the entire problem belongs to the category of 'unprovable' statements. In this case,

it would be utterly impossible to find such an algorithm (at least within OCRONian mathematics).

(Addendum:)

For this, the author has found a truly wonderful algorithm, but the margin is too small to contain it…

-end of thought experiment-

11 PRIME NUMBERS AND THE "MATRIX" SOFTWARE: ARE THERE RULES FOR PRIMES?

11.1 RULES FOR DIFFERENCES OF THE *N*TH ORDER

This chapter examines numbers using the matrix software[59]. For this reason, this software and its functionality should be briefly presented here.
Matrix is an application that makes it possible to create a rule network (i.e. a set of rules) from given data that are somehow arranged causally and that represent a sequence of states of an arbitrary system. This set of rules describes the individual transition probabilities of the system from one state to the next state following.
With the aid of these transitional probabilities, the matrix can produce an arbitrarily long sequence of states of this system. In principle, this is a more general Markov chain with transition probabilities. However, the entire "history" of a process can be included in the calculation of the transition probabilities. In addition, the matrix can also provide 'termination probabilities' (i.e. the probability that a sequence of states ends), as well as the opposite (i.e. the creation of a state 'ex nihilo' at the start of a new state sequence).
In its simplest form, the matrix can also simply be used to store highly dimensionally structured data (hence the name 'matrix'), since it is basically a high-dimensional pointer matrix (with variable dimension length).
The matrix can be applied to almost all systems. The requirements for applicability are very general: the state of the system at a certain starting point must be described by a set of integer (also negative) numbers. There should be enough material about the behaviour of the system. Once the matrix has been fed with data about a system, one can read from the matrix as from an infinite stream. Given a suitable selection of the parameters, this 'stream' will always also provide novel transitions, i.e. reading from the matrix is indeed a creative process. In the matrix itself, no sequences of states are stored; only rules, which are much shorter. When reading from the matrix, the intelligence of the matrix can be adjusted.
Turning on high intelligence, the result will be close to the material with which the matrix was originally fed; with low intelligence, more and more random elements will appear. The result of the output when reading appears much more intelligent than the simplicity of the rules suggests. Somehow the matrix seems to store the knowledge about the behaviour of a system not only locally in the rules but holistically in the totality of all rules. If, for example, you remove part of the rules, the result will not change rapidly.

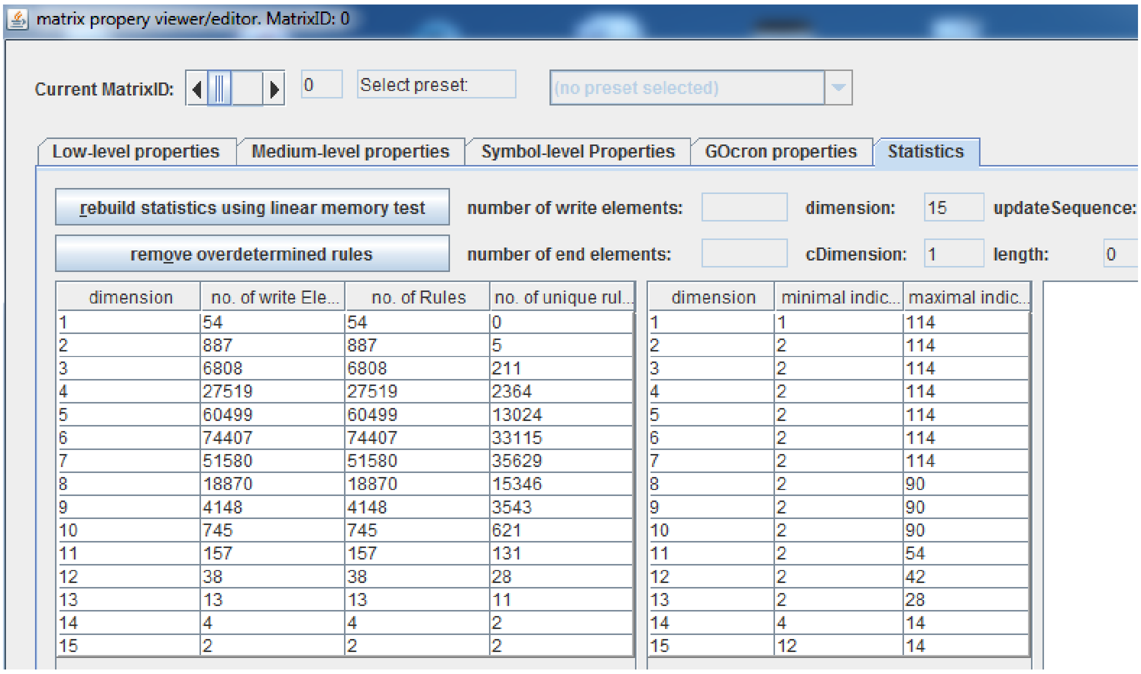
Let's make a first test: we set the maximum rule length to 20 and feed the matrix with the sequence of 1st order differences of the prime number sequence and look at the frequencies of the calculated rule lengths:

For the first 1000 prime numbers (2-7919):
Matrix finds 2581 rules, of which 1093 are unique rules, the maximum of the frequencies is at rule length 5, the longest rule length is 10. The size of the matrix data file is 107 KB.

[59] http://www.kmatrix.eu

For the first 10000 prime numbers (2-104729):
Matrix finds 25092 rules, of which 10590 are unique rules, the maximum of the frequencies is between rule length 5 and 6, the longest rule length is 15. The size of the matrix data file is 1.00 MB.

For the first 100000 prime numbers (2-1299709):
Matrix finds 245731 rules, of which 104032 are unique rules, the maximum of the frequencies is at rule length 6, the longest rule length is 15. The size of the matrix data file is 9.78 MB.



dimension	no. of write Ele...	no. of Rules	no. of unique rul...
1	54	54	0
2	887	887	5
3	6808	6808	211
4	27519	27519	2364
5	60499	60499	13024
6	74407	74407	33115
7	51580	51580	35629
8	18870	18870	15346
9	4148	4148	3543
10	745	745	621
11	157	157	131
12	38	38	28
13	13	13	11
14	4	4	2
15	2	2	2

dimension	minimal indic...	maximal indic...
1	1	114
2	2	114
3	2	114
4	2	114
5	2	114
6	2	114
7	2	114
8	2	90
9	2	90
10	2	90
11	2	54
12	2	42
13	2	28
14	4	14
15	12	14

Figure 106. Matrix: frequency of rule lengths at 1st order difference sequence of the first 100000 prime numbers

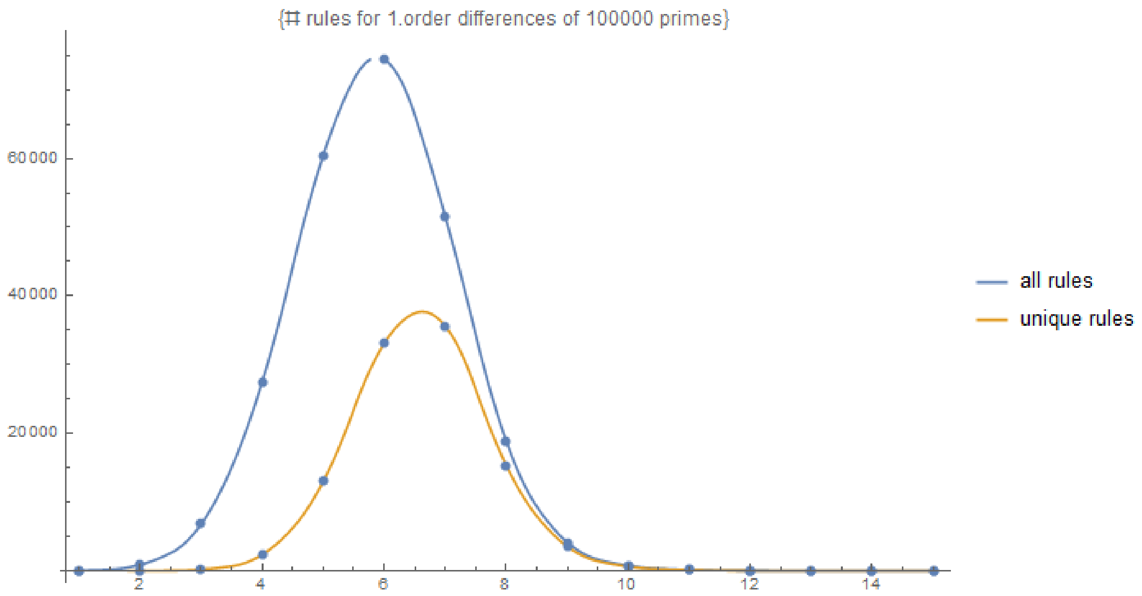
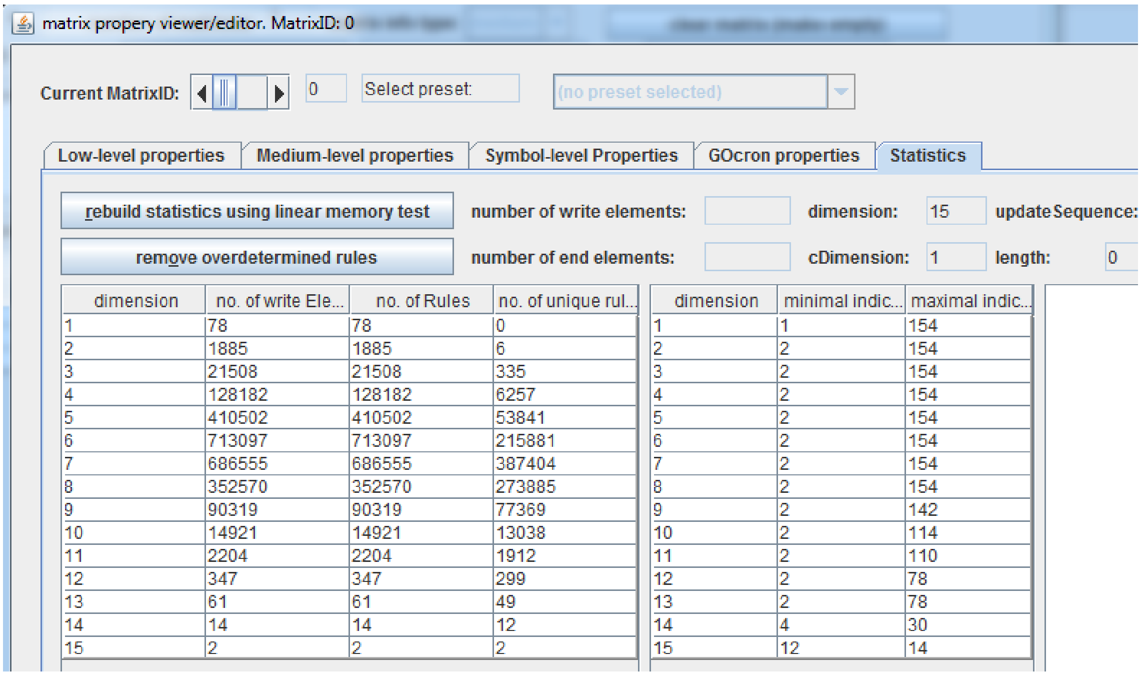


Figure 107. Diagram: frequency of rule lengths for 1st order difference sequence of the first 100000 prime numbers

```
Mathematica:
data=Import["/primes/data_and_Docs/StatisticsPrimesRulesFrom100000PrimesPrepro
c1.txt",{"Data",All,{1,2,3,4}}];
ListLinePlot[{Transpose[data][[2]],Transpose[data][[4]]},AxesLabel-
>Automatic,PlotRange->All,Mesh->Full,InterpolationOrder->2,PlotLegends->{"all
rules","unique rules"},ImageSize->Large]
```

For the first 1000000 prime numbers ($2 - 15485863$):
Matrix finds 2422245 rules, of which 1030290 are unique rules, the maximum of the frequency is between rule length 6 and 7, the longest rule length is 15. The size of the Matrix data file amounts to 96.2 MB.



dimension	no. of write Ele...	no. of Rules	no. of unique rul...
1	78	78	0
2	1885	1885	6
3	21508	21508	335
4	128182	128182	6257
5	410502	410502	53841
6	713097	713097	215881
7	686555	686555	387404
8	352570	352570	273885
9	90319	90319	77369
10	14921	14921	13038
11	2204	2204	1912
12	347	347	299
13	61	61	49
14	14	14	12
15	2	2	2

dimension	minimal indic...	maximal indic...
1	1	154
2	2	154
3	2	154
4	2	154
5	2	154
6	2	154
7	2	154
8	2	154
9	2	142
10	2	114
11	2	110
12	2	78
13	2	78
14	4	30
15	12	14

Figure 108. Matrix: frequency of the rule lengths for 1st order difference sequence of the first 1000000 prime numbers

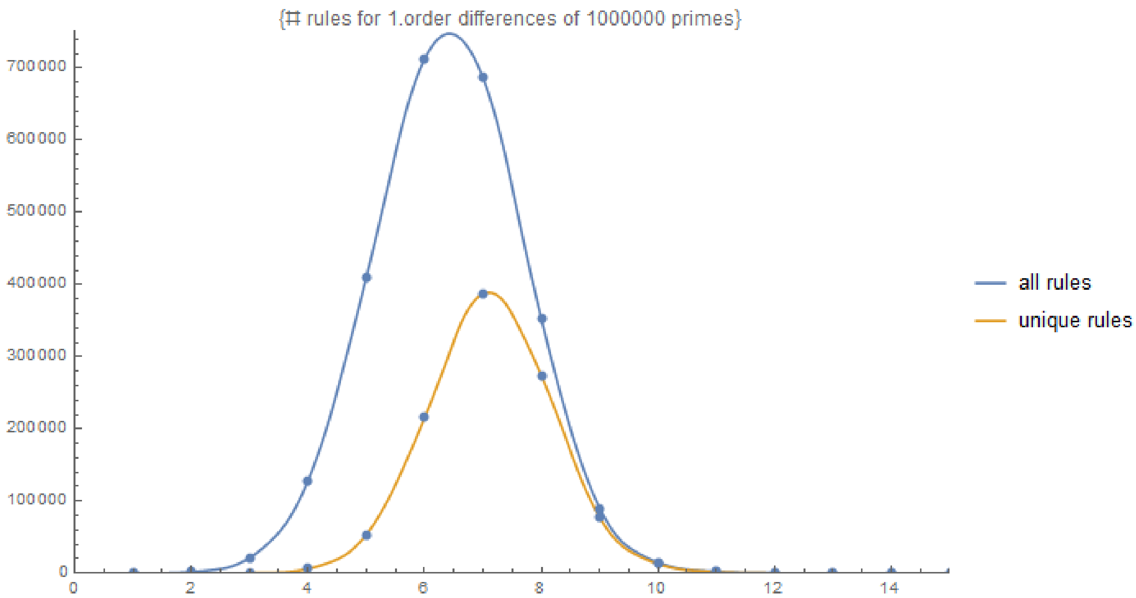


Figure 109. Frequency of rule lengths for 1st order difference sequence of the first 1000000 prime numbers

It looks as though a sequence of 15 consecutive prime numbers is sufficient to compute the 16th succeeding prime number using the matrix rule network.

However, with an increasing number range, the number of rules also increases linearly:

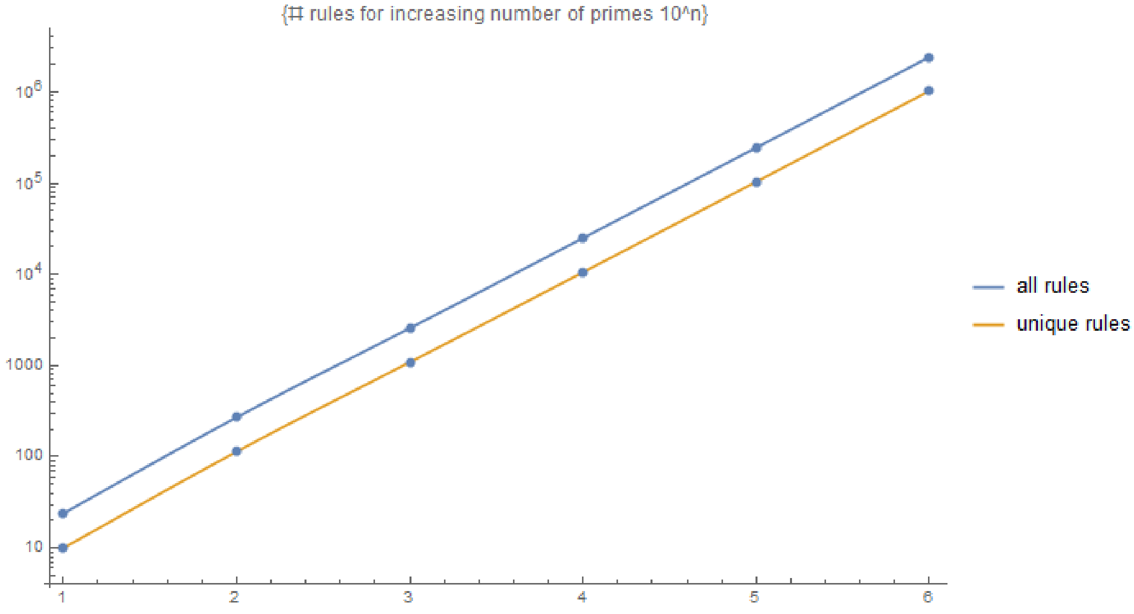


Figure 110. Number of rules calculated from the 1st order differences prime sequence in dependence of the range 10^n

```
Mathematica:
data={{1,24,10},{2,271,114},{3,2581,1093},{4,25092,10590},
{5,245731,104032},{6,2422245,1030290}};
ListLogPlot[{Transpose[data][[2]],Transpose[data][[3]]},AxesLabel-
>Automatic,PlotRange->All,Mesh->Full,Joined->True, InterpolationOrder-
>2,PlotLegends->{"all rules","unique rules"}, PlotLabel->{"# rules for
increasing number of primes 10^n"},ImageSize->Large]
```

For higher order differences, we observe the following behaviour for different orders:

Order of difference	Number of primes	Longest rule	Number of rules	Number of unique rules	Size of file (KB)	Minimum value	Maximal value
1	10	6	24	10			
	100	10	271	114			
	1000	10	2581	1093	107	1	34
	10000	15	25092	10590	1003	1	72
	100000	15	245731	104032	9777	1	114
	1000000	15	2422245	1030290	96206	1	154
2	100	9	254	111			
	1000	9	2506	1074			
	10000	14	24431	10468			
	100000	14	241138	103273			
	1000000	14	2386607	1024963	95478	-148	144
4	100	7	233	101			
	1000	7	2327	1031			
	10000	12	23092	10183			
	100000	12	229102	101283	9064	-332	304
	1000000	12	2282543	1009733	90192	-448	460
8	100	3	189	91			
	1000	4	2247	991			
	10000	8	21252	10002			
	100000	8	211227	100051			
	1000000	8	2225543	1000473	88671	-5962	5638
10	100	3	182	89		-1538	1606
	1000	4	2138	989		-5000	4608

	10000	6	22301	9992		-9488	10238
	100000	6	206803	99993	8512	-14476	15640
	1000000	6	2057953	1000029	85678	-20520	22450
14	100	3	172	85			
	1000	3	1995	985			
	10000	3	20946	9985			
	100000	4	226154	99985	8960	-206992	221570
	1000000	4	2094308	999985	81947	-298794	323414
15	100	2	169	84			
	1000	3	1976	984			
	10000	3	20498	9984	847	-277842	284414
	100000	4	222665	99984	9066	-428562	415348
	1000000	4	2151843	999985	84230	-622208	613240
16	100	2	167	83			
	1000	3	1970	983			
	10000	3	20247	9983			
	100000	3	215514	99983	8843	-843910	790698
	1000000	4	2222938	999984	87189	-1235448	1147684
20	100	2	159	79			
	1000	3	1960	979		-3837740	3855400
	10000	3	19985	9979		-8272220	7732656
	100000	3	2014)90	99979	8655	-12428154	11690554
	1000000	3	2101677	999979	89894	-18210894	17092050

Figure 111. What do the rules for *n*th order difference sequences of prime numbers look like?

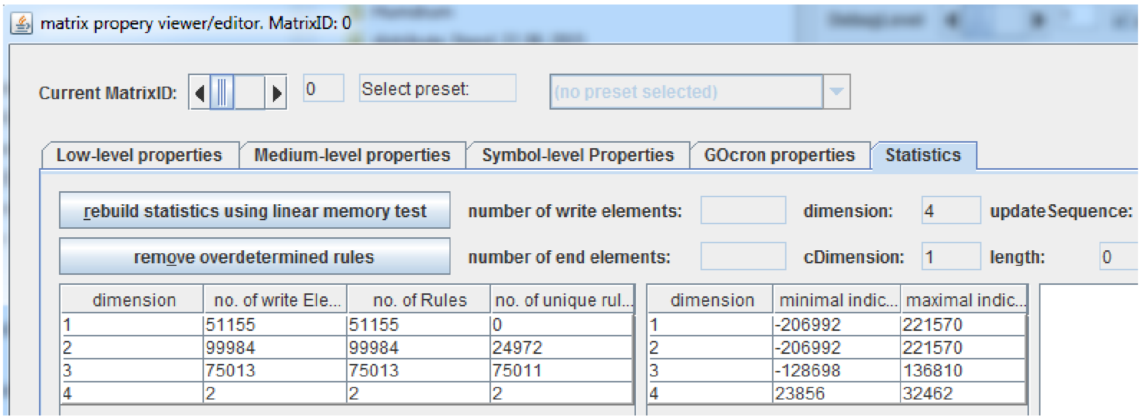


Figure 112. Matrix: frequency of rule lengths for 14. order difference sequences of the first 100000 prime numbers

Also interesting is the dependency of the maximum rule length on the order of the calculated prime difference sequences. The rule length cannot be less than 2 (the 'ex nihilo' rule and the simplest rule that calculates a successor for each value). For high orders of the difference sequences, this value converges to 2, which reflects the fact that each value occurs at most once in the considered difference sequence. This is not surprising and was to be expected.

Here is a diagram describing this dependency in the range of the first 1,000,000 prime numbers (prime difference sequences up to the order 20 were evaluated):

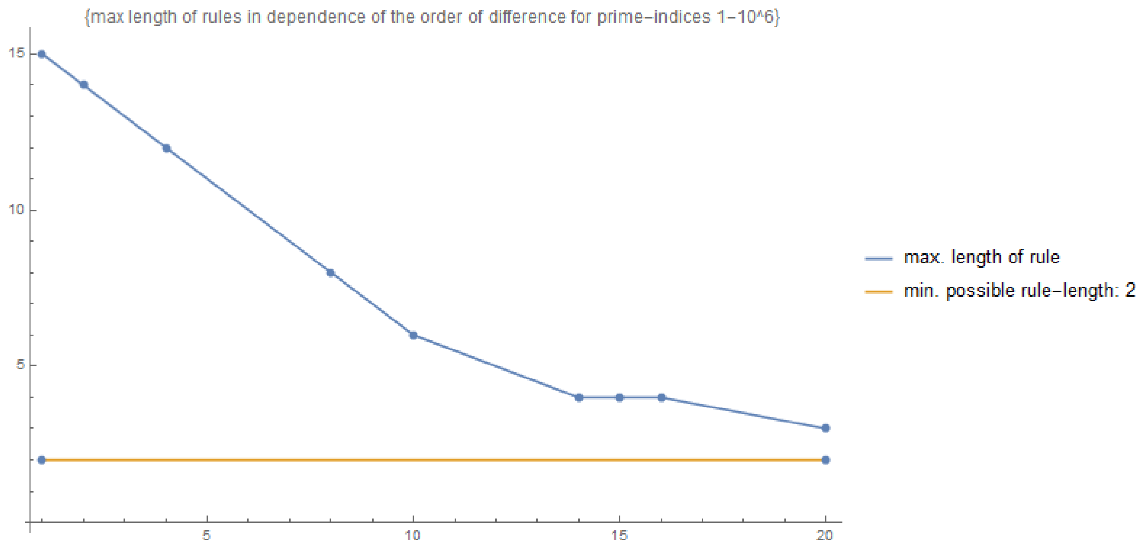


Figure 113. Max. rule lengths in prime difference sequences of order n for the first 10^6 prime numbers

Here are a few statistics for sequences of *n*th order differences.
(Mathematica programs can be found in the Appendix).
The differences of the *n*th order can easily be calculated with Mathematica

Here, for example, the difference sequence of the first 100,000 primes for order 1:

```
range=100000; order=1;
data=Differences[Prime[Range[range]],order];
```

Of the 99999 values, there are actual 54 different values:

```
differents=Union[data]
```

{1,2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48,50,52,54,56,58,60,62,64,66,68,70,72,74,76,78,80,82,84,86,88,90,92,94,96,98,100,106,112,114}

Of the 99999 values, there are 49 values that occur at least twice:

```
doubles=With[{sData=Sort@data},DeleteDuplicates@sData[[SparseArray[Unitize@Differences@sData,Automatic,1]["AdjacencyLists"]]]]
```

{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48,50,52,54,56,58,60,62,64,66,68,70,72,74,76,78,80,82,84,86,88,90,92,96,98,100}

Of the 99999 values, there are 5 values that are unique:

```
{1,94,106,112,114}
```

For the difference sequence of the first 100,000 primes for order 20,
the whole thing looks completely different:

Of the 99999 values, there are 98426 different values:

```
differents=Union[data]
```

{ -12428154,-10525630,-10259274,…, 10253734,11413498,11690554}

Of the 99999 values, there are 1531 values that occur at least twice:

```
doubles=With[{sData=Sort@data},DeleteDuplicates@sData[[SparseArray[Unitize@Differences@sData,Automatic,1]["AdjacencyLists"]]]]
```
{ -5979490,-5554652,-5075372,…, 4158040,5065004,6712100}

Of the 99999 values, there are 96895 values that are unique:
These trends are shown in the following diagrams (the ordered *n*th order differences). Where there are many close-lying values, the curve is flat; where the values occurring are far apart, the curve becomes steep. This is typically the case when the absolute values become large.

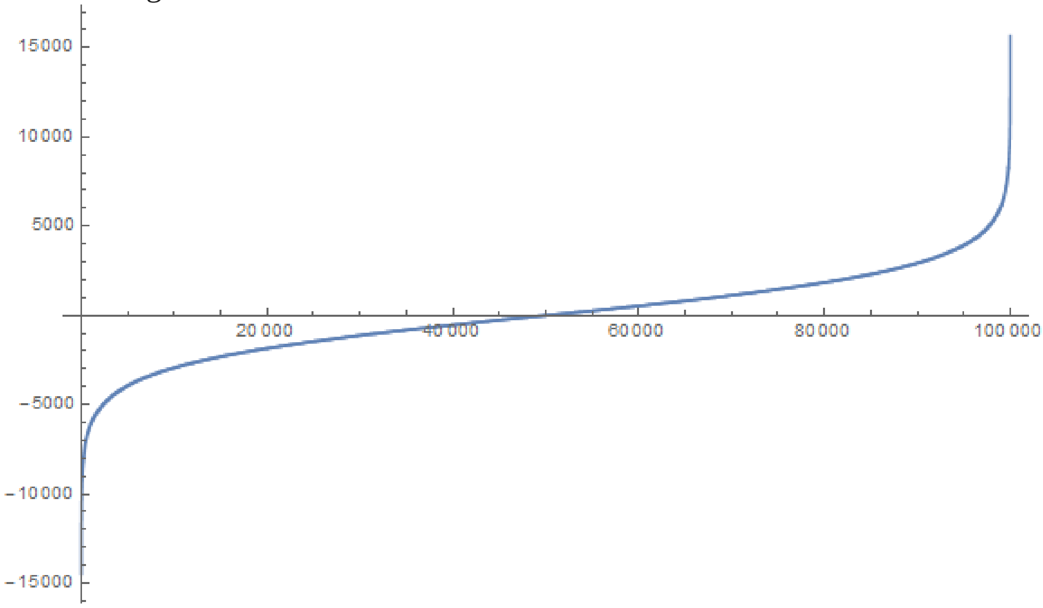


Figure 114. Sorted prime difference values of order 10 of the first 100000 primes

```
Mathematica:
range=100000; data=Sort[Differences[Prime[Range[range]],10]];
ListLinePlot[data,AxesLabel->Automatic,PlotRange->All,InterpolationOrder->0,ImageSize->Large]
```

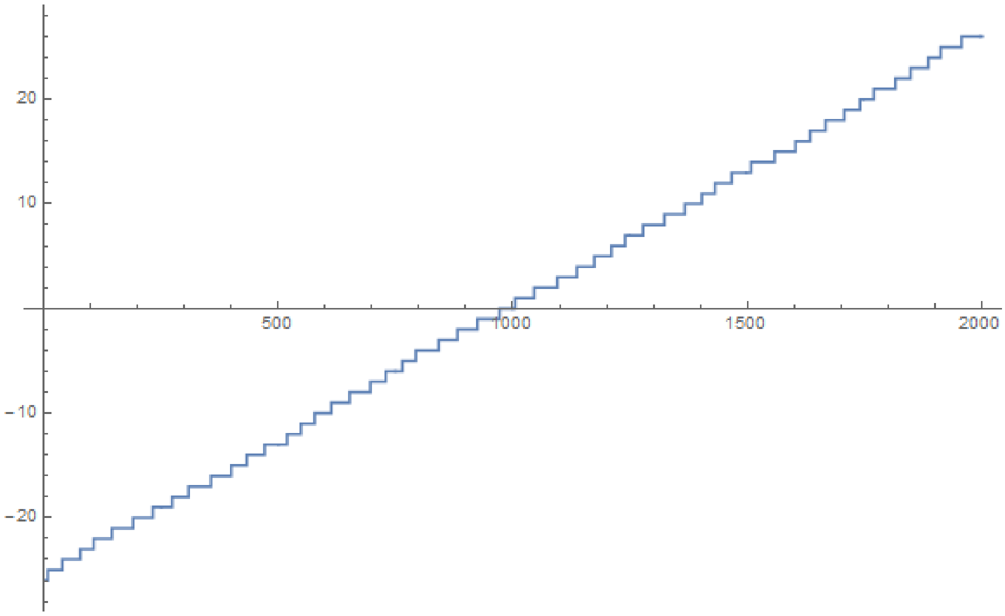


Figure 115. Sorted prime difference values of order 10 in the middle range (2000 values) of the first 100000 prime numbers

```
Mathematica:
range=100000;
data=Sort[Differences[Prime[Range[range]],10]];
ListLinePlot[data[[range/2-1000;;range/2+1000]]/2,AxesLabel-
>Automatic,PlotRange->All,InterpolationOrder->0,ImageSize->Large]
```

It can be seen that the values are dense in the middle region and become thinner in the outer region.

12 THE *ABC* CONJECTURE

12.1 GENERAL

The ‘*abc* conjecture’ is one of the top ten unresolved mathematical conjectures. Many mathematicians are of the opinion that it is presently the most important unsolved problem in number theory. What makes it particularly interesting is that it tries to bring together the two worlds of addition and multiplication. The simplest form of the *abc* conjecture is as follows:

Let $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$ with $a,b,c\in\mathbb{N}$.
Let furthermore a and b be coprime to each other (have no common divisors). For this there are several spellings:
$\boldsymbol{a}\perp\boldsymbol{b}$ or $\mathbf{gcd}(\boldsymbol{a},\boldsymbol{b})=\mathbf{1}$
Due to the addition relation, *a* and *c* as well as *b* and *c* are also mutually coprime. The *abc* conjecture now states that for such additive triplets, the multiplicative structure of the triplets is strongly restricted due to their additive context:
The product of all occurring prime factors, raised to a power which is arbitrarily close to 1: $\left(\mathbf{rad}(\boldsymbol{abc})\right)^{\mathbf{1}+\boldsymbol{\varepsilon}}$ is almost always larger than or equal to the largest number of the triple (c), with $\boldsymbol{\varepsilon}$ being arbitrarily small.
'Almost' in mathematics means: “all but finitely many”.
The so-called strong *abc* conjecture now states that there are only finitely many 'exceptions', so that $\left(\mathbf{rad}(\boldsymbol{abc})\right)^{\mathbf{1}+\boldsymbol{\varepsilon}}<\boldsymbol{c}$, with ε being arbitrarily small.
These exceptions are called *abc* hits. Examples of such *abc* hits are the triples:
{1,8,9}, {5, 27, 32}, {32, 49, 81], etc.
Occasionally, *abc* triples satisfying the slightly ‘softer’ condition $\mathbf{rad}(\boldsymbol{abc})\leq\boldsymbol{c}$ (i.e $\varepsilon=0$) are also referred to as *abc* hits. In this case, however, it has been proved that there are an infinite number of hits.
Depending on how strongly these hits deviate from the prediction of the *abc* conjecture, they are weighted by the value

$$q=\frac{\ln c}{\ln\operatorname{rad}(abc)} \tag{152}$$

This value q is also referred to in the literature as 'quality', 'potency' or 'abc ratio'. It is a measure of the growth of c with the prime content ($\operatorname{rad}(abc)$) of the triple, since q represents the solution of $[\operatorname{rad}(abc)]^q=c$. To date (as of Feb. 2016), only 237 *abc* triples with a 'quality' q> 1.4 have been discovered. There are also other ‘ratings’ of these *abc* hits.[60]
An *abc* hit is called 'unbeaten' (unmatched), if every known *abc* hit with a larger *c* has a smaller quality. The world record (*abc* hit with the highest quality) is (as of Feb. 2016):

$$\{\boldsymbol{abc}\}=\{\mathbf{2},\mathbf{6436341},\mathbf{6436343}\}=\{\mathbf{2},\mathbf{109}\cdot\mathbf{3^{10}},\mathbf{23^5}\},\text{where }\mathbf{rad}(\boldsymbol{abc})=\mathbf{2}\cdot\mathbf{23}\cdot\mathbf{109}=\mathbf{15042}$$

[60] https://de.wikipedia.org/wiki/*abc*-Vermutung#Weitere_Bewertungen_eines_abc-Treffers

If the *abc* hypothesis were to be correct, a whole series of important numerical theorems would follow from its proof (for example, the proof of the famous Fermat hypothesis would be reduced to a few lines).

The Japanese Shinichi Mochizuki published back in 2012 a proof of the *abc* conjecture. The status of his proof within the mathematical community is still undecided. The proof (which runs to 500 pages) is very difficult to understand – even for specialists in this field[61]. The comments of mathematicians on his proof range from 'paper from the future' to 'extraterrestrial'. Here is another mathematically precise formulation of the *abc* conjecture:

For any arbitrarily small $\varepsilon > 0$, there exists a constant C_ε such that for any arbitrary triple of natural numbers a, b, c that are mutually coprime to each other, satisfying the equation $\boldsymbol{a+b=c}$, the following inequality holds (abc conjecture):

$$\mathbf{max}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \leq \boldsymbol{C_\varepsilon} \prod_{\boldsymbol{p|abc}} \boldsymbol{p^{1+\varepsilon}} \tag{153}$$

Note: (as of Feb. 2016) 33.18 million *abc* hits are known. There are only about 5 mathematicians anywhere in the world who can claim to have read the entire proof including all the papers of Mochizuki.[62]

Here are a few graphic representations (created using the following Mathematica program):

```
Mathematica:
rad[n_]:=Times@@First/@FactorInteger[n];
isABC[a_,b_,c_]:=(If[a+b!=c||GCD[a,b]!=1,Return[0]];r=rad[a*b*c];If[r<
c,Return[1],Return[0]]);
isC[c_]:=(For[a=1,a<=Floor[c/2],a++,If[isABC[a,c-
a,c]!=0,Return[1]]];Return[0]);
tab=Select[Range[10000],isC[#]==1&]
ListLinePlot[tab,InterpolationOrder->0,PlotStyle->Black,PlotLabel-
>"abc-conjecture: possible c-values"]
```

[61] http://www.nature.com/news/the-biggest-mystery-in-mathematics-shinichi-mochizuki-and-the-impenetrable-proof-1.18509
[62] https://en.wikipedia.org/wiki/*abc*_conjecture

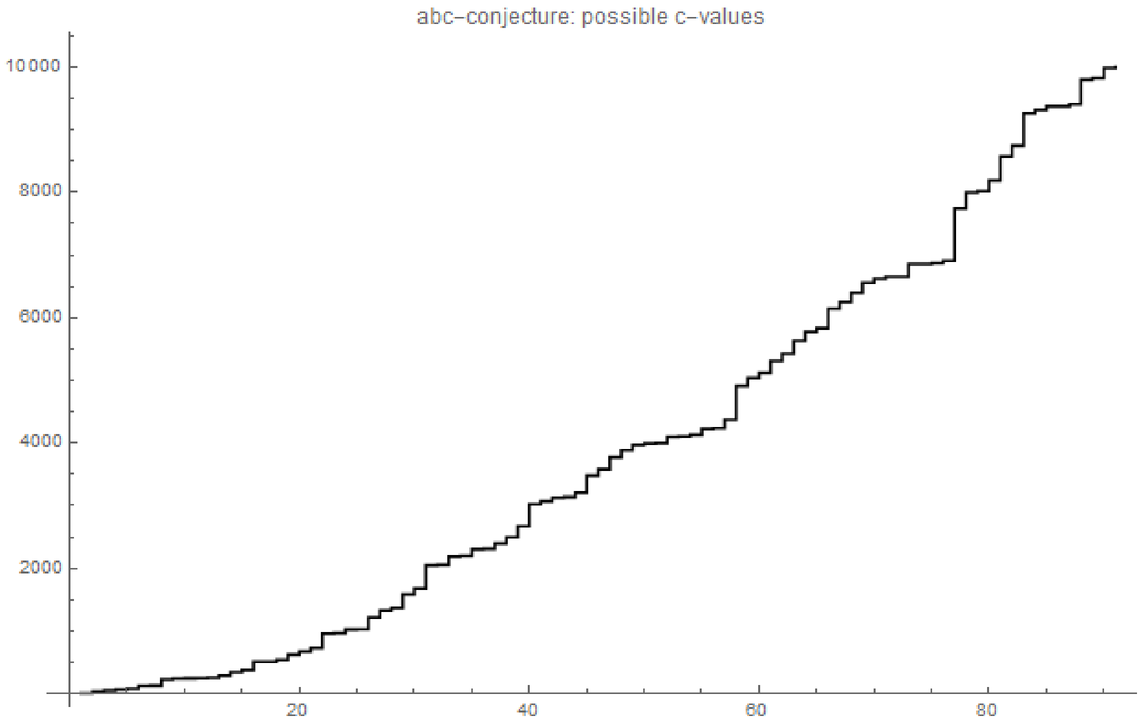


Figure 116. *abc* hits: the first 91 possible c-values (9-10000)

abc hits are very rare. Among the 15.2 million possible *abc* triples up to $n = 10000$, there are only 120 hits, 91 of them are different hits. To date (Feb. 2016), only 237 *abc* hits with a quality $q > 1.4$ have been discovered.
Here are the *abc* hits up to 10000 (only possible c-values, without multiple hits):

```
{2,9,32,49,64,81,125,128,225,243,245,250,256,289,343,375,512,513,539,6
25,676,729,961,968,1025,1029,1216,1331,1369,1587,1681,2048,2057,2187,2
197,2304,2312,2401,2500,2673,3025,3072,3125,3136,3211,3481,3584,3773,3
888,3969,3993,4000,4096,4107,4131,4225,4235,4375,4913,5041,5120,5312,5
427,5632,5776,5832,6144,6250,6400,6561,6625,6655,6656,6859,6860,6875,6
912,7744,8000,8019,8192,8576,8748,9261,9317,9375,9376,9409,9801,9826,9
984,10000}
```

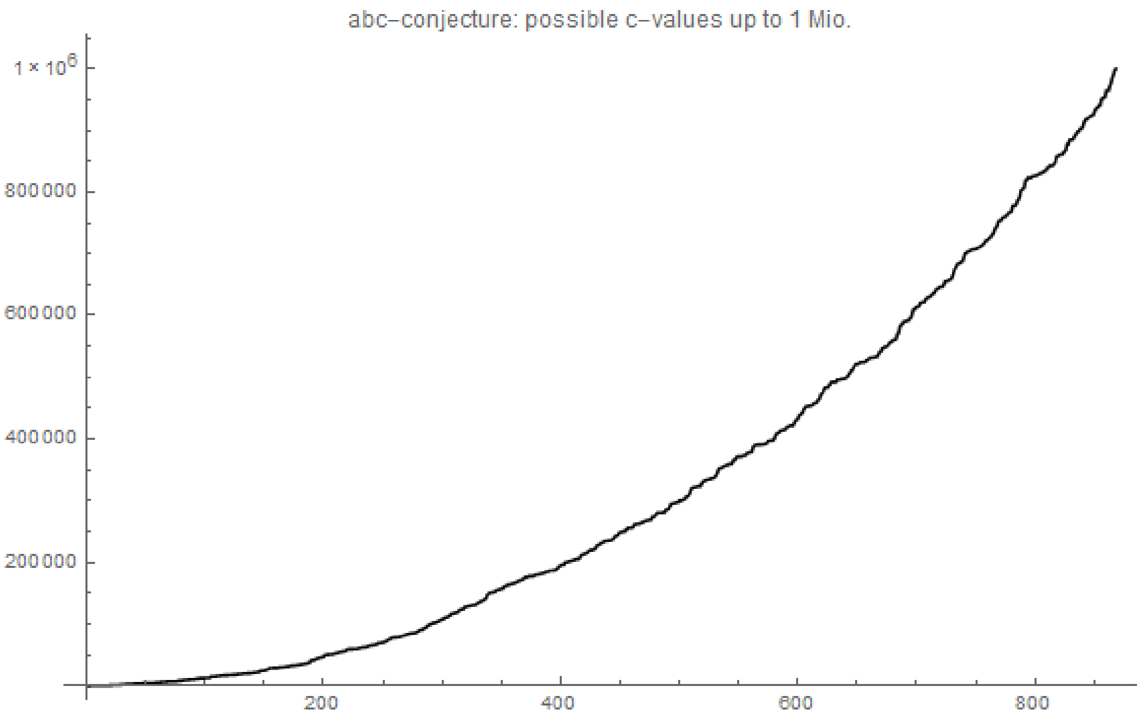


Figure 117. *abc* hits: the first 868 possible c- values (9-1000000)

One can clearly see that the *abc* hits become rarer with increasing size.

Among the 380 million possible *abc* triples below 50000, there are 276 *abc* hits.

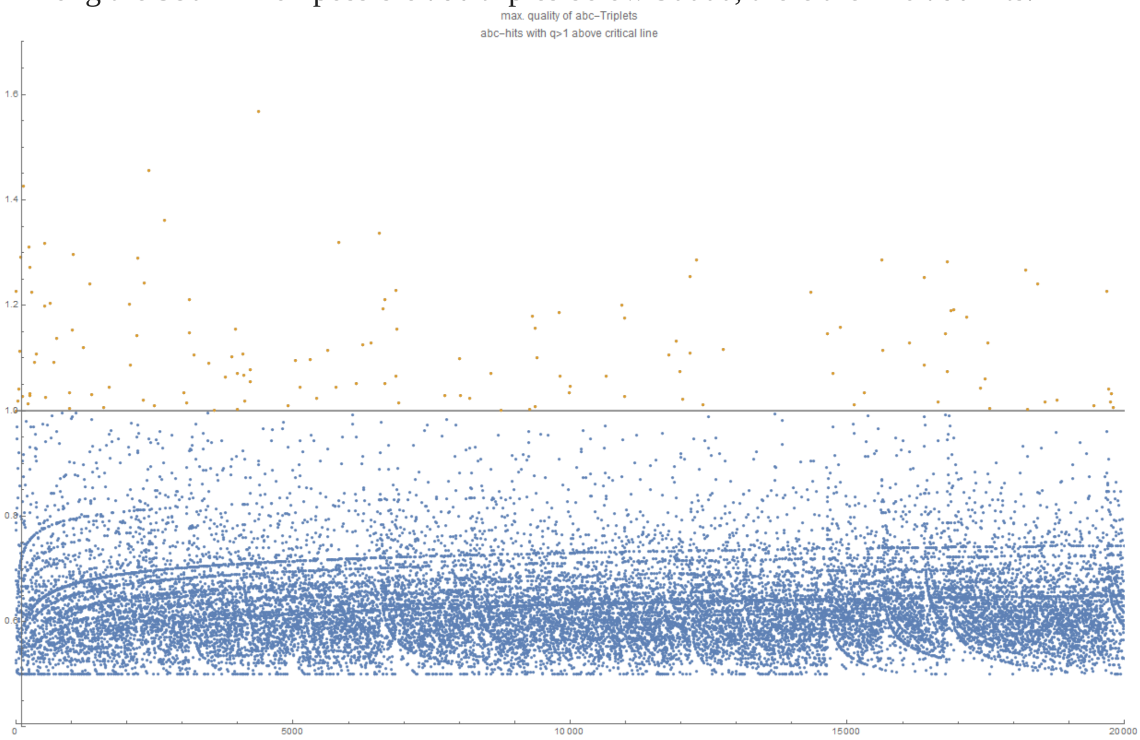


Figure 118. Max. ' quality' of *abc* triples as a function of c (with 138 *abc* hits) in the range up to 20000

Clearly it can be seen that *abc* hits are very rare. In the 'non-critical' range with $q < 1$, clear structures can be seen. One cannot escape the feeling that further surprises are in store…

The first *abc* hits in the range up to 2000 together with their qualities (in the case of multiple *abc* hits for the same c, the corresponding maximum quality was taken) are:

```
{{2,1.},{9,1.22629},{32,1.01898},{49,1.04124},{64,1.11269},{81,1.29203},{125,1
.0272},{128,1.42657},{225,1.0129},{243,1.3111},{245,1.02883},{250,1.03261},{25
6,1.27279},{289,1.22518},{343,1.09175},{375,1.10844},{512,1.19875},{513,1.3175
7},{539,1.02512},{625,1.20397},{676,1.09219},{729,1.13667},{961,1.0048},{968,1
.03443},{1025,1.1523},{1029,1.29721},{1216,1.1194},{1331,1.24048},{1369,1.0299
1},{1587,1.00607},{1681,1.04391}}
```

```
Mathematica program: please contact the author.
```

12.2 THE *ABC* CONJECTURE AND GOCRONS: IS THERE A CONNECTION?

The *abc* conjecture creates a connection between the world of addition and the world of multiplication, in that it predicts that additive operations also have a certain influence on the multiplicative structures of the objects under consideration. What could be more natural, then, than to investigate the *abc* conjecture with objects that perfectly represent multiplicative properties: the OCRONs and GOCRONs (see 10.2.2)? We shall restrict ourselves here to type 4 OCRONs and their extensions (EOCRONs). Since we want to make quantitative statements, we are not using character strings (that is, OCRONs), but their ‘Gödel numbered’ relatives: GOCRONs.

First let’s take a look at our additive structure:

We look for the set of all natural numbers a and b, the sum of which gives a fixed value $\boldsymbol{a+b=c, a, b \geq 1 \text{ and } a, b, < c}$ and especially their GOCRON4 values $g(a), g(b)\ and\ g(c)$, for example $g(a) = \mathrm{nToEGOCRON4}(a)$ (using the function nToEGOCRON4() from the OCRON Mathematica library, which can be found in the Appendix). We shall also investigate the influence of the additional boundary condition of the *abc* conjecture that a and b are to be mutually 'coprime' (i.e. have no common divisor).

We interpret the values $\{g(a), g(b)\}$ as points in the plane, and assign to them a function value $f[g(a), g(b)]$ that forces the whole thing to take on a multiplicative structure $f[g(a), g(b)] = g(a \cdot b)$.

Since GOCRON values can become very large quickly, we prefer logarithmic values (which are better suited to the material in question). Thus, our task can be precisely described as follows. We will search for a set of integer *abc* triples in which an additive structure exists between a and b, but a multiplicative value is assigned to the third value:

$$\boldsymbol{M_{abc} = \{\ln g(a), \ln g(b), \ln g(a \cdot b)\}, \text{where } g(x)\text{: goedelcodes of the OCRONs as well as the boundary condition}: a+b=c \text{ and } a \perp b} \tag{154}$$

If we look at the structure of the set M_{abc}, then we get a surprise, because the three-dimensional points of M_{abc} all lie (with a deviation of about 2 to 3%) on a plane with the incredibly simple equation $z = x + y + \text{const}$, where the value const depends only on c!

$\boldsymbol{M_{abc}}$ can be represented approximately by**: z=x+y+const** (155)

This seems to apply to all $c \in \mathbb{N}$ (the author has not yet found a counterexample). If the boundary condition $\boldsymbol{a} \perp \boldsymbol{b}$ is omitted, then $\boldsymbol{M_{abc}}$ does not lie on a plane for any integer c. The structures seem to be much more complicated for this general case. For the case where c is a prime number, $\boldsymbol{M_{abc}}$ likewise forms a plane, even if we omit the boundary condition a⊥b, since this condition is then automatically fulfilled (the two summands of a prime number are automatically coprime with each other).

Unfortunately, the points of $\boldsymbol{M_{abc}}$ are not exactly on a plane, but exhibit around 2 to 3 per cent ‘noise’. If the relation applied exactly, then we would have found a method of calculating the factorization of a number (in our case c) from the values a and b (or $\ln g(a), \ln g(b)$ respectively). Conversely, the factorization could be calculated solely from c, by a projection of c onto the $x - y$ axis). In general, one could also imagine a method that searched only by evaluating the plane equation by integer values a and b, since the determination of $g\ (a)$ and $g\ (b)$ can demand a great deal of computing time for large a and b.

One assumes that if we were to choose a different, more suitable, GOCRON system (the calculation and the Gödel numbering allow a degree of ‘freedom of choice’ in the selection of parameters), the noise described could be reduced or even eliminated altogether. There is still much work to do here!

Here are a few plots of different sets of $\boldsymbol{M_{abc}}$:

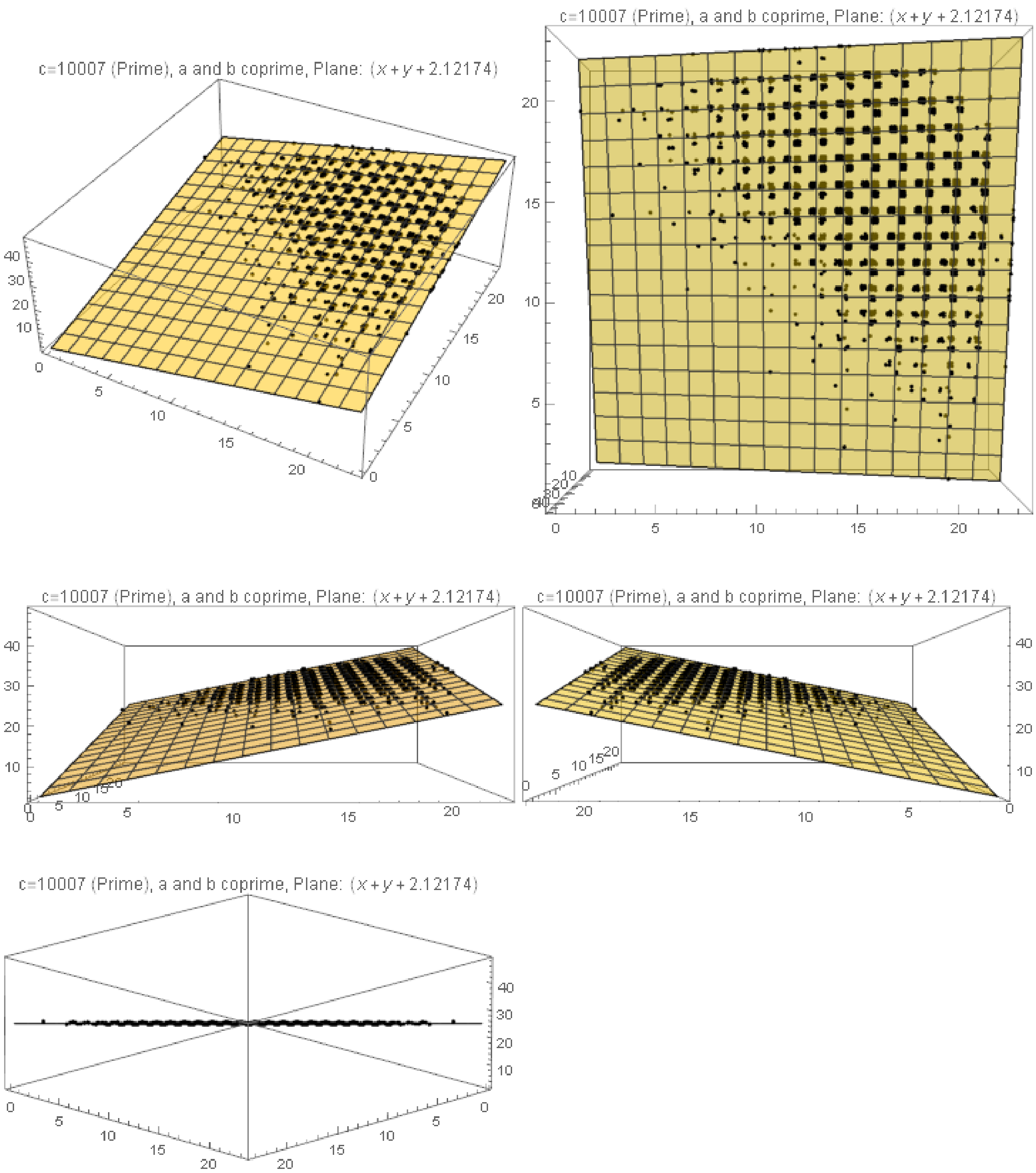


Figure 119. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=10007 (prime number), a and b are coprime (different views).

Note: the plot in Figure 119 does not change if we omit the boundary condition $a \perp b$, because 1007 is a prime number.

`Mathematica program: please contact the author.`

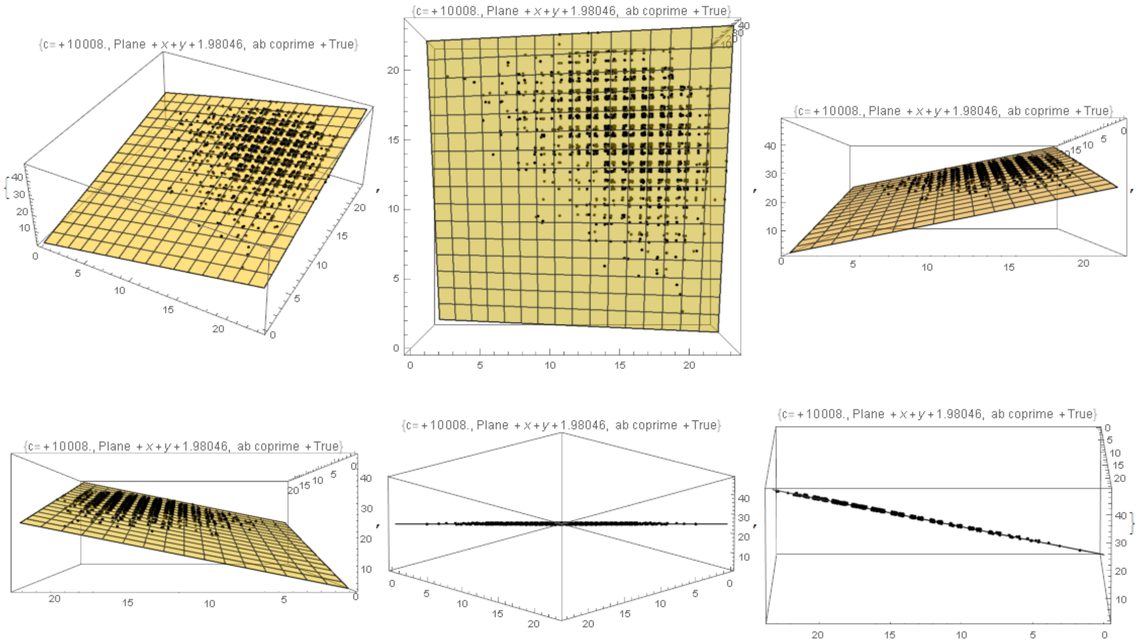


Figure 120. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=10008, a and b are coprime (different views).

(Mathematica programs for the calculations can be found in the Appendix)

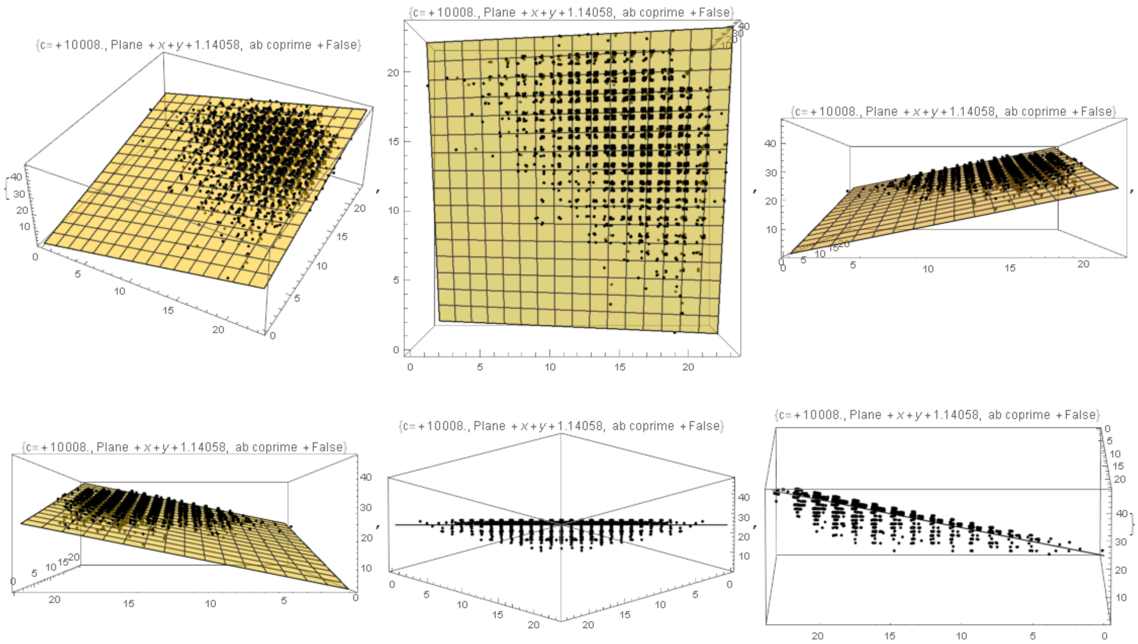


Figure 121. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=10008, a and b not coprime, (different views). The 10004 points are spatially distributed.

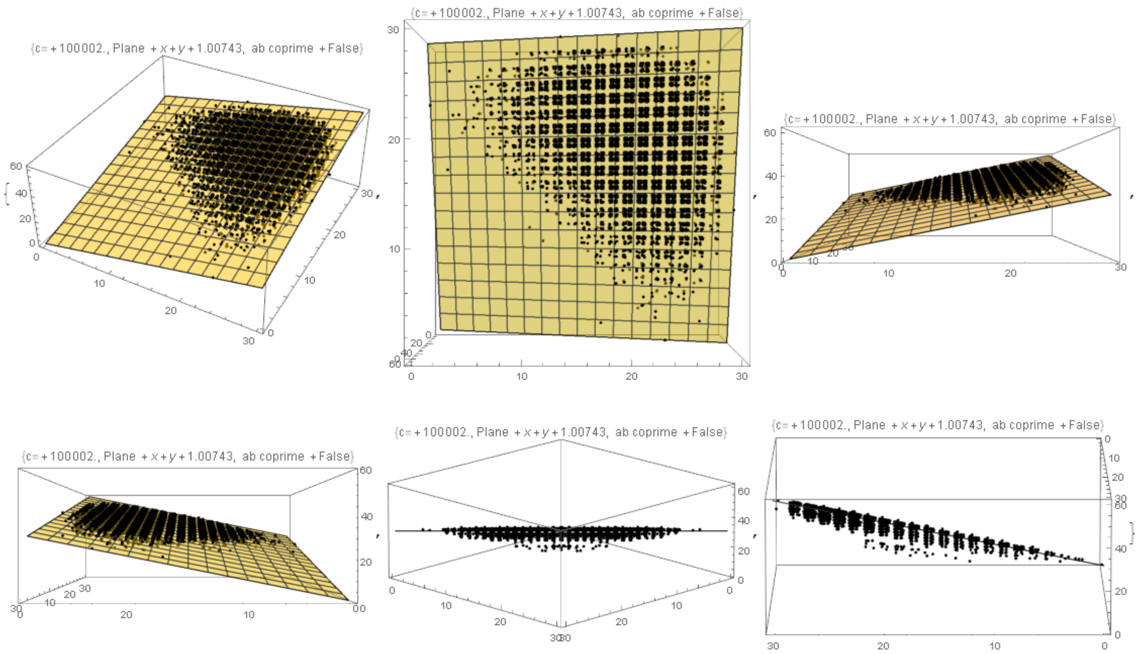


Figure 122. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=100002, a and b not coprime, (different views). The 99998 points are spatially distributed.

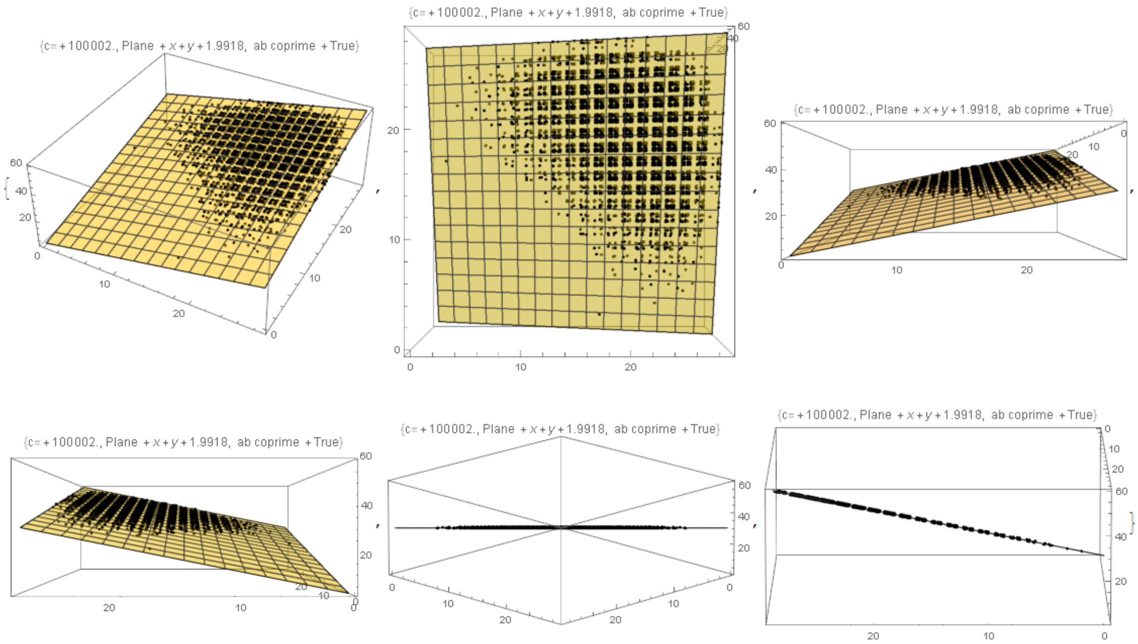


Figure 123. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=100002, a and b coprime (different views). The 28558 points lie approximately on a plane.

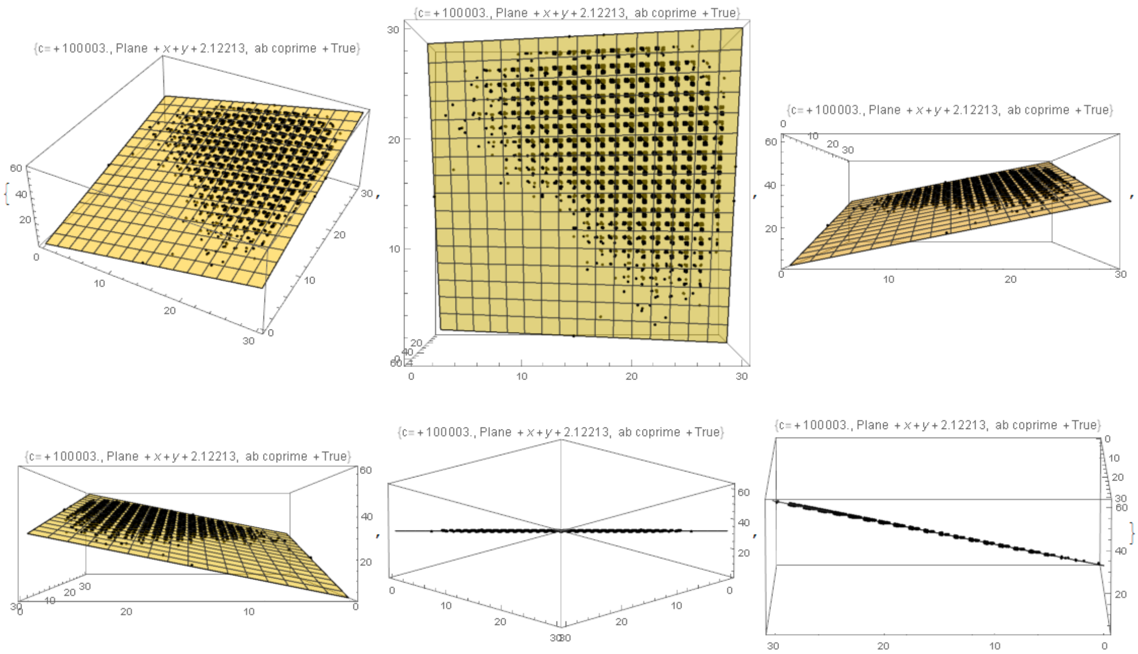


Figure 124. M_{abc}: logarithmic Gödel GOCRON4 codes of abc points. C=10007 (prime number), (different views). The 100003 points lie approximately on a plane.

12.3 THE SET M_{abc} AND ITS PLANE-EQUATION

As we have seen in the last chapter, the points M_{abc} lie approximately on a plane. We will now take a closer look at this issue. At first we notice that the ‘plane principle’ applies to all variants of type 4 GOCRONs: the normal GOCRONs, M2GOCRONs (which belong to the OCRONs, in which the always leading ‘2' has been discarded, as well as EOCRONs (the extended GOCRONs)). The difference between the different types is in the range of values (the normal GOCRONs are at least one order of magnitude larger than their relatives) and in their 'retransformability' into the range of the normal numbers (EGOCRONs can be ‘retransformed’ for any integer value > 2). We now investigate the dependence of the parameters of the corresponding plane-equation on the value c, as well as of various other parameters that might play a role in the calculation of the points M_{abc}. As a criterion for a 'good' parameter choice, we take the 'standard error' that results from the method of least squares applied during the plane calculation from the set M_{abc}. We use the Mathematica function NonliniearModelFit[…] with the model: $z = x + y + c_3$.

Comparing the various Gödel code symbols used in the conversion of OCRONs into GOCRONs, it turns out that, for the (normal) type 4 GOCRONs, the following assignments of the 24 possible permutations of the set of code symbols give the best results:

{“ $*$ ”, “P”, “2”, “^”} $->$ {0,2,3,1} and {“ $*$ ”, “P”, “2”, “^”}$->$ {1,2,3,0}.
The following table shows the results:

Table 25. c=100003. Fit parameter and c_3 of the plane equations for M_{abc} (type GOCRON4) for different sets of Gödel symbols

C	c_3	**Codetable: symbols/values**	**Max. value**	**Standard error**	**t-statistics**

100003	1.85733	1:{*,P,2,^},{0,1,2,3}	37.0455	0.000162446	11433.5
100003	1.48688	2:{*,P,2,^},{0,1,3,2}	37.4114	0.000186537	7970.96
100003	2.34062	3:{*,P,2,^},{0,2,1,3}	36.423	0.000200917	11649.7
100003	**1.44022**	**4:{*,P,2,^},{0,2,3,1}**	**37.3964**	**0.0000736448**	**19556.3**
100003	2.24673	5:{*,P,2,^},{0,3,1,2}	36.3821	0.000422527	5317.36
100003	1.73226	6:{*,P,2,^},{0,3,2,1}	37.0015	0.000153876	11257.5
100003	1.93765	7:{*,P,2,^},{1,0,2,3}	37.0444	0.000332613	5825.54
100003	1.54166	8:{*,P,2,^},{1,0,3,2}	37.4107	0.000303486	5079.82
100003	4.00103	9:{*,P,2,^},{1,2,0,3}	34.4163	0.00246733	1621.61
100003	**1.44418**	**10:{*,P,2,^},{1,2,3,0}**	**37.3804**	**0.0000698268**	**20682.3**
100003	3.79035	11:{*,P,2,^},{1,3,0,2}	34.4462	0.00303396	1249.31
100003	1.738	12:{*,P,2,^},{1,3,2,0}	36.9777	0.000172422	10079.9
100003	2.61489	13:{*,P,2,^},{2,0,1,3}	36.4189	0.000393053	6652.77
100003	1.54569	14:{*,P,2,^},{2,0,3,1}	37.3949	0.000294348	5251.23
100003	4.50595	15:{*,P,2,^},{2,1,0,3}	34.4009	0.00207401	2172.57
100003	1.49486	16:{*,P,2,^},{2,1,3,0}	37.3796	0.000170018	8792.37
100003	3.93228	17:{*,P,2,^},{2,3,0,1}	34.4473	0.00352994	1113.98
100003	2.26812	18:{*,P,2,^},{2,3,1,0}	36.2903	0.000484516	4681.22
100003	2.62617	19:{*,P,2,^},{3,0,1,2}	36.3757	0.00036176	7259.43
100003	1.94952	20:{*,P,2,^},{3,0,2,1}	36.9981	0.000303674	6419.8
100003	4.62145	21:{*,P,2,^},{3,1,0,2}	34.018	0.00222471	2077.33
100003	1.87489	22:{*,P,2,^},{3,1,2,0}	36.9753	0.000122802	15267.6
100003	4.25883	23:{*,P,2,^},{3,2,0,1}	34.0449	0.00320799	1327.57
100003	2.37305	24:{*,P,2,^},{3,2,1,0}	36.288	0.000280365	8464.13

Further tables with different methods of Gödelization and different values of *c* can be found in the Appendix.

An evaluation of these tables shows that code table No. 10 is (albeit narrowly) the 'winner' (with regard to the smallest standard error), if the summation of the OCRONs is performed in the normal order (left to right). In the case of the reverse order, the selection of a best Gödel code set is not so clear. In the following, we will use the Gödel code assignment {'*', 'P', '2', '^' -> {1, 2, 3, 0} for all OCRON4 types and the normal order of symbols (not 'reversed').

The program used for the evaluation can be found in the Appendix.

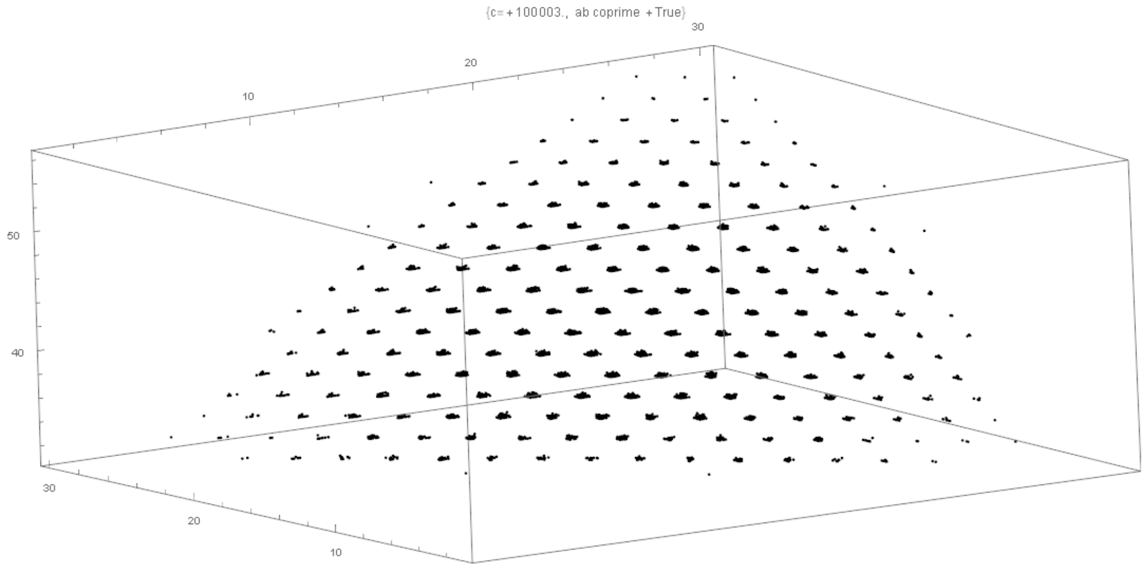


Figure 125. Plane of M_{abc} – points for c = 100003 (prime number)

```
Mathematica program: please contact the author.
```

The next table shows the relationship between c and the plane parameter c_3 with CT: no. of the Gödel code table, GT: GOCRON type (N: normal, M2: without, '2' at the beginning of an OCRON, E: enhanced).

Table 26. Different c_3 -values for different GOCRONs and Gödel symbols

c	c_3(CT:10, GT: N)	c_3 (CT:10, (GT: E)	c_3 (CT:10, GT: M2)	c_3 (CT:4, GT: M2)	c_3 (CT:9, GT: M2)	c_3 (CT:12, GT: M2)
10009	1.44418	**1.65067**	**3.0335**	**3.01507**	4.00661	2.9951
30011	1.44423	**1.65087**	**3.03373**	**3.01535**	4.00661	2.99422
100003	**1.44418**	**1.65062**	**3.03347**	**3.01512**	4.00103	2.99437
1000003			**3.03335**	**3.01497**	4.00219	2.9946

Summary

The points of the set M_{abc} lie (with a deviation of 2-3%) on a plane with the equation $z = x + y + c_3$ (c_3 see above table). Since logarithmic values are taken in the calculation of the points (see (154)), the deviation from the 'fitted' values of the plane is, of course, substantially greater if 'delogarithmized' values are considered. In these deviations from the interpolated values of the equation, there is, so to speak, still a lot of hidden 'structure', which would have to be examined for further regularities. This simple model of the plane equation is not a help in finding a prime factor of a given number. For this, the points would have to lie much more precisely on the plane.

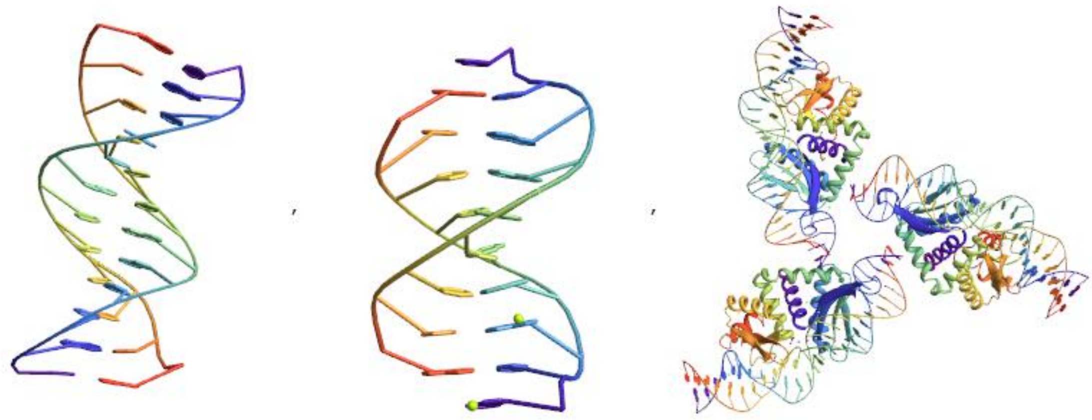
However, the fact that the plane structure only occurs when the relation $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ **and** $\boldsymbol{a} \perp \boldsymbol{b}$ applies is very interesting. If, for example, we don't use the boundary condition $\boldsymbol{a} \perp \boldsymbol{b}$, then there is no plane, but a widely dispersed, complicated spatial structure (see, for example, Figure 121).

This fact indicates a connection with the *abc* conjecture.

13 PRIME NUMBERS IN THE NATURAL SCIENCES

13.1 PRIME NUMBERS IN DNA CODE

Many scientists think that primes also play a role in the construction and design of DNA sequences. Everyone knows the famous double helix of the DNA. Here are a few arbitrarily selected examples:



```
Mathematica:
Import[#,"PDB"]&/@{"http://files.rcsb.org/download/1BNA.pdb","http://files.rcsb.org/download/208D.pdb","http://files.rcsb.org/download/5A0W.pdb"}
```

The following text is essentially the work of J.F. Yan, A.K. Yan and B.C. Yan.[63]

Before dealing with the underlying idea, it is useful to look at a few basic properties of DNA and RNA.

DNA is the carrier of the genetic information that is present in the cell nuclei of all living beings. These are huge chain molecules, which are composed of four different basic building blocks: the nucleotides. Each nucleotide consists of a phosphate group, a sugar (deoxyribose) and one of four possible organic bases (adenine, guanine, cytosine and thymine).

The RNA (ribonucleic acid) is constructed by means of the information in the DNA. For the exact mechanism, the reader is referred to the relevant literature[64]. The RNA (in a similar way to the DNA) is composed of nucleotides, which in turn are composed of four different organic bases (adenine, guanine, cytosine and uracil).

[63] J. theor. Biol. (11991) 151,333-341

[64] e.g. http://www.dna50.org/

The first three bases are present in the RNA and in the DNA; the base thymine occurs only in the DNA and uracil only in the RNA.

The organic bases are usually abbreviated by the letters A, G, C, T and U. These are the smallest information bits of DNA and RNA. The differences in the structure of RNA and DNA (uracil instead of thymine) are irrelevant to the information content we are considering. Therefore, for the sake of simplicity, we will only use the four bases A, G, C, and U (whereas, strictly speaking, it should be 'T' instead of 'U' in the case of the DNA).

Each successive triplet of bases (also called a 'codon') is specifically assigned to a particular amino acid. The converse is not true: several different triplets may be assigned to the same amino acid. These triplets are the smallest information units of DNA and RNA. A few triplets (codons) contain no genetic information. However, they work as so-called 'stop' triplets during replication and copying processes that take place during protein synthesis. There are also start-up triplets, which, however, also contain information: e.g. the triplet 'AUG' (methionine amino acid).

For the four different bases there are $4^3 = 64$ different possibilities for triplet formation. The amino acids occurring practically in nature (canonical, proteinogenic amino acids) are coded with the aid of 61 triplets. The three remaining triplets are 'stop' triplets. A very good representation can be found on the Internet.[65] Some amino acids are encoded only by one triplet, some by 2,3, 4 or 6 different triplets. Note: the number 5 is absent as the number of amino acid forming triplets.

What does all this have to do with primes? How do we get from base triplets to numbers? The answer is similar to that in Chapter 10.2.2.4: by 'Gödelization'. We assign a numerical value to each base triplet (our smallest information unit). There are several possibilities for this:

Klaus Lange used in his work, *Primes in the construction of the DNS* [66] G = 1, A = 3, C = 7, U = 9 and builds the number codes simply by using the decimal system. An example: alanine (= GCA with the code value 173). He then examines the resulting numerical values for their prime factor decomposition and discovers that at least 19 of the 20 canonical amino acids contain a triplet that represents a prime number. It is striking for him that the only amino acid to which no prime can be assigned as a triplet is methionine (Met = AUG). According to his system, this triplet has the value 391. Methionine is the only triplet that works as a so-called 'start signal' (see above).

This number assignment appears to the author somewhat arbitrary. Better is the method of Yan (see above), in which A=0, C=1, U=2 and G=3. From this assignment, a unique set of "nucleotide numbers" (each of which can be assigned to an amino acid) is then constructed. Special cases are the following amino acids:

---

[65] https://de.wikipedia.org/wiki/Genetischer_Code

[66] http://www.primzahlen.de/primzahlen/dns.htm

0: for stop signal (without name, UAA, UAG, UGA)
1: for tryptophan (Trp, also Try), UGG
2: for isoleucine (Ile), AUA, AUC, AUU
3: for methionine (Met), AUG

The principle for constructing a nucleotide number z ($z < 64$) is as follows:

Rule 1: z must be odd or 2. The values 1 and 2 are reserved for AAX triplets

Rule 2: let us call prime numbers of the form $4n + 1$ 'P1 primes' and primes of the form $4n + 3$ 'P3 primes'. From number theory, we know that P1 numbers can always be expressed in a unique way as the sum of two squares. Between 0 and 63 there are a total of 8 P1 primes.
All P1 prime numbers < 64 that can be represented as the sum of 2 squares, are equivalent to the fourfold 'synonym' codons that are generated when we specify the first two bases (of the three possible ones).

The next procedure is to split up all 64 possible codons into 4 groups.

Group 1 are the Diophantine solutions of the equation $z = (2i + 1)^2 + (2j)^2$

Nucleotide numbers	Synonym codons (X=A, C, U or G)	Name of the amino acid
$1^2 + 2^2 = 5$	ACX	Thr
$3^2 + 2^2 = 13$	CCX	Pro
$5^2 + 2^2 = 29$	UCX	Ser
$7^2 + 2^2 = 53$	GCX	Ala

Group 2 are the Diophantine solutions of the equation $z = (2i - 1)^2 + (2j)^2$

Nucleotide numbers	Synonym codons (X=A, C, U or G)	Name of the amino acid
$1^2 + 4^2 = 17$	CUX	Leu
$1^2 + 6^2 = 37$	CGX	Arg
$5^2 + 2^2 = 29$	GUX	Val
$7^2 + 6^2 = 61$	GGX	Gly

Group 3: to determine the values in this group, Yan et. al use some heuristic arguments (borrowed from chemistry) that the reader is welcome to peruse (see references).

Group 4 are the Diophantine solutions of the equation $z = 4(2i + 1) + 3$ as well as $z = 8(2i + 1) + 3$.

Finally, the following code assignment results for all canonical amino acids:

Table 27. Prime number encoding of the canonical amino acids according to Yan et. al

0 (stop)	1 (Try)	2 (Ile)	3 (Met)	5 (Thr)
	7 (Lys)	11 (Asn)	13 (Pro)	17 (Leu)
	19 (Gln)	29 (Ser)	31 (Asp)	37 (Arg)
	41 (Val)	43 (Tyr)	47 (His)	53 (Ala)
	59 (Glu)	61 (Gly)	25 (Phe)	45 (Cys)

Note that Yan uses two codes that are not prime numbers: (25: phe and 45: cys).

Using these codes, Yan et. al derive coding characteristics and strategies that would require a deeper understanding of genetics to explain and which, in any case, would take us too far afield.

13.2 SPECTRAL CHARACTERISTICS OF 'PRIME NUMBER SIGNALS'

Prime numbers show a certain similarity to the statistical data of physical experiments. This similarity probably comes from their 'unpredictability' (not, of course, in the strict mathematical sense). According to the theory of information, we can define a prime number signal as follows:

$x_i = \pi\big((i+1)M\big) - \pi(i \cdot M)$, with a fixed interval length M.

Physicists like to deal with the evaluation of signals. If we examine our 'prime' signal using physical methods, we are not doing 'real' physics, but something akin to a physical thought experiment. Let us imagine, in the search for extraterrestrial life forms by means of radio signals such a signal is received (we shall here not discuss the details of the modulation; we simply assume that it is a digital signal from which the numbers of the sequence x_i were extracted).

Here is an example with $M = 2^{16}$, which gives the following 'signal':

```
{4533,4454,4486,4430,4460,4446,4446,4442,4438,4421,4446,4401,4376,4417
,4358,4384,4435,4386,4355,4344,4360,4258,4337,4354,4394,4283,4339,4343
,4255,4354,4294,4307,4289,4237,4285,4327,4283,4266,4258,4285,4244,4256
,4301,4281,4228,4233,4232,4243,4261,4207,4240,4210,4198,4202,4197,4196
,4188,4221,4239,4217,4128,4220,4157,4226,4209,4128,4148,4195,4230}
```

```
Mathematica:

intervalLength=2^16; startValue=32; endValue=100;
pSignal[j_,m_]:=PrimePi[(j+1)*m]-PrimePi[j*m]
signal=Table[pSignal[k,intervalLength],{k,startValue,endValue}]
```

Let us conduct a 'prime number experiment' by generating a prime number signal and examining it using physical methods, such as spectral analysis. This signal has the following appearance:

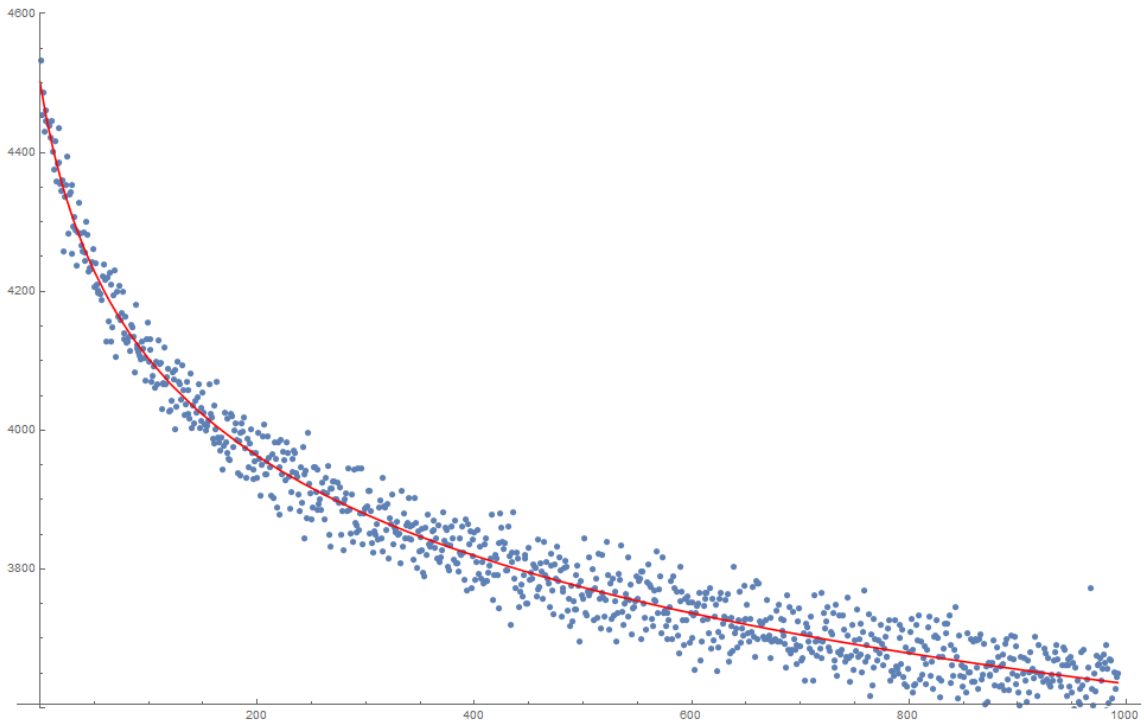


Figure 126. Prime number signal x_i where interval length $\mathrm{M} = 2^{16}$. red: $\frac{M}{\ln Mi}$

From the theory of numbers, we know that the asymptotic behaviour of x_i is as follows: $x_i = \frac{M}{\ln\ Mi}$. The red coloured curve shows the asymptotic behaviour.

We now apply a discrete Fourier transform (DFT) to the signal x_i and obtain in the frequency domain:

$$X_k = \sum_{j=0}^{N-1} x_j e^{-\frac{2\pi i j k}{N}} \tag{156}$$

where N is the length of our prime number signal (e.g. 2^{16}). Now, the physicist is interested in the spectral power density:

$$S_k = |X_k|^2$$

If we look at this spectral power density on a logarithmic scale, then we experience a surprise, because it can be approximated over a wide range by a straight line. This means that the spectral power density of our prime signal behaves like $1/k^{\alpha}$, with a constant exponent α:

$$S_k \sim \frac{1}{k^{\alpha}} \text{ where } \alpha \approx 1.55 \tag{157}$$

The value $\alpha \approx 1.64$ is given in the literature.[67] Computations performed by the author lead to an approximate value of 1.55.

This behaviour is well known to physicists for a group of physical systems, namely, those that are in a so-called ‘self-organized’ critical state. Many other physical systems also exhibit spectral behaviour according to the $\frac{1}{k^\alpha}$ law: for example voltage noise in electronic components (flicker noise).

There are also studies of the statistical behaviour of notes (within our 12-step tonal musical system) that show a statistical 1/f-behaviour for traditionally composed (non-random music). Thus, we have established an (albeit remote) connection between primes and tonal music!

Here is a plot of the spectral power density of a prime number signal:

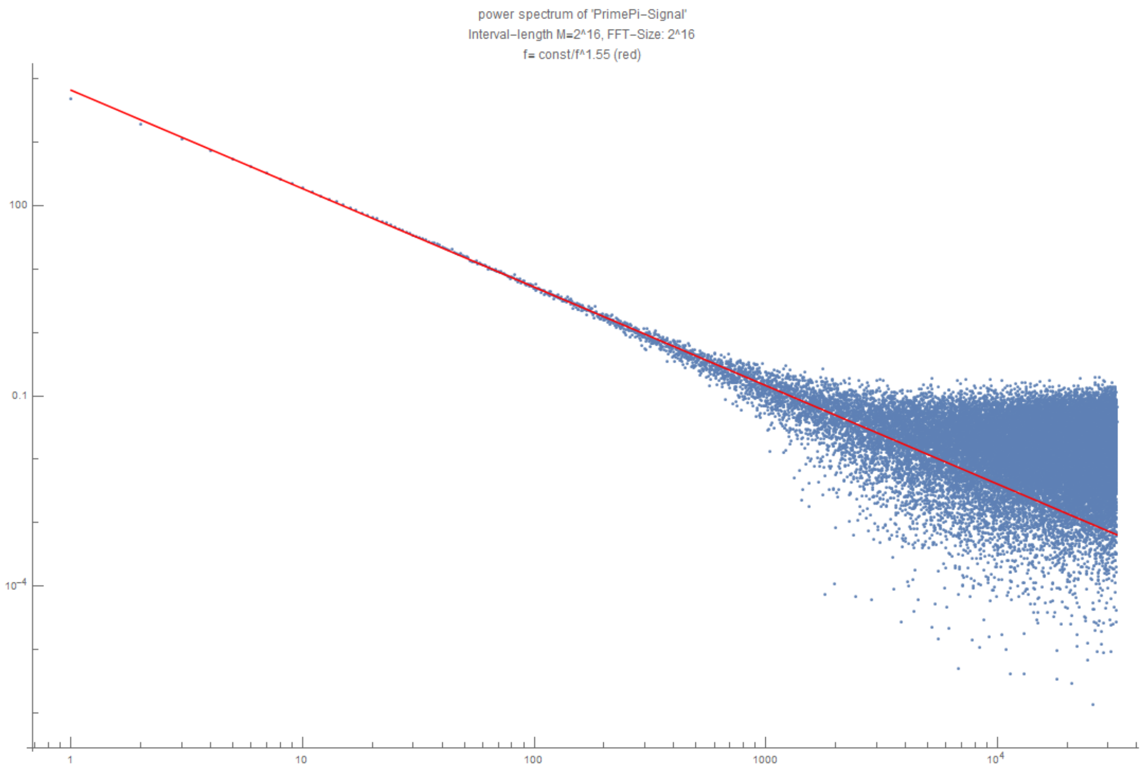


Figure 127. Spectral power density of a prime number signal (red: $\frac{1}{k^\alpha}$ where $\alpha = 1.55$)

`Mathematica program: please contact the author.`

[67] Marek Wolf: *PHYSICA A: Statistical mechanics and its applications* ·January 1997, pp. 493-499

14 PRIME NUMBERS AND ONLINE BANKING

14.1 RSA ENCRYPTION

Prime numbers entered the field of cryptography, the 'science of deciphering', some time ago. In online banking, highly confidential data are constantly being sent back and forth. The number of transactions is so huge that a symmetrical encryption (where both partners have a secret key) would not be feasible, as the secret key would have to be sent on a secure transmission path (e.g. by post) prior to the actual transaction. This is basically impossible. There is, however, a procedure that avoids the complicated sending of keys by post: this is known as 'asymmetrical RSA encryption'.

We will briefly describe the RSA method first. 'RSA' stands for Rivest, Shamir and Adleman, the trio of computer scientists who first implemented the procedure in 1978. The original idea, however, was described by Diffie and Hellman in 1976. This encryption method is called 'asynchronous' because the sender and receiver of encrypted messages use different keys (which are public and secret). The sender uses a public key to encrypt and send, and the recipient uses a top secret private key to receive and decrypt the message. In order for the sender to send such encrypted messages to a receiver, the receiver must first generate a public (non-secret) key and then send it to the sender, who then uses it to send the message. This sending of the public key can, of course, take place in an unencrypted manner.

Since texts are to be encrypted, we do not look at the individual characters, but their ASCII codes.[68] These assign each character a value from 32 (space) to 90 ('Z'), where 65 = 'A', 66 = 'B' etc. Lower case letters are represented by highter numbers but this is irrelevant to an understanding of the procedure.

The text to be encrypted is first translated into a long sequence of digits of these codes using the ASCII code. Thereafter, blocks of a fixed length (e.g. of length 64) are formed from this sequence of numbers. Each of these blocks is now interpreted as a (in this case, 64-digit) number in a numeral system with base 256. The formation of blocks takes place only for reasons of manageability in order to avoid overlarge "numerical monsters". The choice of the base is not important and this can even be smaller if we use a smaller character set.

It is important to understand that we have converted our text into a sequence of very, very large numbers (e.g. 50-digit numbers). It is also possible to use a single block for the whole message. In this case, the original message text to be encrypted consists only of a single (admittedly gigantic) number that we will call m (= 'message'). The number contains all our text. So far it is very easy to restore our original text from m.

Encryption is now coming into play. To recap: the sender has the public key that he has received from the recipient. Only the recipient has the secret private key (which has been created simultaneously with the public key at the recipient's end).

[68] ASCII: 'American Standard Code for Information Interchange'

Now we come to the details. The private and public keys are generated as follows: we are looking for two different, very large prime numbers (typically hundreds of digits long). Primes of this size can be generated using simple mathematical methods, such as probabilistic prime number tests, the Fermat prime number test, the Miller-Selfridge-Rabin test, the APRCL test or the Solovay-Strassen test (the algorithms are clearly set out on Wikipedia).

Primes generated with a good probabilistic method are generally referred to as PRP numbers. They are, as far as it is humanly possible to judge, 'real' primes because the error probability of such large numbers is astronomically low (typically, for example, 10^{-100}). Although there are also exact methods (with a polynomial run time), these are not suitable for the generation of very large primes because of their long running time.

Note: the record for the largest currently known (as of May 2016) PRP number is a so-called Wagstaff prime (see 4.13) and is:

$(2^{13372531}+1)/3$ number of decimal digits: 4025533

For comparison, the largest 'general' prime number found using a method valid for any primes (not primes of a particular form) is (as of 2011):

$$((((((2521008887^3+80)^3+12)^3+450)^3+894)^3+3636)^3+70756)^3+97220$$

This number is the 11th Mills prime number and it has 20562 decimal digits.[69]

It can be seen clearly that for prime numbers of a particular form, primality tests are available for significantly larger numbers. The Lucas-Lehmer test for Mersenne prime numbers still provides the largest prime numbers (over 10 million digits). The largest general prime currently known, with 20562 decimal digits, is rather modest, since it has about 1000 times fewer decimal digits. Back to the RSA procedure:

The methods for generating large primes suitable for the RSA method are not described here. Once again, we rely on the Mathematica software, which provides a set of functions that are used in cryptography:

```
PowerMod[], PowerModList[], PolynomialMod[], RandomPrime[],
Prime[], PrimeQ[],CoprimeQ[], FactorInteger[],
GenerateAsmmetricKeyPair[], Encrypt[], Decrypt[],
PrivateKey[], PublicKey[]
```

[69] Paulo Ribenboim: *Die Welt der Primzahlen* (Springer), p. 118

Mathematica generates 200-digit primes in a fraction of a second (using PRP algorithms):

```
In[3]:= NextPrime[10^200]
```

```
Out[3]=
1000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000357
```

The calculation of a 1000-digit PSP prime number takes about one second:

```
In[5]:= RandomPrime[{10^1000,10^1000+1000}]
```

```
Out[5]=
1000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000453
```

We calculate two different (as large as possible) primes p and q and calculate the product n = $\boldsymbol{n = pq}$. This product is called the 'RSA module'. Only the multiples of p and q are not coprime to n.

The calculation of p and q can be done by the Mathematica function `RandomPrime[]`.

The number of numbers being coprime to n that lie between 1 and n therefore amounts to $\varphi(n) = (p-1)(q-1)$.

Next, we need the so-called encryption exponent k (which is public). The encryption exponent k must be chosen to be coprime to both $p-1$ and $q-1$, which is equivalent to saying that k is coprime to $\varphi(n)$. In addition, the following must hold for k: $3 < k < \varphi(n)$; k can be found with the Mathematica function `CoprimeQ[]`.

Usually, for the sake of efficiency, the 5th Fermat prime number $65537 = 2^{2^{5-1}} + 1$ is chosen (this is a relatively small exponent, the decryption exponent defined below) is usually much larger).

The choice of a prime number for k has the advantage that k is automatically always coprime to $\varphi(n)$ (even with another set of (p, q)), which is a prerequisite. However, k should not be chosen too small (the recommendation is about ¼ of the bit length of the

RSA module), otherwise the system is vulnerable to attack and decryption without the knowledge of p and q.

The module n and the encryption exponent k form the public key.

Next, we need the decryption exponent l. The decryption exponent l is the multiplicative inverse element with respect to $\varphi(n)$. Thus, $k \cdot l \equiv 1 \pmod{\varphi(n)}$. To calculate l, many methods exist, for example, a slightly modified Euclidean algorithm, or Mathematica:

```
l=PowerMod[k,-1,n]
```

The primes p and q as well as the decryption exponent l form the private, secret key.

In fact, only the (secret) decryption exponent l and the (public) module n are needed for decrypting, so that one also speaks of the key pairs (k, n) (public) and (l, n) (private). Once the encryption algorithm has been established, the values $\varphi(n)$ and the primes p and q are no longer required for decryption and can be erased again (for security).

Now we come to the actual processes of encryption and decryption of messages:

Our message is converted to a number m as described above (ASCII codes). **This number m should be less than our modulus n: $m < n$.**
If this condition does not hold, then the message must be split into several blocks m_i, so that $m_i < n$ holds again.

Encryption is done simply by computing $r = m^k \pmod{n}$ using the encryption exponent k. Most implementations use the value 65537 (the 5th Fermat number) for k.

Insufficient values for k reduce the security of the process and make the encryption vulnerable. This encrypted value r is sent to the recipient.

Decrypting is simply done by computing $m = r^l \pmod{n}$ using the (inverse) decryption exponent l.

In the original article of Rivest, Shamir, and Adleman[70], the additional condition is specified that m and n must be coprime to each other (in which case the following relation holds; $m^{\varphi(n)} \equiv 1 \pmod{n}$), but the RSA method also appears to work if $\gcd(m, n) \neq 1$ (i.e. if m and the RSA module n have a common divisor). The assumption that m and n are mutually coprime just simplifies the proof of the validity of the RSA method.

The proof that this algorithm works is simple:

$$\boldsymbol{r^l = \left(m^k\right)^l = m^{kl}, k \cdot l \equiv 1 \pmod{\varphi(n)}}$$

[70] *A Method for Obtaining Digital Signatures and Public-Key Cryptosystems*, p. 7 (1978)

From this, it follows that there is an integer s such that

$$\boldsymbol{k \cdot l = s \cdot \varphi(n) + 1}$$

A few transformations are sufficient to show that encryption of m and subsequent decryption again yields m:

$$\begin{aligned}
&\mathrm{r}^{\mathrm{l}} (\mathrm{mod\ n}) = \mathrm{m}^{\mathrm{kl}} \ (\mathrm{mod\ n}) \\
&= \mathrm{m}^{\mathrm{s}\cdot\varphi(\mathrm{n})+1} \ (\mathrm{mod\ n}) = \mathrm{m}\left(\mathrm{m}^{\varphi(\mathrm{n})}\right)^{\mathrm{s}} (\mathrm{mod\ n}) \\
&= \mathrm{m} \ (\mathrm{mod\ n}), \mathrm{da\ m}^{\varphi(\mathrm{n})} \equiv 1 \ (\mathrm{mod\ n}) \\
&= m
\end{aligned}$$

Commonly used methods employ in addition various padding techniques described in the relevant specifications. Padding means that additional information (possibly also random elements or information about the length of the text) is attached to the text to be encrypted in order to increase the security of the method. Common padding methods are e.g. ‘PKCS#1'', ‘OAEP’ (Optimal Asymmetric Encryption Padding), or ‘SSLV23’. PKCS1 and SSLV23 add 11 additional bytes to the data block to be encrypted, the ‘OAEP’ procedures even adds 41 bytes. Further details will not be given here. If the reader wishes to go into the matter in greater detail, sources from the Internet are recommended.[71] The software Mathematica supports the PKCS1 padding process for encryption and decryption. Note that for the modified message m' it is imperative that $\boldsymbol{m' < n}$.

[71] https://de.wikipedia.org/wiki/RSA-Kryptosystem, http://people.csail.mit.edu/rivest/Rsapaper.pdf, http://www.di-mgt.com.au/rsa_theory.html

14.2 THE SECURITY OF THE RSA METHOD

In the practical implementations of the RSA encryption method, some additional features are built in that make the transfers even more secure. However, the security of the method is based on the fact that the public key (RSA module) with the number n representing the product of two large prime numbers cannot be decomposed into the two prime factors by the currently known factorization algorithms. As long as this is impossible, $\varphi(n)$ and the decryption exponent l cannot be calculated either.

This is where explosives once again are hidden: no one (not even specialists in this field) can say whether a fast factorization algorithm will be found in the future. A factorization algorithm that worked in polynomial time would bring about a total collapse in security. At present (as of May 2016), it is not known whether or not a sufficiently fast algorithm might exist. This is remarkable because, in other fields, mathematicians have succeeded in establishing whether or not fast algorithms are possible for solving entire classes of problems.

In the case of the factorization problem, no such proof has been found, so it is perfectly conceivable that such an algorithm is out there somewhere waiting to be discovered (see 10.3). Our entire online banking system would (literally) collapse if hackers gained access to such an algorithm!

Note: a fast method for calculating $\varphi(n)$ or $\sigma(n)$ would have the same consequences (20.9.3.2). However, the calculation methods known to date (as of May 2016) are of the same complexity as the factorization problem.

Another danger comes from the ever-increasing speed of computer hardware. With the computer hardware we have now, we can rule out the possibility of keys of the length in current use being cracked, as the computing time required would equate to the entire age of the universe. If, however, some day the highly touted quantum computers were to become a reality (to which end an intensive research effort is underway), these would pose a threat to the security of RSA encryption. The author is not aware whether the possibility of increasing the depth of encryption in the RSA algorithm to reduce the risk of ‘hacking’ by quantum computers is also being investigated …

14.3 COMPUTING EXAMPLES OF RSA ENCRYPTION AND DECRYPTION

Here are a few simple examples. The colours indicate the different areas ('public' in blue, 'secret' at the receiver side in red, 'secret' at the receiver and transmitter side in green):

<u>Example 1:</u>
The sender wants to send a secret message to the recipient that only consists of the number 1115:
$\mathbf{mSource = 1115}$
The receiver selects two suitable prime numbers:
$\boldsymbol{p = 47}$ and $\boldsymbol{q = 59}$
This results in the following module, which is communicated unencrypted to the sender:
$\mathbf{pubModulus = pq = 2773}$
The number of numbers being coprime to pubModulus is:
$\text{privModulus} = \varphi(\text{pubModulus}) = (p-1)*(q-1) = 2668$
The receiver chooses a suitable encryption exponent and sends it (unencrypted) to the sender:
$\mathbf{pubExponent = 17}$
The receiver also calculates the multiplicative inverse decryption exponent using φ:
$\mathbf{privExponent} = \text{PowerMod}[\text{pubExponent}, -1, \text{privModulus}] = \mathbf{157}$

The sender encrypts mSource to mCrypt and sends mCrypt to the recipient:
$\mathbf{mCrypt} = \mathbf{mSource}^{\mathbf{pubExponent}} \,(\text{mod } \mathbf{pubModulus}) = \mathbf{1379}$
The recipient finally decrypts mCrypt to mSource:
$\mathbf{mSource} = \mathbf{mCrypt}^{\mathbf{privExponent}} (\text{mod } \mathbf{pubModulus}) = \mathbf{1115}$

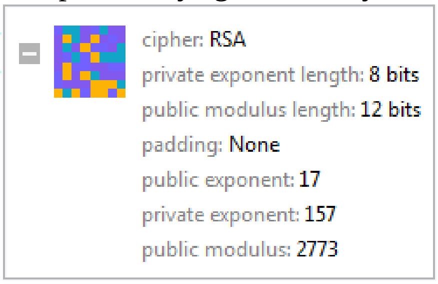
It is easy to crack the decrypting exponent, 'privExponent' by calculating the prime factor decomposition of the module with the factors p and q, and then using privModulus to reconstitute privExponent.

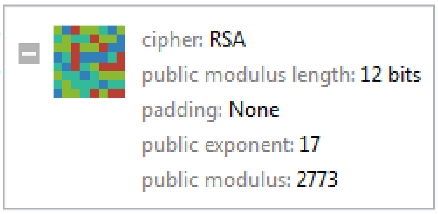
Mathematica:
```
(*very simple example of RSA encryption*)
(*############### implement coding mechanism ######################*)
(*choose two different prime numbers:*)
p=47; q=59;
If[!PrimeQ[p]||!PrimeQ[q],Print["Error: p or q not Prime!"];Exit[];]
(*Compute public module and phi()*)
pubModulus=p*q; privModulus=(p-1)*(q-1); (*=EulerPhi[pubModulus]*)
(*pubExponent can be chosen freely, must be between 3 and privModulus
and coprime to privModulus*)
pubExponent=17;
If[pubExponent >= privModulus||pubExponent<3,Print["Error: pubExponent
> privModulus!"];Exit[];]
If[!CoprimeQ[pubExponent,privModulus],Print["Error: pubExponent not
coprime to privModulus!"];Exit[];]
(*compute private exponent: inverse of public exponent*)
privExponent=PowerMod[pubExponent,-1,privModulus];
(*############## Encode and Decode messages:#####################*)
(*this is the message to be encoded:*)
mSource=1115
If[mSource >=pubModulus,Print["Error: message bigger than module (use
bigger primes)!"];Exit[];]
```

```
(*encode: *)
mCrypt=PowerMod[mSource,pubExponent,pubModulus]
(*decode:*)
mSource1=PowerMod[mCrypt,privExponent,pubModulus]
If[mSource!=mSource1,Print["Error: rSA Coding/Encoding failed"]];
(*########## Hacking the module:##############################*)
Print["Try to hack RSA module…"];
pqHacked=FactorInteger[pubModulus];
pH=First[pqHacked[[1]]];qH=First[pqHacked[[2]]];
privExponentH=PowerMod[pubExponent,-1,(pH-1)*(qH-1)]
If[privExponentH==privExponent,Print["Hack of RSA module
succeeded!"]];
```

Example 2 (similar to example 1)

The sender wants to send a secret message to the recipient that only consists of the number 1115. In this example, however, we use the Mathematica functions `PublicKey[]`,`PrivateKey[]`, `Encrypt[]`, as well as `Decrypt[]`. No padding algorithm has been used (specified by: "None").
The program used can be found in the Appendix (20.11.11).
The private and public keys generated by Mathematica are as follows:

PrivateKey[]

PublicKey[]

The original number reads: **1115**
The prime numbers and module used to generate the keys are:
$\boldsymbol{p = 47}$; $\boldsymbol{q = 59}$; **public Modulus=2773**
The encrypted number reads: **1379**

Example 3

The sender wants to send a secret message to the recipient that only consists of the text "OK". In this example we use Mathematica's built-in functions `GenerateAsymmetricKeyPair[]`, `Encrypt[]`, as well as `Decrypt[]` together with the "PKCS1" padding algorithm using a key length of 97 bits.
The program used can be found in the Appendix (20.11.11).

The private and public keys generated by Mathematica are as follows:

```
⟨|"PrivateKey" → PrivateKey[ ⊟ cipher: RSA
                                 public modulus length: 97 bits
                                 private exponent length: 93 bits
                                 padding: PKCS1
                                 public exponent: 65537
                                 public modulus: 12202433704 ≪7≫ 277596949541
                                 private exponent: 76245427803 ≪5≫ 502629985493 ], "

PublicKey" → PublicKey[ ⊟ cipher: RSA
                          public modulus length: 97 bits
                          padding: PKCS1
                          public exponent: 65537
                          public modulus: 12202433704 ≪7≫ 277596949541 ]|⟩
```

Here in detail are the exact values (output of the program)
Original string to be encoded: “**OK**”
Original string as number**: 20299**
Original text as a byte array including 10 bytes padded by PKCS1 algorithm:
{2, 32, 69, 224, 233, 133, 242, 219, 235, 0, **79, 75**} **“\.02 Eàé…òÛë\.0**0**OK**”
Public Modulus: **122024337043892852277596949541**
Private Exponent: **7624542780333828502629985493**
Private Modulus (phi[publicModulus]): 122024337043892092448561992492
Encrypt. object (data) {**0, 42, 120, 153, 109, 62, 0, 217, 150, 54, 211, 165, 4**}
Encrypt. Number:**13144166048085041547004060932**
Decryption-result (using encrypted Byte data as parameter restores original Bytes:
{**79, 75**}
Decryption result (using encrypted Object as parameter restores original String):
“OK” (20299)

<u>Example 4</u>
A curious hacker is in possession of an encrypted message as well as of the associated public key and would like to decipher the message even though he does not have the private key (with the private exponent). In this example, we use the functions `PrivateKey[]`, as well as `Decrypt[]` implemented in Mathematica together with the "PKCS1" padding method at a key length of 192 bits.
The program used can be found in the Appendix (20.11.11).
Let's suppose someone has generated the following public RSA key (for example, by using the Mathematica function `GenerateAsymmetricKeyPair[]`). Let us suppose further that he has released the module, the public encryption exponent, and an encrypted message:

```
pubExponent = 65537;
pubModulus = 5369695965139088101081485235567478142438728289315726900871;
mCrypt = 1917971481256834478883961041086543933343882914074934636133;
```

Our hacker needs only a few lines of Mathematica program code to hack the key:
He factorizes the module into the prime numbers pH and qH:

```
FactorInteger[53696959651390881010814852355674781424387282893157269008 71]
{{68357071940820194611682396513,1},{78553627484042565312533006567,1}}
```

He calculates phi[] (aka ‘private module’:)
$\text{privModulus} = (\text{pH} - 1) * (\text{qH} - 1)$:
53696959651390881010814852354205674430138655293915114977 92
He calculates the private exponent:
$\text{privExponent} = \text{PowerMod}[\text{pubExponent}, -1, \text{privModulus}]$;
48449918596604924955559678719826115722071335329586073424 01
and finally generates a new private key, with which he can decrypt the message:

```
privKey=PrivateKey[<|"Cipher"->"RSA","Padding"-
>"PKCS1","PublicExponent"->pubExponent,"PrivateExponent"-
>privExponent,"PublicModulus"->pubModulus|>]
```

Finally, he is able to decrypt the message:

```
bCryptArray=ByteArray[IntegerDigits[mCrypt,256]];
decryptedByteArray=Normal[Decrypt[privKey,bCryptArray]];
decryptedString=FromCharacterCode[decryptedByteArray]
```

et voilà! Here is the deciphered message:
“Elvis lives!”

The Mathematica program used can be found in the Appendix (20.11.11).
Note: the computing time is about 30 seconds on a 2.6 GHz quad core computer.

Example 5
A further example of how a private key with a key length of 2048 bits can be hacked and such an encrypted message (about 256 bytes) can be read without knowing the private key can also be found in the Appendix (20.11.11).

15 PRIME NUMBERS IN MUSIC

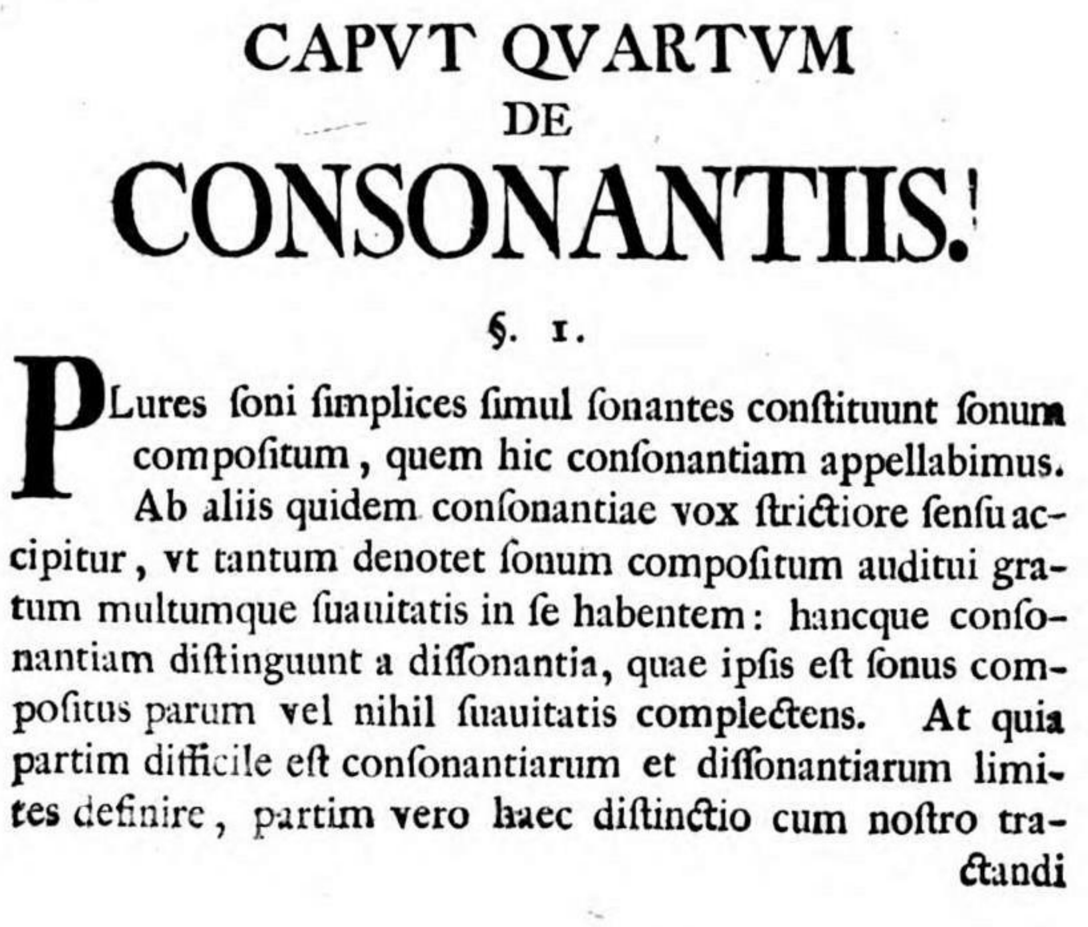
15.1 EULER'S THEORY OF CONSONANCE AND THE *GRADUS SUAVITATIS*

As every musician knows, musical intervals and chords can sound either consonant or dissonant, with a flowing boundary between 'consonant' and 'dissonant' that probably also depends on the taste of the historical epoch of the music. Nevertheless, the mathematician Leonhard Euler (1707-1783) – and the attentive reader will have noticed that this is not the first time we have encountered this gentleman … – was convinced it was possible to formulate a mathematical definition of harmony, or more precisely, of the 'degree' of harmony (aka 'euphony').

Euler found a formula that indicates the degree of euphony as a natural number, and called the number derived from the formula the '*Gradus Suavitatis*'.[72] In the calculation of the *Gradus Suavitatis,* prime numbers (what else?) play a special role. Euler uses the concept of consonance for arbitrary (and not only euphonious) 'composite sounds'.

By 'composite sounds', we mean simultaneously sounding notes, whereby the notes should be tuned in pure temperament and thus have mutually rational (fractions of natural numbers) ratios. Although the *Gradus Suavitatis* can be computed for arbitrary numbers, it was in former times only applied musically to intervals the ratios of which can be described with the primes $2, 3,$ and 5. In his later writings, however, Euler pleads for the introduction of the prime number 7 into 'musical arithmetic'.

[72] Euler, Leonhard:
Tentamen Novae Theoriae Mvsicae Ex Certissimis Harmoniae Principiis Dilvcide Expositae Petropolis 1739



CAPVT QVARTVM
DE
CONSONANTIIS.

§. 1.

PLures ſoni ſimplices ſimul ſonantes conſtituunt ſonum compoſitum, quem hic conſonantiam appellabimus. Ab aliis quidem conſonantiae vox ſtrictiore ſenſu accipitur, vt tantum denotet ſonum compoſitum auditui gratum multumque ſuauitatis in ſe habentem: hancque conſonantiam diſtinguunt a diſſonantia, quae ipſis eſt ſonus compoſitus parum vel nihil ſuauitatis complectens. At quia partim difficile eſt conſonantiarum et diſſonantiarum limites definire, partim vero haec diſtinctio cum noſtro tractandi

Figure 128. Opening of the 4th chapter of Leonhard Euler's book *Tentamen novae theoriae musicae*

In contrast to Pythagorean tuning, in which all the scales occurring are constructed by using fifths (which are projected into the octave space, if necessary), 'pure' tuning only uses the number ratios 5/4 and 6/5 for the major and minor thirds. If we wish to assign a 'pure' numeral ratio to all the twelve semitones of the octave, we discover a certain ambiguity exists for the intermediate notes, because only the root, fourth, fifth, major / minor third, and the major / minor sixth are precisely defined. If we take the root note C, the seven precisely defined notes are:

C, Eb, E, F, G, Ab, A (having the ratios $1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}$).

The remaining notes can be chosen in different ways, depending on the note from which they are constructed. Here is an example: the note Bb can be constructed by starting from F: the two consecutive fourths starting from the C then give the value $\frac{4}{3} \cdot \frac{4}{3} = \frac{16}{9}$. Bb can, however, also be constructed starting from G. Fifth and minor third starting from C then result in $\frac{3}{2} \cdot \frac{6}{5} = \frac{9}{5}$.

The most commonly used pure scale that is closest to the tempered tone scale, is:

Chromat. scale	**C**	**Db**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**c**
Freq. ratio	1/1	16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	5/3	9/5 (16/9)	15/8	2/1

From this ambiguity, we recognize that the application of the *Gradus Suavitatis* to the 'equal temperament' tuning system we use these days (in which the frequency ratios of the chromatic scale are simply the values of a geometric sequence with the factor $q = \sqrt[12]{2}$) is problematic.

If we define an interval in pure tuning as the frequency ratio $\frac{p}{q}, p, q \in \mathbb{N}$ (see table), and build the so-called Euler exponent $E = \mathrm{lcm}(\mathrm{p}, \mathrm{q})$, then the ***Gradus Suavitatis G*** is defined as follows:

$$G(E) = 1 + \sum_{i=1}^{\boldsymbol{\omega(E)}} e_i\,(p_{\boldsymbol{n_i}} - 1), \text{where } E = \mathrm{lcm}(p,q) = \prod_{i=1}^{\omega(E)} {p_{n_i}}^{e_i} \tag{158}$$

Here $\mathrm{lcm}(p,q)$ is the least common multiple of p and q; ${p_{n_i}}^{e_i}$ are the occurring prime factors with their multiplicities (as exponents), n_i the indices of the occurring primes; $\omega(E)$ denotes the number of different primes. Sometimes the *Gradus Suavitatis* is also referred to in the literature with the symbol $\Gamma(p,q)$. We prefer to use the symbol G to avoid confusion with the gamma function $\Gamma(\mathrm{x})$. The *Gradus Suavitatis* can also easily be applied to chords consisting of three notes ('triads') or more. In this case, E is simply calculated as $E = \mathrm{lcm}(q_1, q_2, q_3, \ldots)$ where the integers $q_1, q_2, q_3, \ldots$, represent the frequency ratios with respect to the lowest occurring note.

The *Gradus Suavitatis* can thus also be calculated for combinations of arbitrarily many notes. For a single argument n, $E = n$. In this case, the *Gradus Suavitatis* is merely an arithmetical function depending solely upon the positive integer variable n. Furthermore, we define: $G(1) = 1$. Note: we also assume that the ratios $(q_1, q_2, q_3, \ldots)$ are minimal, i.e. 'simplified' as far as possible, since otherwise the *Gradus Suavitatis* would yield higher values, thus the notation $G\left(\frac{3}{2}\right)$ is somewhat imprecise.

Since in this case we want to calculate the *Gradus Suavitatis* for a two-note chord (or 'harmonic interval') with the frequency ratios 1 (root) and $\frac{3}{2}$ (fifth), it would be more precise mathematically to write:

$G\left(1, \frac{3}{2}\right) = G(2,3)$.

In order to avoid confusion, it is best to calculate first the Euler exponent E for the chord or interval to be examined, and then the *Gradus Suavitatis* for the integer argument E.

Here are a few examples:

$G(2) = 1 + 1 \cdot (2-1) = 2$, $G(3) = 1 + 1 \cdot (3-1) = 3$, $G(4) = 1 + 2 \cdot (2-1) = 3$

$G\left(\frac{3}{2}\right) = G(3,2) = G(6) = 1 + 1 \cdot (2-1) + 1 \cdot (3-1) = 4$ (fifth)

$G\left(\frac{2}{3}\right) = G(2,3) = G(6) = 4$ (fourth)

$G\left(\frac{12}{5}\right) = G(12,5) = G(60) = 1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 4 = 9$ (minor tenth)

Major triad: $q_1 = 1, q_2 = \frac{5}{4}, q_3 = \frac{3}{2}, \Rightarrow E = \mathrm{lcm}(4,5,6) = 60, G(60) = 9$

1st inversion: $q_1 = 1, q_2 = \frac{6}{5}, q_3 = \frac{8}{5}, \Rightarrow E = \mathrm{lcm}(5,6,8) = 120, G(120) = 10$

2nd inversion: $q_1 = 1, q_2 = \frac{4}{3}, q_3 = \frac{5}{3}, \Rightarrow E = \text{lcm}(3,4,5) = 60, G(60) = 9$
Minor triad: $q_1 = 1, q_2 = \frac{6}{5}, q_3 = \frac{3}{2}, \Rightarrow E = \text{lcm}(10,12,15) = 60, G(60) = 9$
1st inversion: $q_1 = 1, q_2 = \frac{5}{4}, q_3 = \frac{5}{3}, \Rightarrow E = \text{lcm}(12,15,20) = 60, G(60) = 9$
2nd inversion: $q_1 = 1, q_2 = \frac{4}{3}, q_3 = \frac{8}{5}, \Rightarrow E = \text{lcm}(15,20,24) = 120, G(120) = 10$

The *Gradus Suavitatis* is greater, the greater the dissonance of the intervals considered. Smaller Gradus-Suavitatis values mean a higher degree of consonance (greater euphony)…

However, a clear unique assignment from the *Gradus Suavitatis* to the categories 'consonant' and 'dissonant' is problematic, according to the original text:

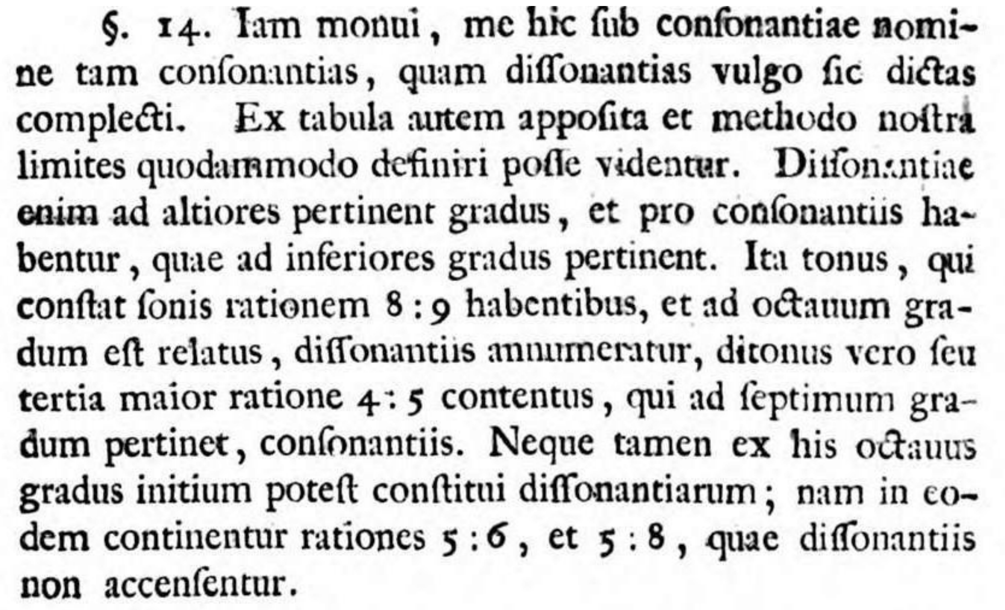

§. 14. Iam monui, me hic ſub conſonantiae nomine tam conſonantias, quam diſſonantias vulgo ſic dictas complecti. Ex tabula autem appoſita et methodo noſtra limites quodammodo definiri poſſe videntur. Diſſonantiae enim ad altiores pertinent gradus, et pro conſonantiis habentur, quae ad inferiores gradus pertinent. Ita tonus, qui conſtat ſonis rationem 8 : 9 habentibus, et ad octauum gradum eſt relatus, diſſonantiis annumeratur, ditonus vero ſeu tertia maior ratione 4 : 5 contentus, qui ad ſeptimum gradum pertinet, conſonantiis. Neque tamen ex his octauus gradus initium poteſt conſtitui diſſonantiarum; nam in eodem continentur rationes 5 : 6, et 5 : 8, quae diſſonantiis non accenſentur.

Figure 129. Chapter 4, §14 from Leonhard Euler's book *Tentamen Novae Theoriae musicae*

Translation into German (Mitzler):

> *"…Die Dissonanzen gehören zu höheren Graden, und für Konsonanzen werden diejenigen gehalten, die zu tieferen Graden gehören. So wird der Ganzton, der aus Tönen im Verhältnis 8 : 9 besteht und zum achten Grad gehört, zu den Dissonanzen gezählt, der Ditonus aber (die große Terz), der im Verhältnis 4 : 5 enthalten ist, welcher zum siebten Grad gehört, wird zu den Konsonanzen gezählt. Und trotzdem kann aus diesem achten Grad nicht der Anfang der Dissonanzen festgelegt werden; denn in demselben sind die Verhältnisse 5 : 6 und 5 : 8 enthalten, welche nicht zu den Dissonanzen gerechnet werden."*

Here are a few Mathematica programs that show calculations of the *Gradus Suavitatis*:

```
(*Computing Euler's Gradus Suavitatis *)
(*works for any number of arguments n >1, arguments can be rational:*)
eulerExp[q__]:=Module[{exp,qList},
qList=List[q];
```

```
If[Length[qList]==1&&IntegerQ[qList[[1]]],
exp=qList[[1]], exp=Apply[LCM,qList]/Apply[GCD,qList]];
Return[exp];
];
(*works for 1 argument. argument can be rational:*)
eulerExp[r_]:=eulerExp[Numerator[r],Denominator[r]];
(*works for 1 integer argument, number-theoretic version:*)
gas[n_]:=Module[{s=FactorInteger[n]},1+Sum[s[[k,2]]*(s[[k,1]]-
1),{k,Length[s]}]];
(*works for 1 or 2 arguments can be rational:*)
gs2[p_,q_]:=gs[LCM[p,q]/GCD[p,q]];
gs2[x_]:=gs2[Numerator[x],Denominator[x]];
(*works for any number of arguments n > 1, arguments can be
rational:*)
gsn[q__]:=Module[{ exp, retValue},
exp=eulerExp[q];
s=FactorInteger[exp];
retValue=1+Sum[s[[k,2]]*(s[[k,1]]-1),{k,Length[s]}];
Return[retValue];
];
```

15.1.1 MATHEMATICAL PROPERTIES OF THE *GRADUS SUAVITATIS*

1) $G(1) = 1$
2) $G(p) = p$, if $p \in \mathbb{P}$
3) $G(pq) = G(p) + G(q) - 1$ ('quasi-logarithmic')
4) $G(2^n) = n + 1$
5) $G(p^n) = (p-1)n + 1$
6) $G\left(\frac{p}{q}\right) = G\left(\frac{q}{p}\right)$ or $G(p,q) = G(q,p)$
7) $G(q_1, q_2, q_3, \dots q_n) = G\left(q_{i_1}, q_{i_2}, q_{i_3}, \dots q_{i_n}\right)$, where $\mathrm{i}_1, i_2, i_3, \dots i_n$

   represent all possible permutations of $1,2,3, \dots n$

8) $G\left(\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}, \dots \frac{1}{q_n}\right) = G(q_1, q_2, q_3, \dots q_n)$

For major and minor triads (let X be the three frequency ratios of a major triad, Xm the three ratios of a minor triad, the indices 1 and 2 each represent the first and second inversions, respectively):

9) $G(X) = G(X_2) = G(Xm) = G(Xm_1)$
10) $G(X_1) = G(Xm_2)$

According to Euler, a sequence of frequency ratios (which build up a chord) can be continued in such a way that the *Gradus Suavitatis* value remains the same. Euler calls this the 'complete consonance'. For the major triad the complete consonance gives the following frequency ratios:

$$1:2:3:\mathbf{4}:\mathbf{5}:\mathbf{6}:\mathbf{10}:\mathbf{12}:\mathbf{15}:20:30:60\ \dots$$

Note that in this complete consonance there are major triads (**4: 5: 6**, for example, corresponds to C-E-G) and minor triads (**10: 12: 15** corresponds to E-G-H). Note also that up to the sixth term, this sequence is identical to the overtone sequence (harmonics).

This corresponds quite well to the modern convention of jazz harmony whereby the major triad is routinely accompanied by a major seventh.

Here is a plot of the *Gradus Suavitatis* that bears a striking similarity to Figure 43 (integer logarithm):

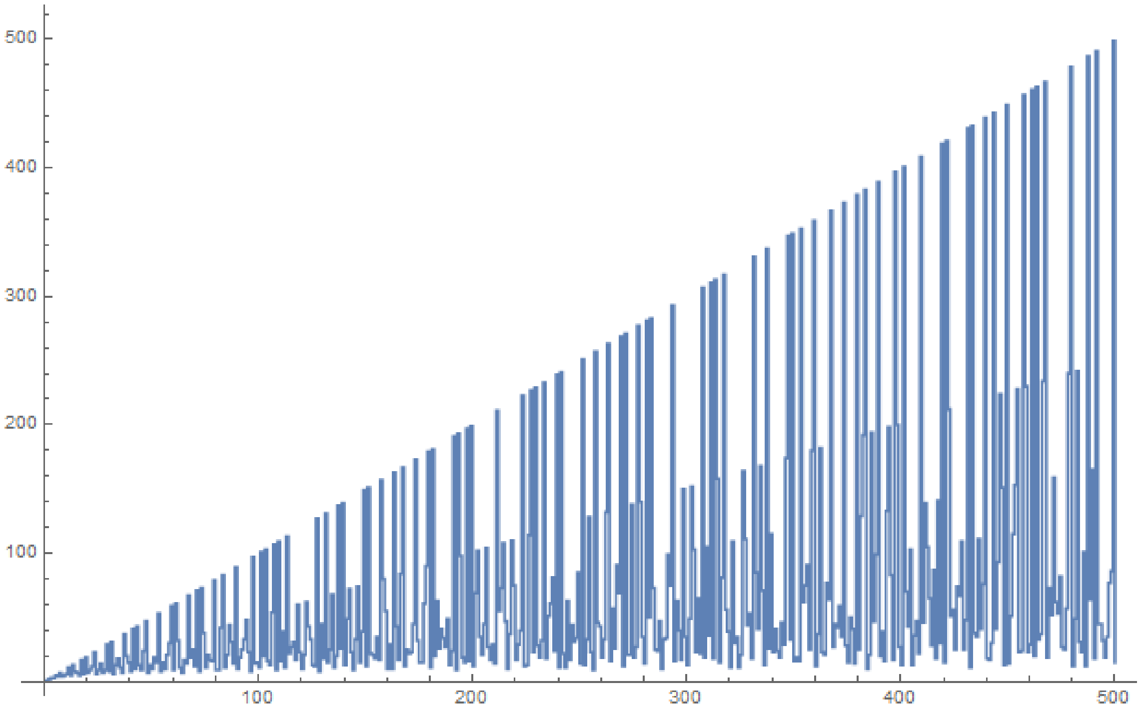


Figure 130. Leonhard Euler's *Gradus Suavitatis* in the range 1 to 500

15.1.2 'ADJUSTED LISTENING' TO COMPLEX OR IRRATIONAL INTERVALS

The method of the *Gradus Suavitatis* is, of course, only a rough approximation and reflects the actual sense of hearing (consonant or dissonant) only in a limited way, and only for proportions in which small numbers (composed of the primes 2,3 and 5) occur.

This is evident from the fact alone that inaudibly 'small' detune values in the calculation of the GS (the calculation requires, of course, an approximation by rational numbers) would lead to huge *Gradus Suavitatis* values, although they would still be perceived as 'pleasant'. As an example, we take an A major triad with 440 Hz for the note A:

The frequency ratios of the A major triad (A-C#-E) are 440:550:660. The *Gradus Suavitatis* for this major triad is (as we have seen):

$G(440{,}550{,}660) = G(4{,}5{,}6) = 9$. Let us now consider the note E, which is inaudibly detuned by 1 Hz, at 661 Hz instead of 660 Hz. There is no change perceptible by the human ear, but the *Gradus Suavitatis* of the slightly detuned triad is $G(440.550.661) = 682$!

Euler was therefore of the opinion that the human soul has the capacity to 'justify' such small discrepancies: in other words, a tempered fifth with an irrational frequency ratio of $\left(\sqrt[12]{2}\right)^7$ will simply be adjusted by our souls to approximately the same value of $\frac{3}{2}$.

It's fortunate that $\left(\sqrt[12]{2}\right)^7 = 1.49831$ is so close to the value of the pure fifth ($\frac{3}{2} = 1.5$)!

If this were not the case, we would not be able to make music (at least Western music) using the tempered 12-note scale! Which in turn raises the question whether it really is purely due to the coincidence that the tempered fifth is so close to the pure fifth. Some people may not be entirely comfortable with the idea that we have (random) chance to thank for anything as sublime as the works of J. S. Bach.

15.2 PRIME NUMBERS AS RHYTHMIC PATTERNS

If we translate the differences between the prime numbers into temporal differences, we get a rhythmic pattern.

We use the Sieve of Eratosthenes, 'sieve' the first 50 prime numbers (2 to 229) and interpret the X-axis as the time axis. The Y-axis is interpreted as the pitch. In order to reach an 'audible' range, we multiply the relevant prime numbers by the frequency factor with the value 110 Hz so that the lowest notes (which create a 2-beat rhythm) are located at 220 Hz.

This corresponds to A3 (employing the international convention) or small 'a' (employing the convention used in German-speaking countries). The highest notes in this representation then lie at approximately 20000 Hz. Thus the following diagram, in which each 'prime number rhythm' is marked by a different colour, results:

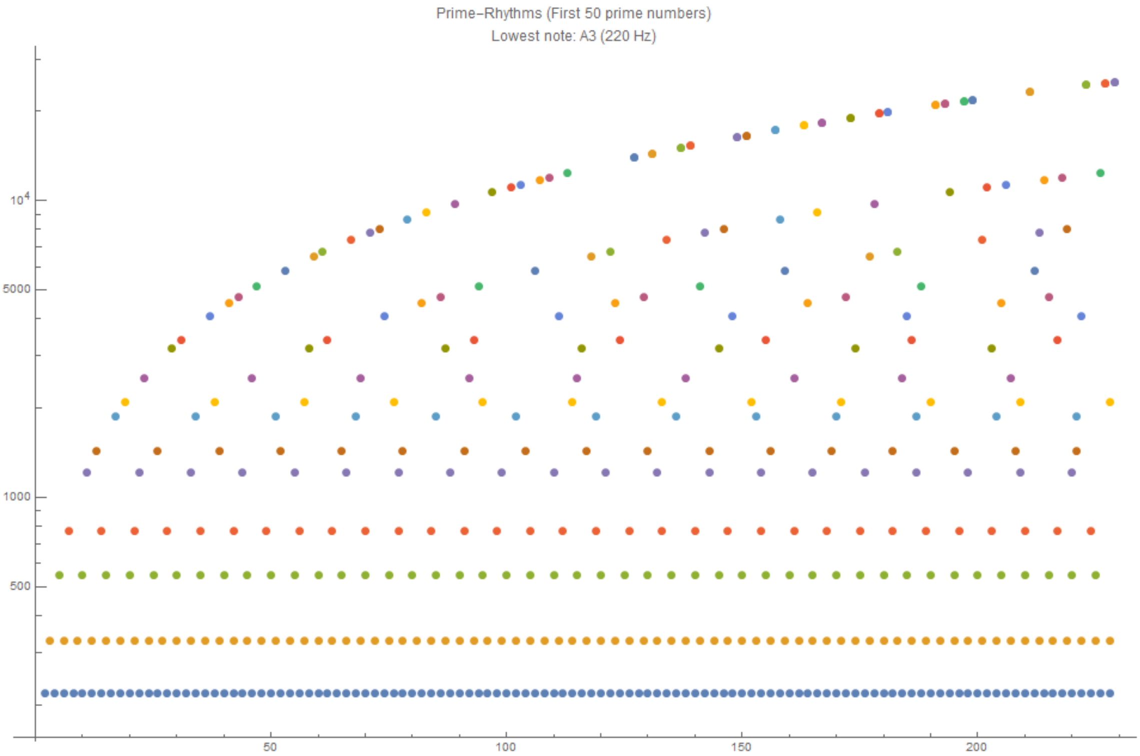


Figure 131. Prime number rhythms, from the first 50 prime numbers with A3 as lowest note

```
Mathematica:

(*Generate a list with 50 different sequences of
(frequency,Primenumber)-Pairs, using A2=110Hz as the base-frequency*)
tab=Table[{j,Table[{i,110*Prime[j]},{i,Prime[j],Prime[50],Prime[j]}]},
{j,1,50}];
ListLogPlot[Table[tab[[k]][[2]],{k,1,50}],PlotLabel->"Prime-Rhythms
(First 50 prime numbers)\nLowest note: A3 (220 Hz)", ImageSize->Large]
```

With Mathematica, it is very easy to translate these tables into music. We use the table of Figure 131 and generate a ‘prime number’ song 46 seconds in duration:

```
Mathematica:
(*needs the prime sound-library, to be found in the Appendix *)
tab=Table[{j,Table[{i,110*Prime[j]},{i,Prime[j],Prime[50],Prime[j]}]},
{j,1,50}];
sortedTab=Sort[Flatten[Drop[tab,None,1],2]];
noteList=createNoteListFromSortedTable[sortedTab];
song1=Sound[{"Percussion",Table[SoundNote[noteList[[k]][[2]]-
10,0.2],{k,1,Length[noteList]}]},{0,46}];
song2=Sound[{"Marimba",Table[SoundNote[noteList[[k]][[2]],0.2],{k,1,Le
ngth[noteList]}]},{0,46}];
primenumberSong=Sound[{song1,song2}];
Export["C:/primes/Sounds/primenumberSong46Sec.mid",primeNumberSong];
```

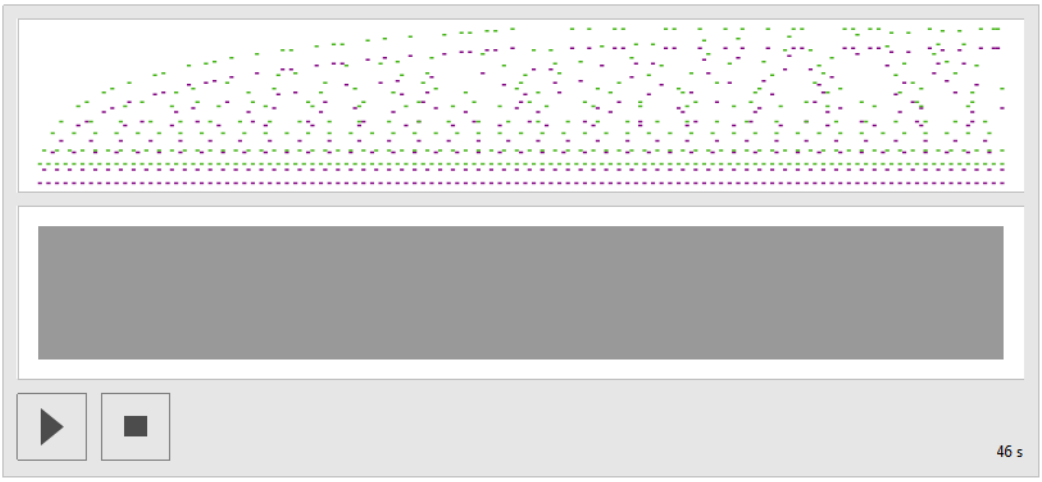


Figure 132. Mathematica sound object (prime number song)

The idea of interpreting prime numbers as rhythmic patterns was originally that of Peter Neubäcker, head of the company Celemony and inventor of the music software 'Melodyne'.[73]

With Melodyne it is also very easy to create 'songs' with primes. Melodyne has the advantage over Mathematica that the result can be exported as real sound (and not merely in MIDI format). In addition, the pitch of the generated note events can be set much more precisely.

Here is a screen shot of Melodyne with a prime number arrangement:

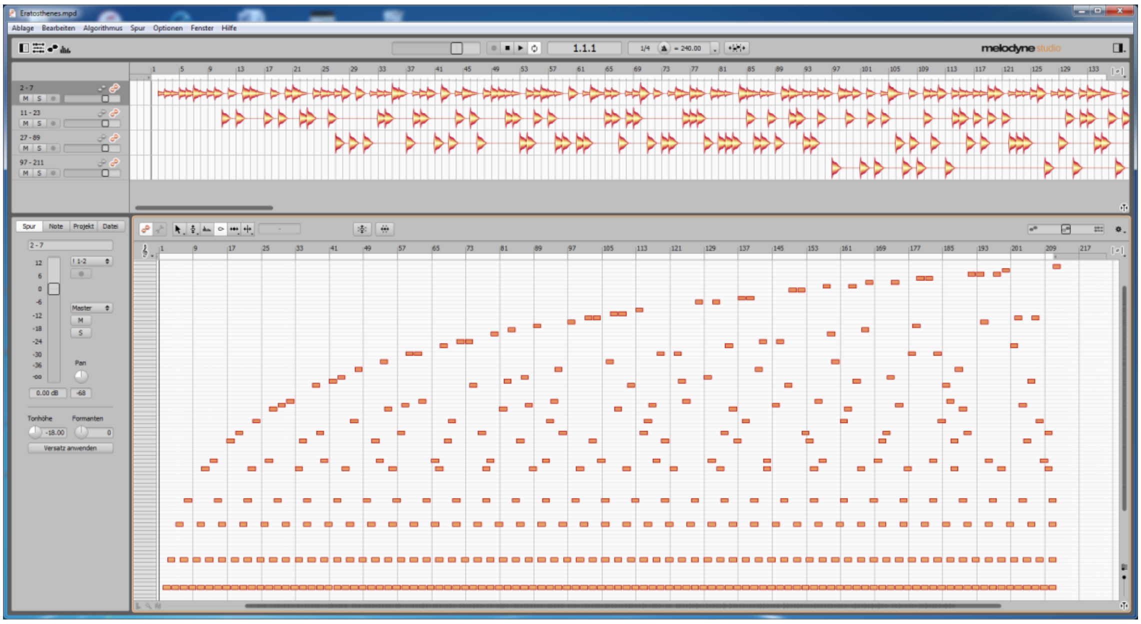


Figure 133. Melodyne creates prime number rhythms using the Sieve of Eratosthenes

73 http://www.celemony.com

16 PRIME NUMBERS IN POETRY

16.1 HAIKUS AND TANKAS

荒海や
佐渡によこたふ
天河

ura umi ya
sado ni yokotau
ama no gawa

Tosende See.
nur die Milchstraße reicht
zur Insel Sado hinüber.

Turbulent the sea—
across to Sado stretches
the Milky Way

古池や
蛙飛び込む
水の音

furu ike ya
kawazu tobikomu
mizu no oto

Der alte Weiher:
Ein Frosch springt hinein.
Oh! Das Geräusch des Wassers

Ah! The ancient pond
As a frog takes the plunge
Sound of the water

Matsuo Bashô (松尾芭蕉), 1644-1695

道のべに清水流るる柳影
しばしとてこそ立ちどまりつれ

Michi no be ni
Shimizu nagaruru
Yanagikage
Shibashi tote koso
Tachidomaritsure

Wo am Wegerand ein Bach
Fließt mit glasklarem Wasser
Und eine Weide steht,
Da würde ich gerne noch bleiben:
“Ach, nur ein Weilchen”

Along the road
A pure stream flows
In the shade of a willow
Wanting to rest
I paused – and have not left

Saigyō (1118-1190)

What do these three Japanese poems from the 12th and 17th centuries have to do with prime numbers (the first two are haikus; the third one, a tanka)?

It is the form that is reflected in the number of syllables (so-called ‘mores’). This form unfortunately exists only in Japanese, since the translation into other languages results in a different number of syllables.

A haiku consists of three lines (word groups) each with (5-7-5) syllables, thus 17 syllables altogether.

A tanka consists of five lines (word groups) each with (5-7-5-7-7) syllables, a total of 31 syllables.

All numbers of syllables occurring are prime numbers! Haikus and tankas have no rhyme and act mostly (but not always) from nature. Haikus and tankas are meant to convey feelings and moments of experience. In contrast to the outer structure of the rhymes that prevails in occidental poetry, it is the prime number of the syllables that allows each poem to have an individual, exceptional structure.

Daniel Tammet has followed up this subject extensively in his book *Thinking in numbers*[74]. Tammet writes:

"Prime numbers contribute to the haiku form's elemental simplicity. Each word an image calls out for our undivided attention. The result is an impression of sudden, striking insight, as if the poem's objects had been put into words for the very first time…As I think of the complicity between poems and primes, perhaps the only surprise is that we should even find it surprising. Viewed one way, the relationship makes a perfect kind of sense. Poetry and prime numbers have this in common: both are as unpredictable, difficult to define and multiple-meaning as in life…Poems and primes

[74] *Thinking in numbers* p. 189 (United Kingdom, 2012)

are tricky things to recognize. A glance will usually not suffice to tell us if such-and-such a number has factors, or whether a given text contains much meaning…"

Daniel Tammet is one of an estimated 100 'savants' alive today.

He learns new languages within a week and calculates almost as fast as a computer. He also holds the European record for reciting from memory the digits of the number *pi* (22514 decimal digits, as of June 2016).

16.2 SESTINAS

Another type of poem that long ago passed into oblivion, and in which primes also loomed large, is the sestine. The sestine is a verse form comprising six stanzas of six lines each with a final stanza of three lines. The name comes from the word 'sesto' (six). The inventor of the sestine was the French troubadour Arnaut Daniel, who lived in Provence from 1150 to 1200.

Similar to the Japanese haiku, the sestine is not held together by devices like rhyme or symbolism. Unlike the haiku, however, the number of syllables or words per line does not play a big role. In German, however, the iambic verse meter is preferred.

The structure that holds a sestine together is the following: each sestine has a 'core' of six words. The last word of a line must be one of these 6 core words, in fact alternating, until all six core words have been used up, which obviously occurs after six lines. For the next group of 6 lines, the same principle is applied, but with a different order of the core words, each at the end of the line.

It can be seen that the 'power' and the almost musical appeal of the poems lie in this repetition. In the course of the 36-line poem, each core word occurs exactly 6 times. The order in which the core words are permuted within a group of six is complicated and reminiscent of throwing a dice. Let us suppose that our core words are numbered from 1 to 6; then the (ending) core words in the entire poem appear in the following order:

stanza 1:	1, 2, 3, 4, 5, 6
stanza 2:	6, 1, 5, 2, 4, 3
stanza 3:	3, 6, 4, 1, 2, 5
stanza 4:	5, 3, 2, 6, 1, 4
stanza 5:	4, 5, 1, 3, 6, 2
stanza 6:	2, 4, 6, 5, 3, 1

Final stanza: (1, 2), (3, 4), (5, 6)

Note: in the three-line final stanza, there are two key words per line (one at the end and one within the line). The order of the key words in the final stanza can, however, also be different; in the ending lines, every core word must occur exactly once.

Here is an example of a sestine (in German) that the author found on the Internet:[75]

[75] http://www.leselupe.de/lw/titel-Gedichtstrophen--Die-Sestine-100219.htm

Martin Opitz, in "Schäfferey von der Nimfen Hercinie"

Wo ist mein Auffenthalt, mein Trost und schönes Liecht?
Der trübe Winter kömpt, die Nacht verkürtzt den Tag;
Ich irre gantz betrübt umb diesen öden Waldt.
Doch were gleich jetzt Lentz und Tag ohn alle Nacht
Und hett' ich für den Wald die Lust der gantzen Welt,
Was ist Welt, Tag und Lentz, wo nicht ist meine Zier?

Ein schönes frisches Quell giebt Blumen ihre Zier,
Dem starcken Adler ist nichts liebers als das Liecht,
Die süsse Nachtigal singt frölich auff den Tag,
Die Lerche suchet Korn, die Ringeltaube Waldt,
Der Reiger einen Teich, die Eule trübe Nacht;
Mein Lieb, ich suche dich für allem auff der Welt.

So lange bist du mir das liebste von der Welt.
So lange Pales hegt der grünen Weide Zier,
So lange Lucifer entdeckt das klare Liecht.
So lange Titans Glantz bescheint den hellen Tag,
So lange Bacchus liebt den Wein und Pan den Waldt,
So lange Cynthia uns leuchtet bey der Nacht,

Die schnelle Hindin sucht den Hirschen in der Nacht,
Was schwimmt und geht und kreucht, liebt durch die gantze Welt,
Die grimme Wölffin schätzt den Wolff für ihre Zier,
Die Sternen leihen uns zum Lieben selbst ihr Liecht;
Ich aber gehe nun allhier schon manchen Tag,
O Schwester, ohne dich durch Berge, Wildt und Wald.

Was ist, wo du nicht bist? So viel der kühle Waldt
Ein Sandfeldt übertrifft, der Morgen für der Nacht
Uns angenemer ist, der Mahler dieser Welt,
Der Lentz, für Winterlufft, so viel ist deine Zier,
Die Schönheit, diese Lust mir lieber, o mein Liecht,
Als das, so weit und breit bestralt wird durch den Tag.

Der Trost erquickt mich doch, es komme fast der Tag,
Da ich nicht werde mehr bewohnen Berg und Wald,
Da deine Gegenwart und die gewünschte Nacht
Der Treu noch lohnen soll; in dessen wird die Welt
Vergessen ihrer selbst, eh' als ich deiner Zier,
Mein höchster Auffenthalt, mein Trost und schönes Liecht.

Laß wachsen, edler Wald, mit dir mein treues Liecht,
Die liebste von der Welt; es schade deiner Zier,
O Baum, kein heisser Tag und keine kalte Nacht.

The permutated core words at the end of the lines resemble the permutations (periods) of the digits of a cyclic number. Cyclic numbers are generated by division by prime numbers. For example, the following six cyclic numbers 1,4,2,8,5,7 are generated if the number 1 is divided by the prime number 7:

$$1/7 = 0{,}142857\ 142857\ 142857\ \ldots.$$

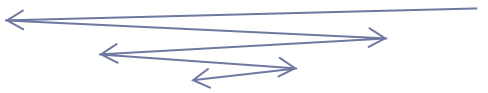
The permutations of these 6 numbers 1,4,2,8,5,7 are generated by multiplying the number 142857 by all the integers $1 \leq n < 7$:

$$
\begin{aligned}
142857 \cdot 1 &= 142857 \\
142857 \cdot 2 &= 285714 \\
142857 \cdot 3 &= 428571 \\
142857 \cdot 4 &= 571428 \\
142857 \cdot 5 &= 714285 \\
142857 \cdot 6 &= 857142
\end{aligned}
$$

The whole thing bears a striking resemblance to the permutations of the core words of the sestine!

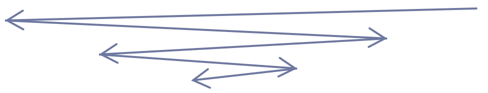
Once again, we go back to the sequence of terminating core words: the construction algorithm of the permutations, which appeared to be complicated at first, turns out in the end to be quite simple and is illustrated by the following zigzag scheme:

1 2 3 4 5 6



yields:

6 1 5 2 4 3



yields:

3 6 4 1 2 5

etc.

The question now arises as to why the number of six verses had such an importance in poetry, and not poems of four stanzas (‘tetrine’) or seven stanzas (‘septine’). The beauty of the sestine, which is based on its form, is that after six iterative applications of the zigzag scheme to the original ordered sequence 1,2,3,4,5,6 the same order 1,2,3,4,5,6 as in the beginning appears again and that the respective core word occurs in every stanza at another line number. We now generalize and demand that the same principle should apply to a 'beautiful n-tine’:

Let our initial sequence of core words be: $1,2,3, \ldots, n$.

If the zigzag scheme is applied $(n-1)$ times, the index of the core word should be different for each iteration step (i.e. a core word may not occur in two different stanzas in the same row (e.g. the 5th row)).

But this is exactly the case for some values of n. For example, a 'septine' would lead to unpleasant ‘word accumulations’ of the respective core word in the 5th line (here the schema of the core word indexes, the number in the i th column indicates the corresponding line in the verse):

stanza 1: 1, 2, 3, 4, **5**, 6, 7
stanza 2: 7, 1, 6, 2, **5**, 3, 4

stanza 3: 4, 7, 3, 1, **5**, 6, 2
stanza 4: 2, 4, 6, 7, **5**, 3, 1
stanza 5: 1, 2, 3, 4, **5**, 6, 7

In addition, the scheme repeats itself after four permutations and not as before, only after 7 permutations.

With the demands of the principle of 'beautiful' n-tines, a more exact investigation gives the following results for values from 3 to n:

'n-tines' are 'beautiful ', if $n \cdot 2 + 1$ is a prime number. That is why 'tritines', 'quintines', sestines, or '11-tines' are beautiful, but not 'quartines' or 'decines' This condition is satisfied for 31 numbers $n < 100$.

Note: a simpler version of the sestine is the *ghazel* verse form of the Arabic world, in which there is only one core word (the last word of a line) that is repeated for every second line.[76]

[76] https://de.wikipedia.org/wiki/Ghasel

16.3 MATTER FOR REFLECTION

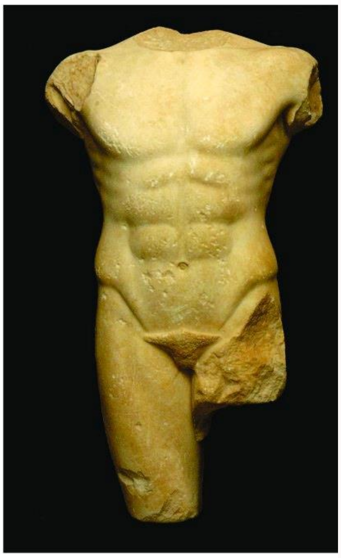
Archaic Torso of Apollo

We cannot know his legendary head
with eyes like ripening fruit. And yet his torso
is still suffused with brilliance from inside,
like a lamp, in which his gaze, now turned to low,

gleams in all its power. Otherwise
the curved breast could not dazzle you so, nor could
a smile run through the placid hips and thighs
to that dark center where procreation flared.

Otherwise this stone would seem defaced
beneath the translucent cascade of the shoulders
and would not glisten like a wild beast's fur:

would not, from all the borders of itself,
burst like a star: for here there is no place
that does not see you. You must change your life.



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(Rainer Maria Rilke)

This is one of Rilke's most beautiful, but also most puzzling, poems. The reader may wonder what it has to do with prime numbers…

Already from the form of the poem (sonnet), we see something far more is being attempted here than a simple textual message. It touches us in a way that one can only describe as 'mysterious'.

Interpretations of this poem differ widely. Some people read into it the basic, philosophical questions of human life: "Who am I?", "What should I do?" and find answers in Rilke's poem (at least hints of answers…).

A very nice interpretation can be found in Victor Zuckerkandl's book *Vom musikalischen Denken*[77].

In this book, Zuckerkandl describes how the observer and thing observed suddenly reverse roles: the work of art I am contemplating becomes the observer and looks at me, making me the thing observed, and its wordless gaze is transmuted into the command: "You must change your life".

[77] Victor Zuckerkandl: *Vom musikalischen Denken* (p.151), Rhein-Verlag Zürich, 1964

For the further interpretation, Zuckerkandl cites the story of the butterfly's dream of Zhuangzi[78], which is so beautiful that it is also quoted here:

The dream of the butterfly

Once Zhuang Zhou dreamed he was a butterfly, a fluttering butterfly. What fun he had, doing as he pleased! He did not know he was Zhou. Suddenly he woke up and found himself to be Zhou. He did not know whether Zhou had dreamed he was a butterfly or a butterfly had dreamed he was Zhou. Between Zhou and the butterfly there must be some distinction. This is what is meant by the transformation of things.

This deeply meaningful parable (like Rilke's poem) shows that both levels have the same quality of 'reality' and are coequal to each other. Every work of art has its own life; indeed, it *is* alive. Therefore, it can also look at me. For the artist and the process of creation, this means that a work of art (as soon as a certain threshold is passed during the process of creation) begins in effect to 'come alive' and therefore to have a will of its own. The work of art 'communicates' with the artist and wants to participate in its 'creation'; it demands to be realized…

The verse form, i.e. the numerical structure, in which prime numbers play a role, as well as the 'meter' of the poem and, of course, the actual text comprise a complex network in which the concept of 'aesthetics' first becomes meaningful.

That this immanent sense cannot always be rationally and logically put into words, but nevertheless is "understood" by our aesthetic feeling, reminds us very much of the haiku and other Japanese verse forms discussed in the last chapter.

Here, too, the two (supposedly independent) levels of meaning, "form" and "content", seem to be inseparably merged, or, as Ludwig Wittgenstein put it:

Ethics and aesthetics are One[79].

A more profound examination of Wittgenstein's argument is needed if we wish to understand exactly what he means by this. Similarities to many koans from Zen Buddhism are obvious. Here's an example:

"The eye with which I see God,
Is exactly the same eye with which God looks at me. "
"Show me this eye!"

In these poems the reader interested in mathematics immediately finds the element of self-reference and recurrence or recursion. Recursion therefore appears to be not only a very powerful instrument in mathematics, but also a means in philosophy and poetry to 'give voice' to things that cannot be expressed by words: insights into a higher level of truth?

---

[78] Dschuang Dsi: *Das wahre Buch vom südlichen Blütenland*: Eugen Diederichs Verlag München (1988)
[79] From *Tractatus logico-philosophicus* by Ludwig Wittgenstein

17 PRIME NUMBERS AND EXTRATERRESTRIAL LIFE FORMS

If, in the not-too-distant future, we should happen to receive electromagnetic signals from extraterrestrial civilizations, a discussion will be launched as to what form communication with intelligent extraterrestrial beings should take. The search for such signals has been going on for years and is mainly run by the project 'SETI' ('Search for Extraterrestrial Intelligence') in Mountain View near San Francisco.[80]

First of all, one must be aware that the nearest extraterrestrial planet inhabited with intelligent living things will not be found in our immediate neighbourhood, but will most likely be several hundreds, if not thousands, of light years away from the Earth. Communication could only take place over a period of many centuries. In what 'interstellar language' should we send messages, or do we expect interstellar messages?

Basically, both communication partners must find something that is common to both. This is, on the one hand, the transmission path using electromagnetic waves, which, according to our current knowledge, is the only practicable method. One can assume that extraterrestrial life forms, by the time they are engaged in a search for 'cosmic' neighbours, will have mastered this technique.

The second known class of waveforms capable of dissemination over enormous distances are the recently discovered gravitational waves. It is conceivable, theoretically, that aliens might employ these as a medium of communication. At the moment, however, we do not have any technology that would allow signals to be detected in modulated gravitational waves.

The language that is probably mastered by all intelligent life forms in the universe is the language of mathematics, which pervades every realm of our existence. All our physics is written in the language of mathematics. Thus, we can be sure that the language of mathematics is "understood" throughout the universe. Note: there are also theories assuming a variety of universes that may also have a completely different physics and perhaps even a different mathematics (Tegmark, 2015). If such universes exist, they would not be 'physically' accessible to our universe anyway. We restrict ourselves here to consideration of our universe, which is the only one observable by us.

Prime numbers are perfect for such interstellar messages, since a sequence of such numbers would almost certainly be the product of intelligent design rather than pure chance. Our universe contains very well modulated electromagnetic signals, mostly from rotating neutron stars or other physical processes (wherever charged objects are strongly accelerated). However, what is common to all these previously observed signals is that they are more or less periodic, and therefore contain no information.

Since the formal language in which mathematics is carried on at different places in our universe will also differ, it would make sense to keep the message as simple as possible. The simplest option mathematics offers are the prime numbers. Therefore, all experts to have considered the question of extraterrestrial communication agree that transmitting the prime numbers – let's say, up to 100 – on as many interesting frequencies as possible (e.g. the absorption frequency of hydrogen) would be an excellent means of interstellar

[80] http://www.seti.org

communication. The information could, for example, be wrapped in pulsed 'packets', the temporal distances between the pulses being proportional to the differences of the prime numbers.

This is exactly what happens in the movie 'Contact'[81] from 1997, starring Jodie Foster.

81 https://web.archive.org/web/20071125172406/http://www.cisci.net/film.php?lang=2&display=5&topic=Astronomie%20und%20Astrophysik&seq_id=42&film=26

17.1 THE ARECIBO MESSAGE

On 16 November 1974, scientists at the Arecibo Observatory in New Mexico sent a message from mankind into space — directed specifically at the globular cluster M13, which is 25,000 light years from Earth and known to astronomers by the name NGC 6205. This spherical star cluster (aka globular cluster) is visible on a clear night with bare eyes and is located in the constellation of Hercules between the stars η and ξ, above ξ Herculis:

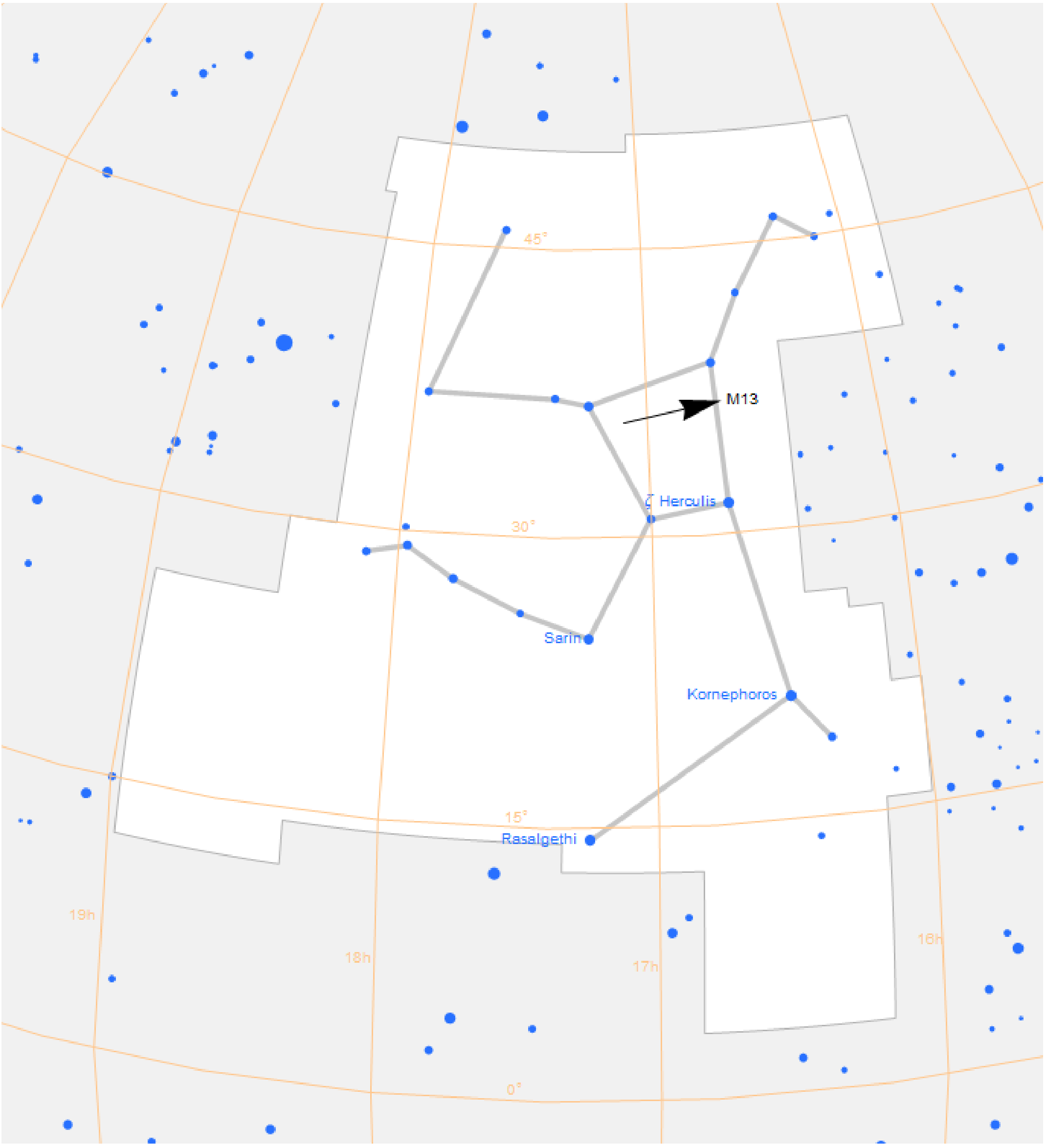


Figure 134. Constellation Hercules, with globular cluster M13 (destination of the Arecibo message)

```
Mathematica:
ConstellationData["Herculis","ConstellationGraphic"]
```

The message was digital and consisted of 1679 zeros and ones.

Since the globular cluster is (using an 'astronomical scale') relatively close to our solar system and has a high star density (it consists of about 300000 single stars), it seemed to be the ideal destination for a message to extraterrestrial life forms. An extraterrestrial receiver would first have to recognize the length of the message (1679 bits) as the product of the primes 23 and 73 and thus interpret as a binary image with the dimensions 73 * 23. Then the following picture results from the binary sequence:

The message can only be deciphered if the sequence is represented as an image with the dimensions of the prime factor assignment: At the top, the basic 'alphabet' of the binary coding of the numbers 1 to 10 is set out. These symbols are, so to speak, instructions as to how to read the following illustrations. In the picture, you will find information about our chemical elements, amino acid nucleotides, DNA structure, mankind, the planet Earth, etc…

More detailed information can be found on the Internet.

The Mathematica program code is contained in the Appendix.

A reply to the message from the star cluster M13 or its "cosmic neighborhood" would be expected in about 50,000 years.

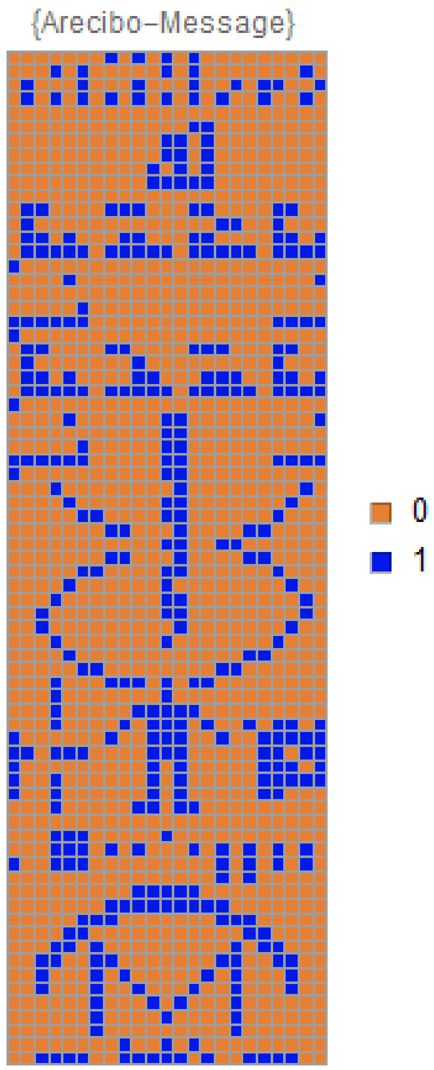


Figure 135: the Arecibo message

18 MISCELLANY

18.1 THE NUMBER 12

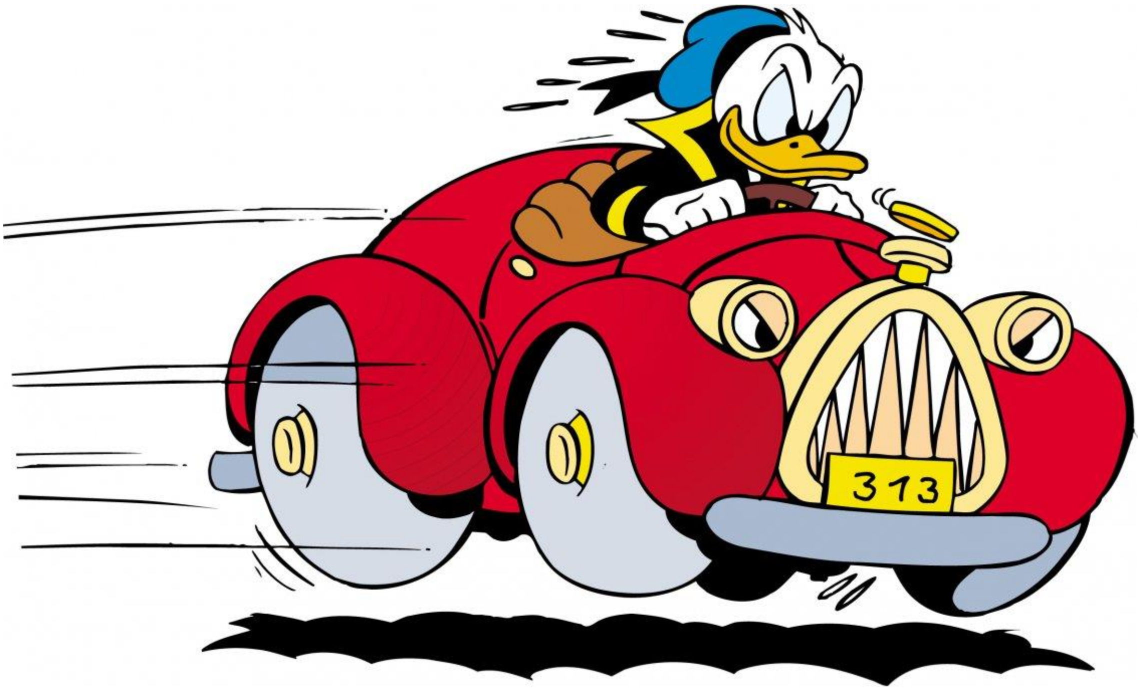
The number 12 has many special properties:

1) It was prevalent in units of measurement and counting systems until quite recently – a case in point being the British (twelve-penny) shilling, which was only abandoned in 1971 – and, even today, eggs are still sold by the dozen.
2) The year has 12 months; the day has 2 x 12 hours.
3) Jesus had 12 apostles; Israel had 12 tribes.
4) The octave has 12 semitone steps.
5) It is the smallest 'abundant' number (abundant numbers are numbers, whose 'true' sum of divisors is larger than the number itself …)
6) It is the 3-dimensional kissing number.[82]
7) There are 12 signs of the zodiac.
8) There are 12 Olympic gods.
9) The $12th$ Fibonacci number is (among the infinitely many Fibonacci numbers) the only square in this sequence and it has the value $12^2 = 144$.
10) 12 is the smallest 'sublime' number (at least two sublime numbers are known). Note: a sublime number is a number where the sum of its divisors and the number of its divisors are perfect numbers. The second such number known is 6086555670238378989670371734243169622657830773351885970528324860512791691264
11) It plays a special role in music: the 12-bar blues scheme.
12) The number 12 is the only number n for which the notable relation $n = \frac{r_4(n)}{8}$ applies (where $r_4(n)$ is the number of four-dimensional lattice points of a squared radius of n).
13) The world of the physicist Burkhard Heim contains exactly 12 dimensions.
14) It appears in Ramanujan's magical formula: $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$
15) The 12th Mersenne prime number exponent has the striking short OCRON representation "2PPPP" which contains neither the ^ - nor the * - operator (see chapter 10.1)

[82] https://de.wikipedia.org/wiki/Kusszahl

18.2 THE NUMBER 313

There are also primes in comics! Who does not know Donald Duck's bright red duck convertible ("1934 Belchfire Runabout") with the license plate number 313?



313 is a special number in several respects:

- It is a 3-digit palindromic prime number (gives the same value read forwards and backwards)
- It is in binary representation (100111001) palindromic and 100111001 decimal interpreted also gives a palindromic prime number!

The website https://primes.utm.edu/curios/page.php/313.html lists another 40 special features of this number …

18.3 PRIME NUMBERS AND THE ARTS

The following figure shows the primes in the Gaussian plane after the two-colour (red / yellow) representation has been Fourier transformed, numerically integrated in the frequency domain by division with the frequencies, and then retransformed. A gimmick, though one can see that using scant mathematical resources, landscape-like graphics can be generated from prime numbers.

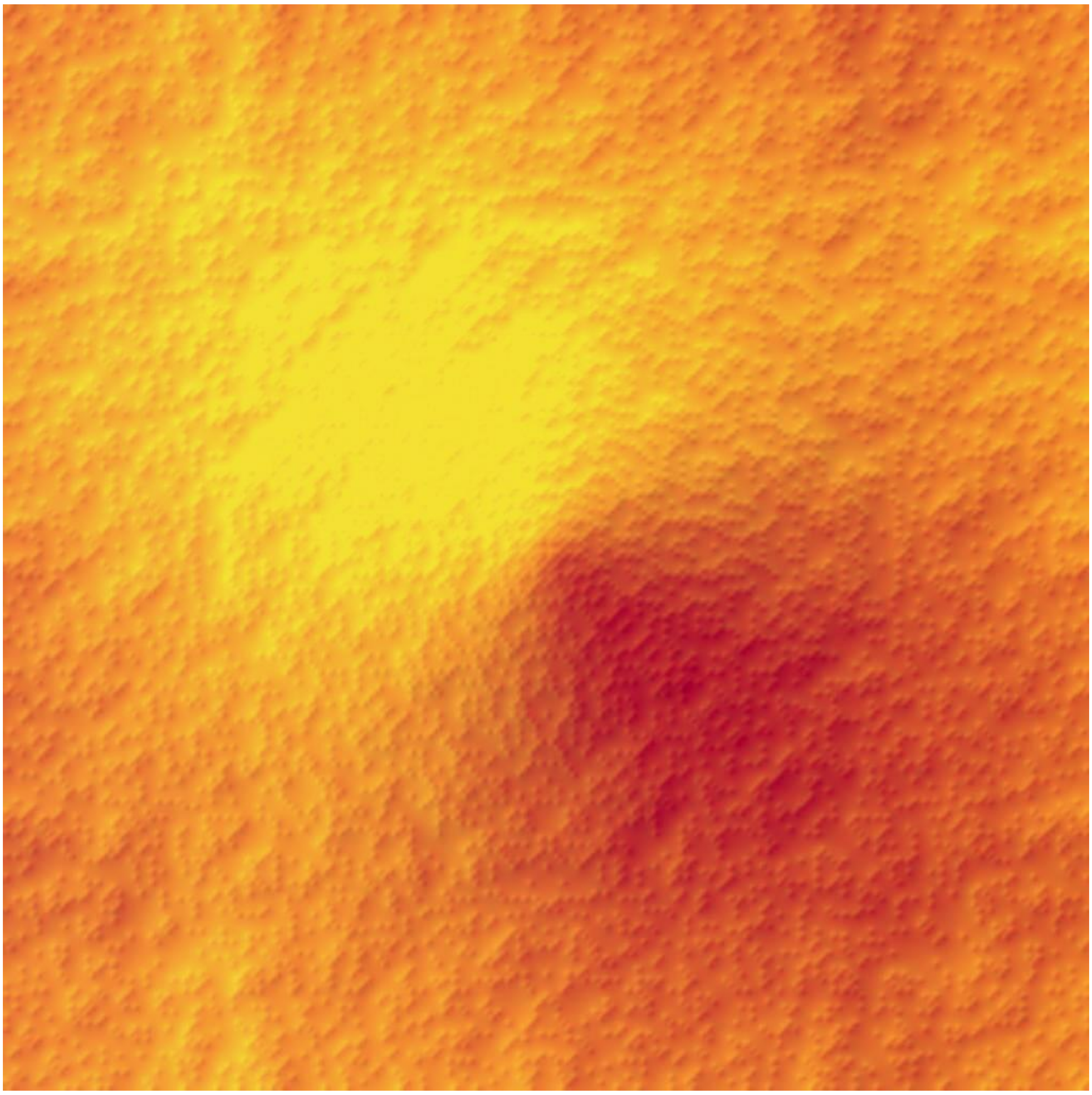


Figure 136. Gaussian prime numbers, filtered with Fourier transforms

19 CONCLUSION

Prime numbers have been a source of fascination to us for as long we have been studying mathematics. Although we know a lot about them, they have lost none of their mystique. They pervade many areas of all possible sciences and also inhabit the realms of culture, such as poetry, as well as economic life.
We have come to understand a great deal about them, but not the true, the *real* "message" that is hidden in them.

There remain many secrets still to be uncovered!

20 APPENDIX

20.1 CATALAN'S CONJECTURE

Catalan's conjecture states that there are no integer powers of natural numbers that differ exactly by the value 1, with one exception:

$$2^3 = 8 \text{ and } 3^2 = 9$$

In other words: the only integer solution of the equation

$$\boldsymbol{m^p - n^q = 1}, \text{where } \boldsymbol{m, n, p, q > 1} \text{ is } \boldsymbol{m = 3, n = 2, p = 2, q = 3} \tag{159}$$

It was proved in the year 2002 by Preda Mihăilescu.

.

The proof was obtained with the help of 'double Wieferich primes' (see 4.14).

20.2 STATISTICAL ANOMALIES OF THE LAST DIGITS IN THE PRIME NUMBER SEQUENCE

What statistical anomalies come to light if we include still more of the preceding prime numbers in our investigation? Here are the results if we consider not only the predecessors but also the pre-predecessors:

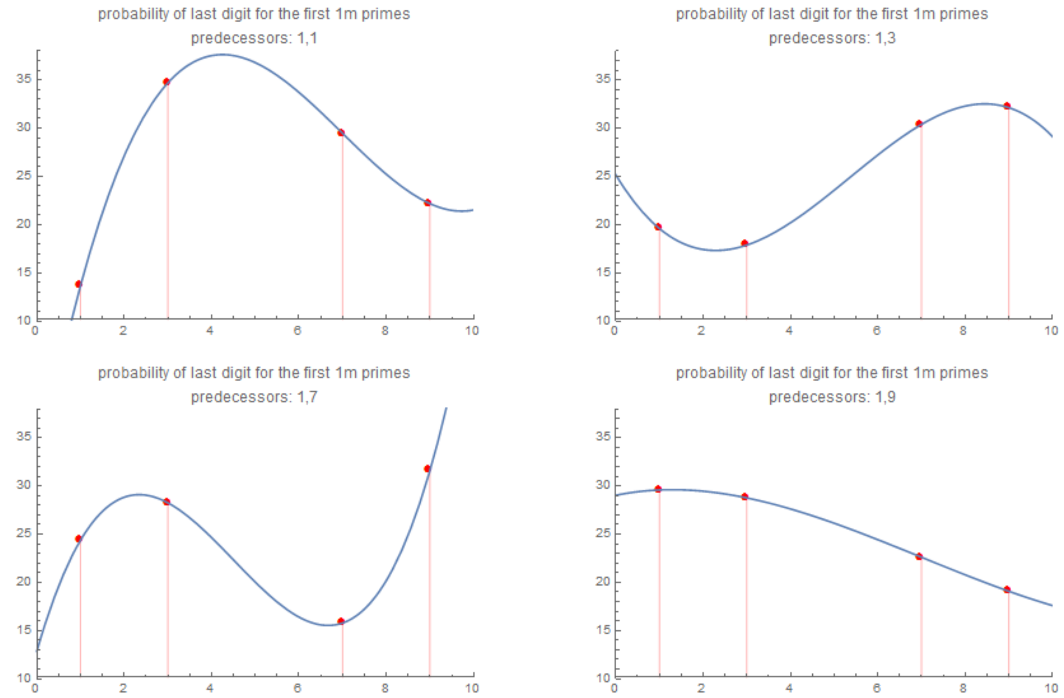


Figure 137. Incidence of final digits repeating in the prime number sequence (predecessor:1,x)

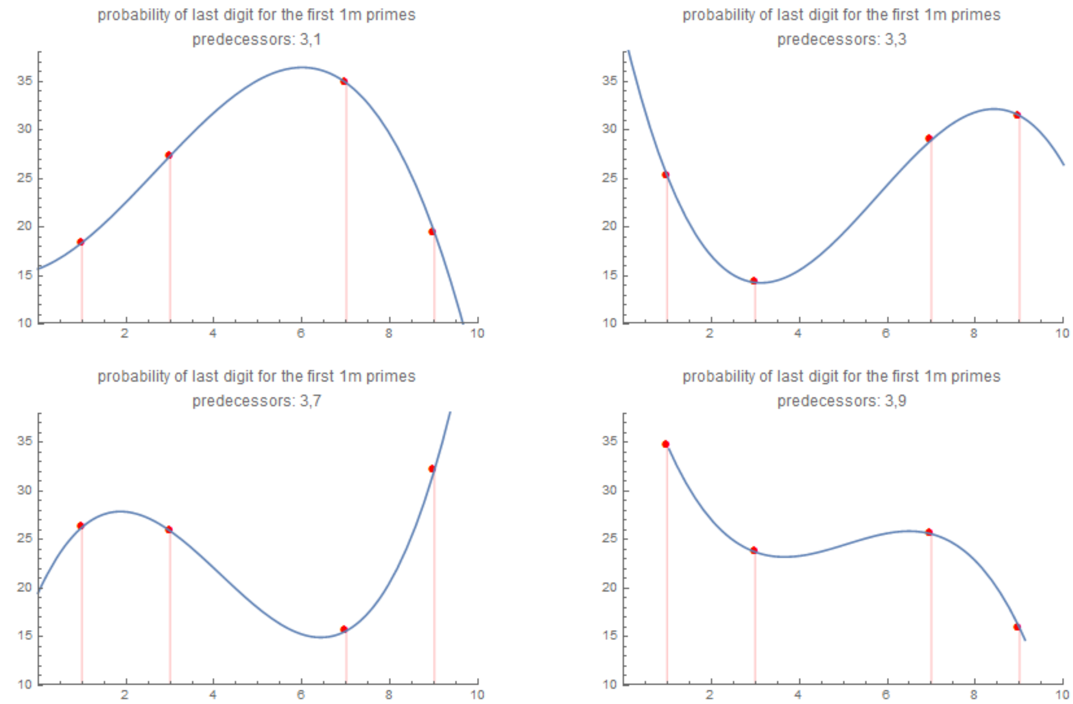


Figure 138. Incidence of final digits repeating in the prime number sequence (predecessor:3,x)

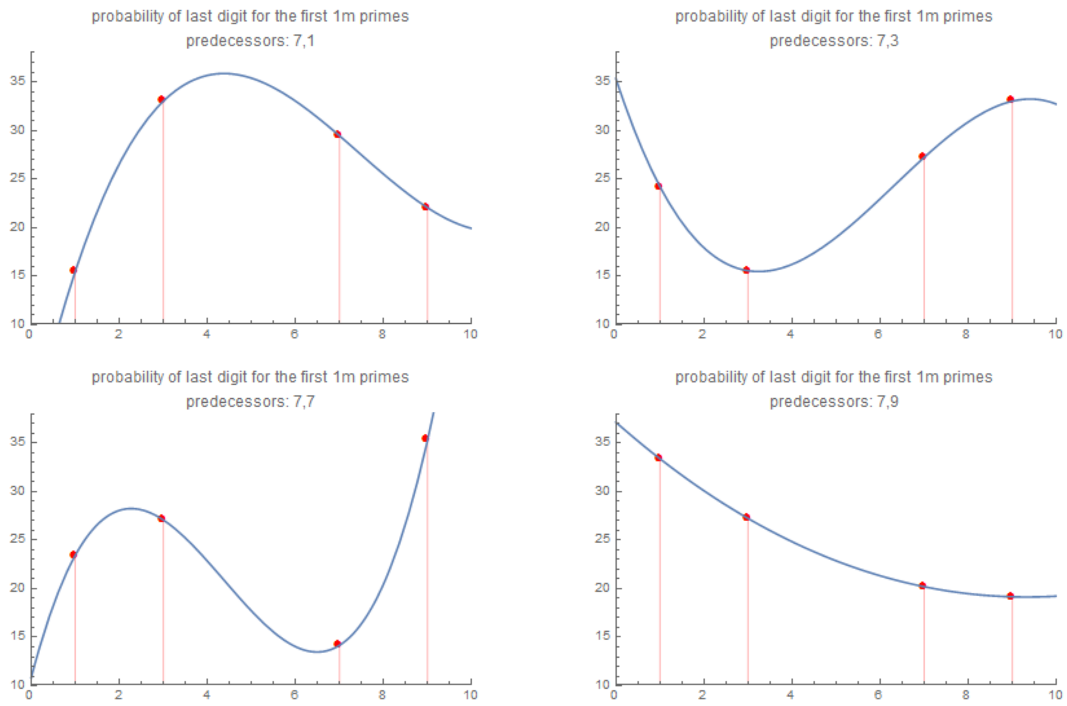


Figure 139. Incidence of final digits repeating in the prime number sequence (predecessor:7,x)

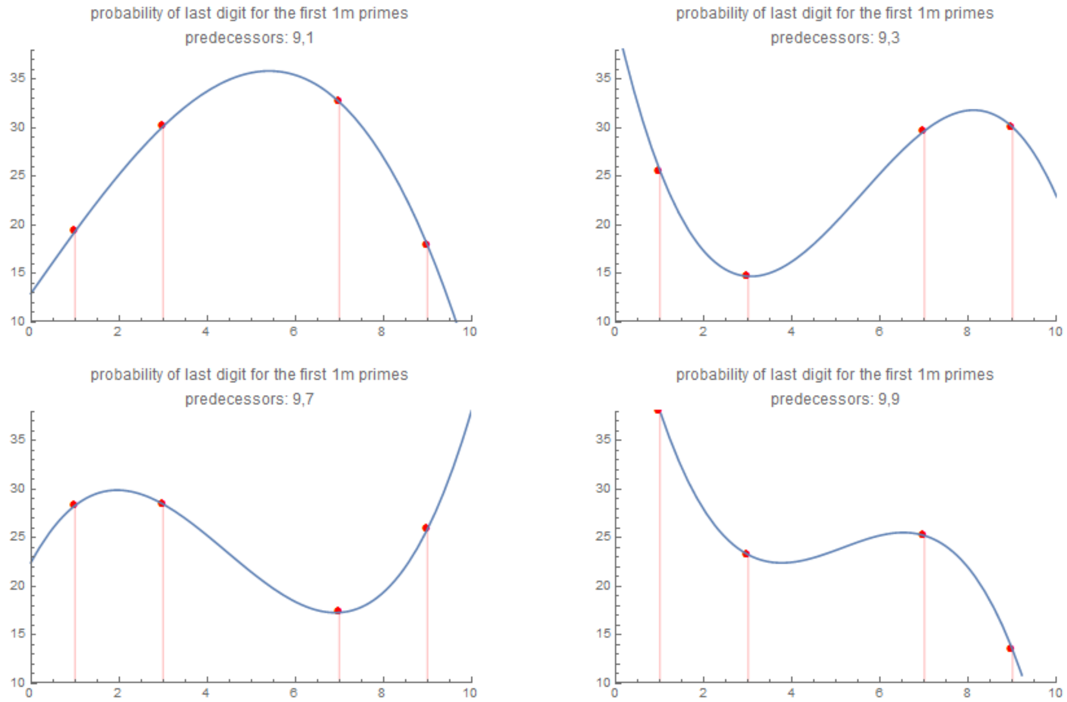


Figure 140. Incidence of final digits repeating in the prime number sequence (predecessor:9,x)

It can be seen that the tendency of the end digits not to repeat themselves increases still further. For example, the probability that the next prime number again has a last digit 9 after two prime numbers with the final digit 9 is only 13.48 %.

20.3 AN INTERESTING SEQUENCE: THE PERRIN SEQUENCE

The Perrin sequence (also referred to as the Skiponacci sequence) is an interesting curiosity:
Its recursive definition is:

$$a(n) = a(n-2) + a(n-3), \text{where } a(0) = 3, a(1) = 0, a(2) = 2 \tag{160}$$

It was actually discovered by Édouard Lucas in 1878. The peculiarity is that the pth sequence member is divisible by p (or yields mod p 0) if p is a prime number.

```
Mathematica-Code:
LinearRecurrence[{0, 1, 1}, {3, 0, 2}, 50]

{0,2,3,2,5,5,7,10,12,17,22,29,39,51,68,90,119,158,209,277,367,486,644,
853,1130,1497,1983,2627,3480,4610,6107,8090,10717,14197,18807,24914,33
004,43721,57918,76725,101639,134643,178364,236282,313007,414646,549289
,727653,963935}
```

or (replacing the prime numbers with ‘O’ for improved visibility):

```
reduced=Mod[LinearRecurrence[{0,1,1},{3,0,2}, {2,50}], Range[2,50]-1]

{0,0,0,2,0,5,0,2,3,7,0,5,0,9,8,10,0,14,0,17,10,2,0,13,5,15,12,23,0,20,
0,26,25,19,12,2,0,21,3,5,0,33,0,2,32,2,0,21,7}
```

Let’s take the ‘0’ positions:

```
Flatten[Position[reduced, 0]]
{1,2,3,5,7,11,13,17,19,23,29,31,37,41,43,47}
```

Anyone who now thinks that this method always provides prime numbers is, unfortunately, mistaken. The first counter example $271441 = 521^2$ for a composite number is, however, already very large and is a long time in coming (in place of the red ‘0’, there should be a value > 0).

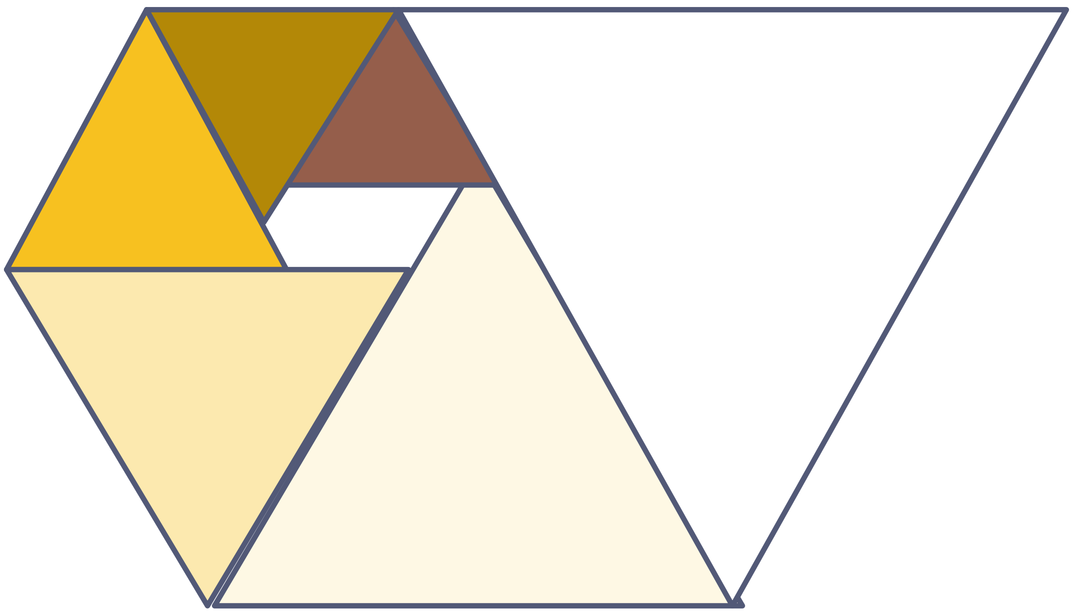
```
Mathematica:
Mod[LinearRecurrence[{0,1,1},{3,0,2},{271440,271445}],{271439,271440,2
71441,271442,271443,271444}]

{107778,199578,0,135723,3,112577}
```

The 'composite numbers' of the Perrin sequence for which n is a divisor of P_n are called Perrin pseudoprimes. At present 658 of these are known, the smallest being $271441 = 521^2$ (as of Dec. 2015).
It is assumed that there are infinitely many Perrin pseudoprimes.[83]

The Perrin sequence is closely related to the sequence of geometrically increasing equilateral triangles:

83 http://www.pseudoprime.com/pseudo3.pdf



The sides of the equilateral triangles follow the Perrin sequence as well as a second recursion sequence: $a(n) = a(n-1) + a(n-5)$
The characteristic polynomial of the Perrin sequence is thus:

$$(x^3 - x - 1) \text{ or } (x^5 - x^4 - 1)$$

The zero of the first polynomial can be written as a nested infinite expression of 3rd roots:

$$r = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}} = 1.324717957244746$$

The Perrin sequence can also be written as a closed expression:
if

$$\Theta = \operatorname{acos}\left(\frac{-r^{\frac{3}{2}}}{2}\right) \text{ then}$$

$$s_n = r^n + 2\frac{\cos(n\Theta)}{r^{\frac{n}{2}}}$$

In many respects the Perrin sequence appears even more interesting than the Fibonacci sequence. It possesses many other remarkable properties that we cannot go into here. The reader can find further information on the Internet.[84]

[84] http://www.mathpages.com/home/kmath345/kmath345.htm

20.4 MORE CONJECTURES ABOUT PRIME NUMBERS

The Goldbach conjecture

The Goldbach conjecture states that any natural even number $n > 2$ can be written as the sum of two prime numbers. The conjecture has been verified numerically for all $n < 4 \cdot 10^{18}$ (as of Oct. 2015). The 'extended Goldbach conjecture' gives an estimate for the number of representations R_g of a number n as the sum of 2 prime numbers:

$$R_g(n) \approx 2\Pi_2 \prod_{\substack{k=2 \\ p_k|n}} \frac{p_k - 1}{p_k - 2} \int_2^n \frac{dx}{(\ln x)^2} = 2\Pi_2 \prod_{\substack{k=2 \\ p_k|n}} \frac{p_k - 1}{p_k - 2} \left[li(x) - \frac{x}{\ln(x)} \right]_2^n \tag{161}$$

```
Mathematica program (from oeis.org):
a[n_] := Length @ Select[PowersRepresentations[2 n, 2, 1], (#[[1]] ==
  1 || PrimeQ[#[[1]]]) && (#[[2]] == 1 || PrimeQ[#[[2]]]) &]; Array[a,
  98] (* Jean-François Alcover, Apr 11 2011 *)

nn = 10^2; ps = Boole[PrimeQ[Range[2*nn]]]; ps[[1]] = 1;
  Table[Sum[ps[[i]] ps[[2*n - i]], {i, n}], {n, nn}] (* T. D. Noe, Apr
  11 2011 *)
```

20.5 PRIME *N*-TUPLETS: CONSTELLATIONS OF PRIME NUMBERS

Let us assume that we have an arbitrarily large prime *p* at index 1. **Then p+1 can certainly be divided by 2**. Hence, as in the **Sieve of Eratosthenes**, we delete all the following numbers divisible by 2:

p	2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2

The next possible prime is at p+2 at index 3(twin).

p	2	p	2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2

The group p-2-p must contain at least one number divisible by 3. This can only be the 2 at the 2nd position. So we delete all parts divisible by 3 (there are, of course, multiple deletions at the positions divisible by 6):

p	2 3	p	2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2

The next possible prime is at p+6 at index 7: (triplet).

p	2 3	p	2	3	2	p	2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2

For the next sifting operation (divisibility by 5), one cannot find any unique requirement (it cannot be determined which of the first 5 numbers has to be divisible by 5 since we still have ambiguity (candidates: positions no. 4 and 5). **Therefore the next possible prime number is at position p+8 at index 9: (quadruplet).**

p	2 3	p	2	3	2	p	2 3	p	2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2	3	2		2 3		2

The group p-2-p-2-3 must contain at least one number divisible by 5. This can only be at the 5th position with '3'. Reason: the 2th position $\frac{2}{3}$ can be excluded, since then also the 7th position would have to be divisible by 5. This position, however, is already occupied by p. The same holds true for the 4th position '2', since the 9th position would then also be divisible by 5, which is also already occupied by p. Thus the 5th position with the '3' remains as the only possibility. We delete all positions divisible by 5:

p	2 3	p	2	3 5	2	p	2 3	p	2 5	3	2		2 3	5	2	3	2		2 3 5		2	3	2	5	2 3		2	3	2 5		2 3		2	3 5	2		2 3		2 5

The next possible prime is at p+12 at index 13: (pentuplet).

p	2 3	p	2	3 5	2	p	2 3	p	2 5	3	2	p	2 3	5	2	3	2		2 3 5		2	3	2	5	2 3		2	3	2 5		2 3		2	3 5	2		2 3		2 5

For the next sifting operation (divisibility by 7), we cannot find any unique requirement (it cannot be determined which of the first 7 numbers has to be divisible by 7 since we still have ambiguity (candidates: positions no. 4 and 5). **Therefore the next possible prime number is at position p+18 at index 19: (6-tuplet).**

p	2 3	p	2	3 5	2	p	2 3	p	2 5	3	2	p	2 3	5	2	3	2	p	2 3 5		2	3	2	5	2 3		2	3	2 5		2 3		2	3 5	2		2 3		2 5

The group p-$\frac{2}{3}$-p-2-$\frac{3}{5}$-2- p must contain at least one number divisible by 7. This can only be at the 4th position with '2'. Reason: the 2th position $\frac{2}{3}$ can be excluded, since then also the 9th position would have to be divisible by 7. This position, however, is already occupied by p. The same holds true for the 5th position $\frac{2}{3}$, since the 19th position would then also be divisible by 7, which is also already occupied by p. The same holds true for the 6th position '2', since the 13th position would then also be divisible by 7, which is also already occupied by p Thus the 4th position with the '2' remains as the only possible. We delete all positions divisible by 7:

p	2 3	p	2 7	3 5	2	p	2 3	p	2 5	3 7	2	p	2 3	5	2	3	2 7	p	2 3 5		2	3	2	5 7	2 3		2	3	2 5		2 3 7		2	3 5	2		2 3	7	2 5

The next possible prime is at p+20 at index 21: (7-tuplet)

p	2 3	p	2 7	3 5	2	p	2 3	p	2 5	3 7	2	p	2 3	5	2	3	2 7	p	2 3 5	p	2	3	2	5 7	2 3		2	3	2 5		2 3 7		2	3 5	2		2 3	7	2 5

For the next sifting operation (divisibility by 11), one cannot find any unique requirement (it cannot be determined which of the first 11 numbers has to be divisible by 11 since we still have ambiguity (candidates: positions no. 4, 5, 6, 11)). **Therefore the next possible prime number is at position p+26 at index 27: (8-tuplet).**

p	2 3	p	2 7	3 5	2	p	2 3	p	2 5	3 7	2	p	2 3	5	2	3	2 7	p	2 3 5	p	2	3	2	5 7	2 3	p	2	3	2 5		2 3 7		2	3 5	2		2 3	7	2 5

For the next sifting operation (divisibility by 11), one cannot find any unique requirement (it cannot be determined which of the first 11 numbers has to be divisible by 11 since we still have ambiguity (candidates: positions no. 4, 6, 11)). **Therefore the next possible prime number is at position p+30 at index 31: (9-tuplet).**

p	2 3	p	2 7	3 5	2	p	2 3	p	2 5	3 7	2	p	2 3	5	2	3	2 7	p	2 3 5	p	2	3	2	5 7	2 3	p	2	3	2 5	p	2 3 7		2	3 5	2		2 3	7	2 5

Etc.

20.6 EXPLICIT SOLUTIONS FROM CHAPTER 4.10.1

Here are some explicit solutions of the recurrence equations from Table 10:

Perrin sequence:

$$\begin{aligned} P_n = {} & 2^{-n/3}3^{-2n/3}(\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}})^n \\ & + 2^{-4n/3}3^{-2n/3}(i(\sqrt{3}+i)\sqrt[3]{9-\sqrt{69}}+(-1-i\sqrt{3})\sqrt[3]{9+\sqrt{69}})^n \\ & + 2^{-4n/3}3^{-2n/3}((-1-i\sqrt{3})\sqrt[3]{9-\sqrt{69}}+i(\sqrt{3}+i)\sqrt[3]{9+\sqrt{69}})^n \end{aligned} \tag{162}$$

Complementary Perrin sequence:

$$\begin{aligned} P_n{}^* = {} & \left(\frac{3}{-1+\sqrt[3]{\frac{1}{2}(25-3\sqrt{69})}+\sqrt[3]{\frac{1}{2}(25+3\sqrt{69})}}\right)^{-n} \\ & + \left(-\frac{1}{3}+\frac{1}{6}i(\sqrt{3}+i)\sqrt[3]{\frac{1}{2}(25-3\sqrt{69})}-\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(25+3\sqrt{69})}\right)^n \\ & + \left(-\frac{1}{3}-\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(25-3\sqrt{69})}+\frac{1}{6}i(\sqrt{3}+i)\sqrt[3]{\frac{1}{2}(25+3\sqrt{69})}\right)^n \end{aligned} \tag{163}$$

Padovan sequence:

$$\begin{aligned} P_n = {} & 2^{-n/3}3^{-2n/3}(\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}})^n + \frac{1}{23}(23+\sqrt[3]{\frac{23}{2}(437-51\sqrt{69})} \\ & + \sqrt[3]{\frac{23}{2}(437+51\sqrt{69})})2^{-4n/3}3^{-\frac{2n}{3}-1}((-1-i\sqrt{3})\sqrt[3]{9-\sqrt{69}} \\ & + i(\sqrt{3}+i)\sqrt[3]{9+\sqrt{69}})^n + \frac{1}{23}(92+i2^{2/3}(\sqrt{3} \\ & + i)\sqrt[3]{23(437-51\sqrt{69})}+2^{2/3}(-1 \\ & - i\sqrt{3})\sqrt[3]{23(437+51\sqrt{69})})2^{-\frac{4n}{3}-2}3^{-\frac{2n}{3}-1}(i(\sqrt{3}+i)\sqrt[3]{9-\sqrt{69}} \\ & + (-1-i\sqrt{3})\sqrt[3]{9+\sqrt{69}})^n \\ & + (\frac{1}{276}(92+2^{2/3}(-1-i\sqrt{3})\sqrt[3]{23(437-51\sqrt{69})}+i2^{2/3}(\sqrt{3} \\ & + i)\sqrt[3]{23(437+51\sqrt{69})}))^n \end{aligned} \tag{164}$$

20.7 MORE ILLUSTRATIONS OF RG SEQUENCES

Here are a few examples of type 4 RG sequences (EGOCRON4):

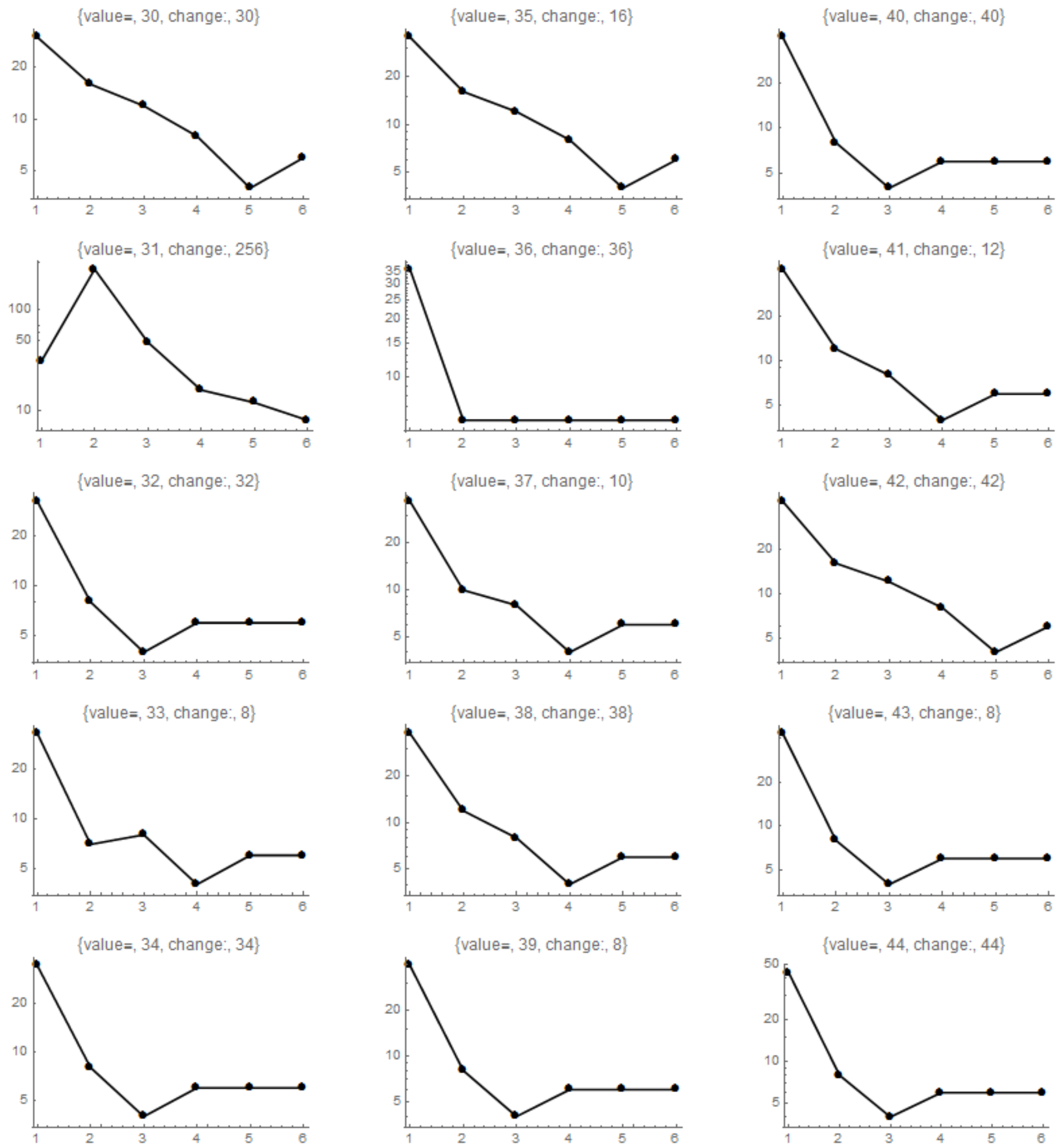


Figure 141. RG sequences of type 4 EGOCRONs in the direction of positive indices (values 30-44)

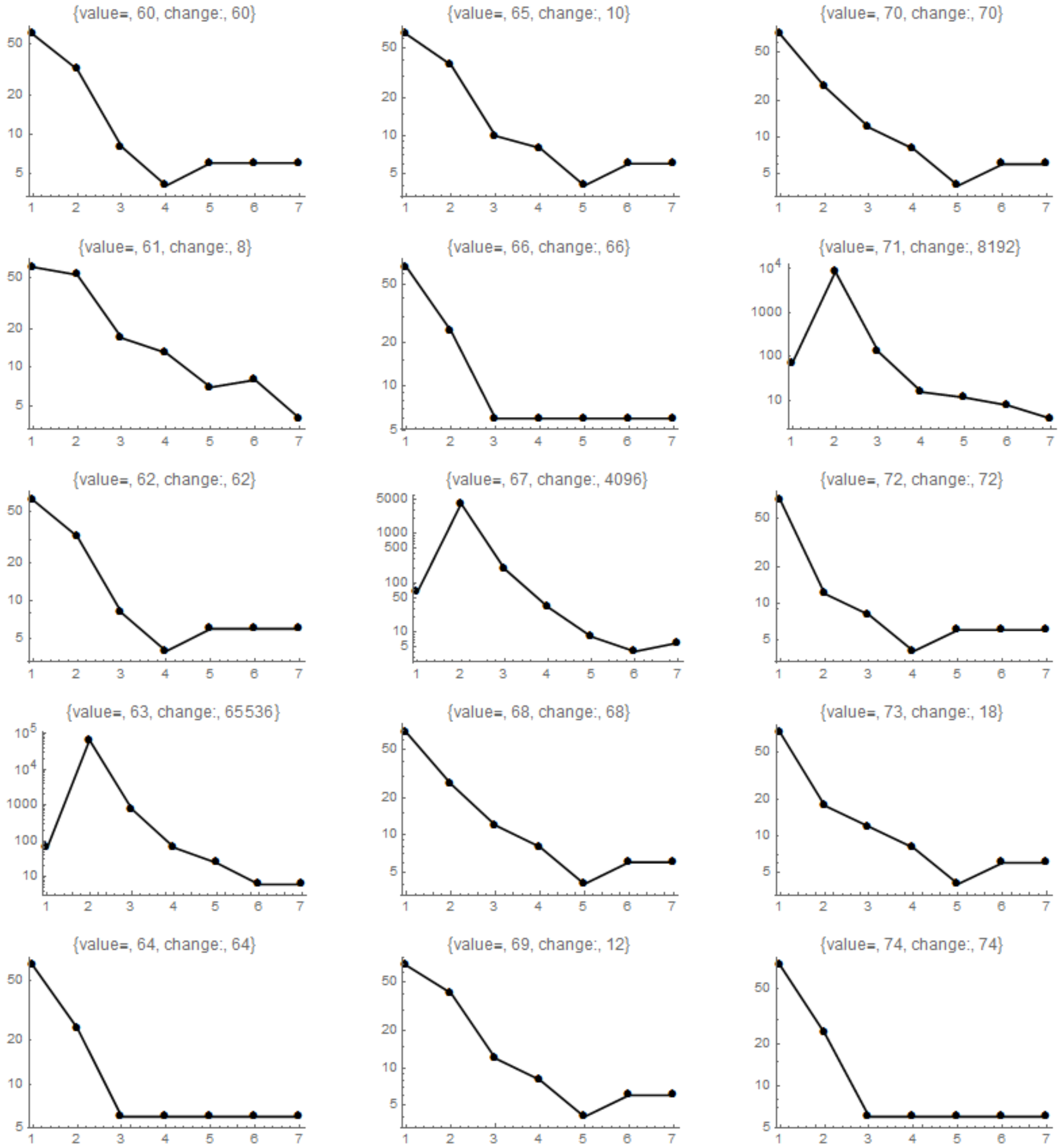


Figure 142. RG sequences of type 4 EGOCRONs in the direction of positive indices (values 60-74)

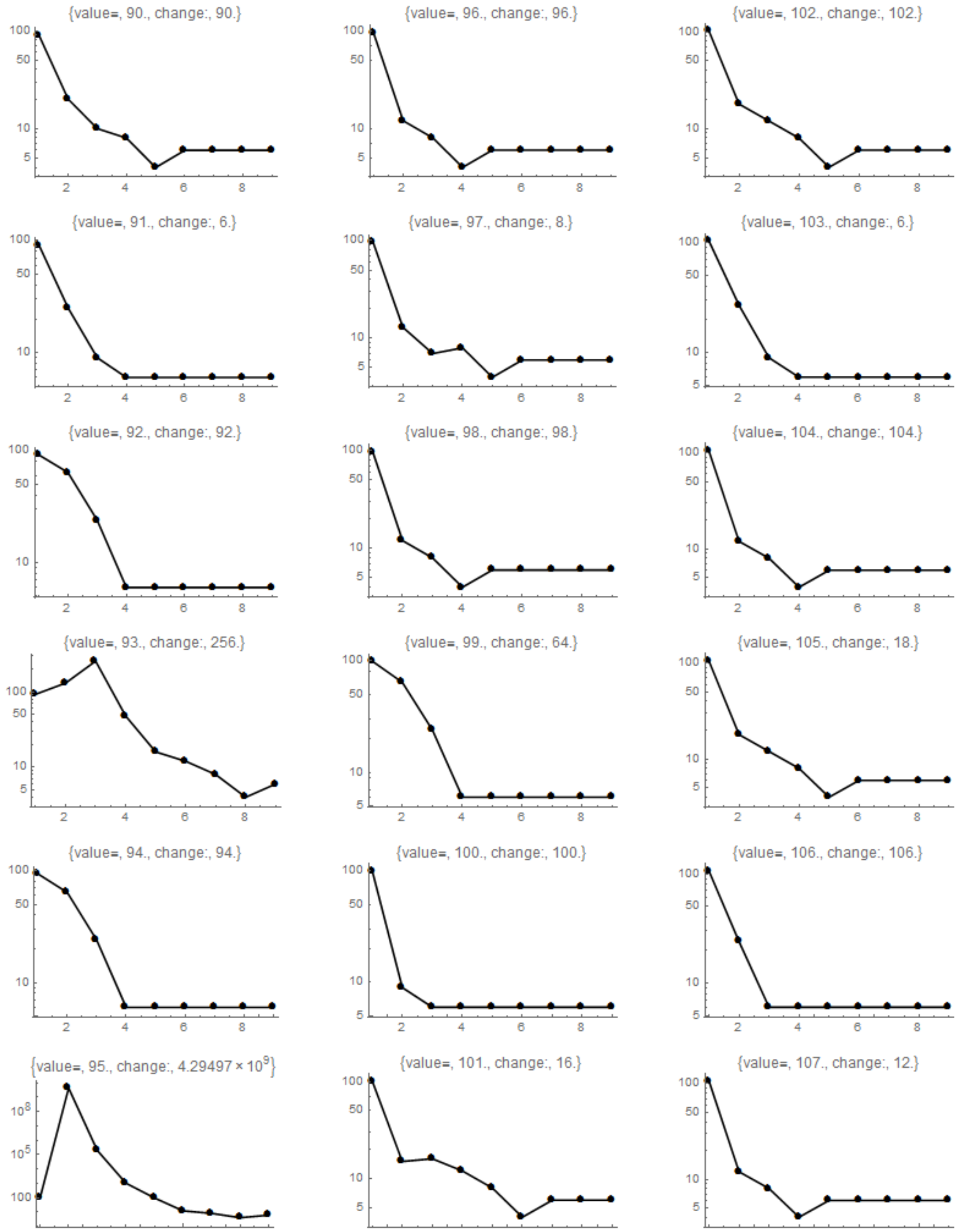


Figure 143. RG sequences of type 4 EGOCRONs in the direction of positive indices (values 90-107)

20.8 VIRTUAL OCRONS

If we look at type 4 OCRONs, we see that there are OCRONs in which the symbol '*' does not occur, i.e. that only consist of the symbols "2", "P" and "^". These are prime numbers or powers of prime numbers. For the following studies we also need the OCRONs raised to a power for the base 2. As the reader knows by now, an OCRON raised to a power for a base 2 is created by prepending a "2" and appending "^" (below in green). Here are a few examples of OCRONs and their ‘powered’ versions (the 'unpowered' part in black or blue):

```
2 (4):          22^
3 (8):          22P^
4 (16):         222^^
5 (32):         22PP^
6 (64):         22P2*^, 22P^2^
7 (128):        222^P^
8 (256):        222P^^
9 (512):        22P2^^
10 (1024):      22PP2*^, 22PP^2^
11 (2048):      22PPP^
12 (4096):      22P22^*^, 222^^2P^, 22^2P^2^
13 (8192):      22P2*P^ (no “*”-free OCRON representations)
```
13 (2^{8192}): `222P2*P^^, 222PP^^222P^^^, 222^^22PPP^^`
```
14 (16384):     222^P2*^, 22^22^P^
15 (32768):     22PP2P*^, 22P^2PP^
16 (65536):     2222^^^
17 (131072):    222^PP^
18 (262144):    22P2^2*^, 22^2P2^^
19 (524288):    222P^P^
```

The OCRONs shown in blue are "*"-free OCRONs, which can be processed easily according to the OCRON rules. The OCRONs shown in red are "non-well-formed", i.e. they do not make sense as OCRONs of type 4. However, if they are raised to a power with the base 2 (or even ‘powered’ twice in the case of the number 13), they represent well-formed, interpretable OCRONs. We wish to call the red, ‘non-raised’ OCRONs', "virtual OCRONs", since they only make sense if they are raised to a power with a base of 2, one or more times. In order to obtain the value of a virtual OCRON, the numerical value of the n-times ‘powered’ OCRON must be ‘logarithmized’ again by applying n times with the logarithm for base 2.

Finding equivalent, '*'-free OCRONs is a non-trivial task, because the whole set of degenerated OCRONs belonging to this OCRON must be searched for '*'-free OCRONs.

The following theorem is a still unproven conjecture:

Each OCRON type 4 representation of a natural number $n \geq 2$ is either "*"-free, or there are equivalent, degenerated '*'-free OCRON representations in the higher 'raised-to-OCRON-power' levels of the OCRON.

If this conjecture is true, we would have an OCRON representation of all natural numbers ≥ 2 consisting only of the OCRON symbols "2", "P", and "^". This would be a description without the "multiplicative" operator "*".

Virtual OCRONs have interesting properties. Here is a table with some degenerated, virtual OCRONs in the range 2 to 40:

Table 28. Degenerated virtual OCRONs. Primes and prime powers in red (order: ord)

N	GC(P=1, 2=2, ^=0), OCRON, (ord)	N	GC(P=1, 2=2, ^=0), OCRON, (ord)
1	-	21	1774, 2P^22^P (1) 1978, 22^P^2P (1) 13834, 2^^222P^P (2)
2	2, 2 (0)	22	553, 2^2PPP (1) 605, 2PPP^2 (1)
3	7, 2P (0) 1484, 2^^^222 (3) 2375, P^^2^222 (3)	23	208, 2P2^P (0) 164832, 22P^P^^222^ (2)
4	24, 22^ (0) 20, 2^2 (1) 170, 2^^22 (2) 13928, 2^P^^22P2 (3) 15388, 2P^^^222P(3) 24569, P^2^2^^222 (3) 25541, P^22^^^222 (3) 28619, PPP^^2^222 (3) 40049, 2^^^22P^22(3) 40903, 2^^2^^222P (3)	24	1776, 2P^22P^ (1) 2032, 22P^^2P (1) 4921, 2^2^2^2P (1) 4925, 2^2^2P^2 (1) 4961, 2^2P^2^2 (1) 4965, 2^2P^22^ (1) 5029, 2^22^^2P (1) 5285, 2P^2^2^2 (1) 5289, 2P^2^22^ (1) 5321, 2P^22^^2 (1)
5	22, 2PP (0) 511, 2^^22P (2) 575, 2P^^22 (2) 1520, 2^^2^22 (2) 8980, PP^^22P2P (3) 33611, P2^P^^22P2 (3) 41783, 2^P^^22PP2 (3)	25	204, 2PP2^ (0) 1804, 2PP^2PP (1) 13816, 2^^22P2^P (2)
6	61, 2^2P (1) 65, 2P^2 (1) 1532, 2^^22^2 (2) 1536, 2^^222^ (2) 1628, 2^2^^22 (2) 1726, 2P^^22P (2) 1952, 22^^^22 (2) 4561, 2^^2^22P (2) 5161, 2P^^2^22 (2)	26	46621, 2P^^22P2^P (2)
7	73, 22^P (0) 1534, 2^^22PP (2) 1790, 2PP^^22 (2) 4885, 2^2^^22P (2) 5177, 2P^^22^2 (2) 5181, 2P^^222^ (2) 5857, 22^^^22P (2) 7252, P^^22PP2P (3) 13682, 2^^2^22^2 (2) 13686, 2^^2^222^ (2)	27	210, 2P2P^ (0) 1770, 2P^2P2^ (1) 1870, 2P2^^2P (1) 5299, 2P^2P^2P (1) 13812, 2^^22PP2^ (2)
8	75, 22P^ (0) 182, 2^2^2 (1) 186, 2^22^ (1) 218, 22^^2 (1) 4597, 2^^22^2P (2)	28	4933, 2^2^22^P (1) 5033, 2^22^P^2 (1) 5905, 22^^22^P (1) 5933, 22^P^2^2 (1) 5937, 22^P^22^ (1)

	4601, 2^^22P^2 (2) 4949, 2^2P^^22 (2) 5179, 2P^^22PP (2) 5273, 2P^2^^22 (2) 5371, 2PP^^22P (2)		
9	69, 2P2^ (0) 196, 2P^2P (1) 4609, 2^^222^P (2) 5921, 22^P^^22 (2) 14656, 2^2^^22PP (2) 14848, 2^2P^^22P (2) 15532, 2P^^22^2P (2) 15536, 2P^^22P^2 (2) 15820, 2P^2^^22P (2) 16112, 2PP^^22^2 (2)	29	13818, 2^^22P2P^ (2) 17018, 2P2P^^^22 (2)
10	184, 2^2PP (1) 200, 2PP^2 (1) 4611, 2^^222P^ (2) 6083, 22P^^^22 (2) 13790, 2^^22^2^2 (2) 13794, 2^^22^22^ (2) 13826, 2^^222^^2 (2) 14750, 2^2^2^^22 (2) 15074, 2^22^^^22 (2) 15074, 2^22^^^22 (2)	30	4963, 2^2P^2PP (1) 4975, 2^2PP^2P (1) 5287, 2P^2^2PP (1) 5303, 2P^2PP^2 (1) 5407, 2PP^2^2P (1) 5411, 2PP^2P^2 (1)
11	67, 2PPP (0) 4605, 2^^22P2^ (2) 5597, 2P2^^^22 (2) 13804, 2^^22P^2P (2) 15546, 2P^^222P^ (2) 15884, 2P^2P^^22 (2) 18250, 22P^^^22P (2} 41061, 2^^2^222P^ (2) 43975, 2^2^^222^P (2)	31	202, 2PPPP (0) 153168, 2P2P^^^222^ (2)
12	547, 2^2^2P (1) 551, 2^2P^2 (1) 587, 2P^2^2 (1) 591, 2P^22^ (1) 655, 22^^2P (1) 13792, 2^^22^2PP (2) 13808, 2^^22PP^2 (2) 14912, 2^2PP^^22 (2)	32	228, 22PP^ (0) 1680, 2^22^2^ (1) 1692, 2^222^^ (1) 2000, 22^2^^2 (1) 2108, 222^^^2 (1) 4935, 2^2^22P^ (1) 5051, 2^22P^^2 (1) 5907, 22^^22P^ (1) 6095, 22P^^2^2 (1) 6099, 22P^^22^ (1)
13	4603, 2^^22PPP (2) 5435, 2PPP^^22 (2) 48351, 2PP^^222P^(2) 52719, 22^^^22P2^(2)	33	1768, 2P^2PPP (1) 1816, 2PPP^2P (1) 13810, 2^^22PPPP (2)
14	559, 2^22^P (1) 659, 22^P^2 (1)	34	1678, 2^22^PP (1) 1982, 22^PP^2 (1) 13836, 2^^222PP^ (2)
15	589, 2P^2PP (1) 601, 2PP^2P (1)	35	5419, 2PP^22^P (1) 5935, 22^P^2PP (1)
16	222, 22^2^ (0) 234, 222^^ (0) 561, 2^22P^ (1) 677, 22P^^2 (1) 1640, 2^2^2^2 (1) 1644, 2^2^22^ (1) 1676, 2^22^^2 (1) 1964, 22^^2^2 (1) 1968, 22^^22^ (1)	36	4929, 2^2^2P2^ (1) 4997, 2^2P2^^2 (1) 5609, 2P2^^2^2 (1) 5613, 2P2^^22^ (1) 5901, 22^^2P2^ (1) 14776, 2^2^2P^2P (1) 14884, 2^2P^2^2P (1) 14888, 2^2P^2P^2 (1)
17	220, 22^PP (0) 41413, 2^^22P^2PP (2) 41425, 2^^22PP^2P (2)	37	166288, 22PP^^^22PP (2)
18	555, 2^2P2^ (1)	38	1684, 2^22P^P (1)

	623, 2P2^^2 (1) 1654, 2^2P^2P (1) 1762, 2P^2^2P (1) 1766, 2P^2P^2 (1) 13830, 2^^222^2^ (2) 13842, 2^^2222^^ (2)		2036, 22P^P^2 (1)
19	226, 22P^P (0) 13828, 2^^222^PP (2)	39	498871, 22PP^^^222^P (2)
20	1642, 2^2^2PP (1) 1658, 2^2PP^2 (1) 1802, 2PP^2^2 (1) 1806, 2PP^22^ (1) 1966, 22^^2PP (1)	40	5421, 2PP^22P^ (1) 6097, 22P^^2PP (1) 14764, 2^2^2^2PP (1) 14780, 2^2^2PP^2 (1) 14924, 2^2PP^2^2 (1) 14928, 2^2PP^22^ (1)

Let us call the number of times '2' was raised to the power of an OCRON the 'power level' or the 'order' of the (virtual) OCRON and let us call the OCRON on which the operation 2 to the power of … has been performed n times the OCRON 'exposed' by a power level n. Let us also call the process of 2 raised to the power of OCRON 'exposure'. We call virtual OCRONs with an associated number n of exposures 'virtual' OCRONs of order n. From the power laws

$(2^a)^b = 2^{a*b}$ *as well as* $(2^{2^a})^{2^b} = 2^{2^{a+b}}$, the following rules for virtual OCRONs of order 1 and 2 can be found:

- **A number n that can be represented as a '*'-free OCRON is (simultaneously) a virtual OCRON of order 0. This applies to all primes and prime powers if the prime number has a '*'-free representation.**
- **Each composite number that can be written as a product of different, '*'-free factors can be represented as a virtual OCRON of order 1 and 2, but not as a virtual OCRON of order 0.**
- **Prime numbers correspond to either virtual OCRONs of order 0 or 2.**
- **Prime powers (with powers $\geq$ 2) can be represented as virtual OCRONs of orders 0, 1 and 2.**
- **Virtual OCRONs of order 0, 1 or 2 always start with the symbol "2".**
- **From order 3 onwards, virtual OCRONs can also start with the symbol "P".**

Theorem: every natural number can be represented as a virtual OCRON of order 0, 1 or 2.

Translated into "everyday mathematical language", this theorem reads:

Any natural number $n > 1$ can be represented by the use only of the number 2, the functions `Prime()` and `Log()` (to the base 2), as well as raising to a power.

Note that the arithmetic operations "$*$" and "$+$" are not required!
The proof is clear, since every natural number can be represented either by a product of two or more '*'-free factors, or by a sum of two or more '*'-free summands, in which we want to understand those factors or summands as '*'-free prime numbers or powers of prime numbers.

20.9 MORE UNSOLVED MATHEMATICAL PROBLEMS

20.9.1 THE EUCLID-MULLIN SEQUENCE

This sequence is defined very simply:

Let be $a_1 = 2$, then a_n is the smallest prime factor in the decomposition:

$$\prod_{i=1}^{n-1} a_i + 1$$

The first elements of the Euclid-Mullin sequence are:

```
2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17,
5471, 52662739, 23003, 30693651606209, 37, 1741, 1313797957, 887, 71,
7127, 109, 23, 97, 159227, 643679794963466223081509857, 103,
1079990819, 9539, 3143065813, 29, 3847, 89, 19, 577, 223, 139703, 457,
9649, 61, 4357,
87991098722552272708281251793312351581099392851768893748012603709343,
107, 127, 3313,
2274326891085895327549849150757748483866714395682604207544149407807612
45893,59, 31, 211
```

Mathematica:

```
f[1]=2;f[n_]:=f[n]=FactorInteger[Product[f[i],
{i,1,n-1}]+1][[1,1]];Table[f[n],{n,1,43}]
```

It is not known whether the Euclid-Mullin sequence runs through all prime numbers. It is also not known whether the problem of finding out whether a given prime is contained in the sequence belongs to the set of computable problems[85]. For example, it is still unclear whether the number 41 is an element of the Euclid-Mullin sequence.

[85] https://de.wikipedia.org/wiki/Berechenbarkeit

20.9.2 ALIQUOT SEQUENCES

20.9.2.1 GENERAL

Aliquot sequences are recursively defined sequences defined in the domain of the natural numbers:

$$n, s(n), s(s(n)), s\left(s(s(n))\right), \dots \text{ where } s(n) = \sigma(n) - n, n \in \mathbb{N} \qquad (165)$$

In the process, $\sigma(n)$ is the sum of divisors function (see Chapter 7.5). (Note: $\sigma(n)$ is the simplified notation of the generalized sigma function $\sigma_k(n)$ for $k = 1$: $\sigma(n) = \sigma_1(n)$). $\sigma(n)$ counts and sums all the divisors (including 1 and n itself). $s(n)$ sums all divisors, but without n itself. $s(n)$ is therefore sometimes called the sum of the proper divisors of n. Occasionally, also the term 'numerical content' can be found for $s(n)$.

Here are a few examples of aliquot sequences for different starting values:

```
{4,3,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
{6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6}
{7,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
{10,8,7,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
{11,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
{12,16,15,9,4,3,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
{28,28,28,28,28,28,28,28,28,28,28,28,28,28,28,28,28,28,28,28}
{220,284,220,284,220,284,220,284,220,284,220,284,220,284,220,284}
{276,396,696,1104,1872,3770,3790,3050,2716,2772,5964,10164,19628,19684
,22876,26404}
{496,496,496,496,496,496,496,496,496,496,496,496,496,496,496,496}
{562,284,220,284,220,284,220,284,220,284,220,284,220,284,220,284}
{790,650,652,496,496,496,496,496,496,496,496,496,496,496,496,496}
{12496,14288,15472,14536,14264,12496,14288,15472,14536,14264,12496,142
88,15472,14536,14264,12496}
```

(Cycles are marked in red, OE sequences ('open end': blue))

```
Mathematica:

(*06:*)Join[NestList[DivisorSigma[1,#]-#&,6,25],PadRight[{},0,0]]
(*10:*)Join[NestList[DivisorSigma[1,#]-#&,10,4],PadRight[{},21,0]]
(*11:*)Join[NestList[DivisorSigma[1,#]-#&,11,2],PadRight[{},23,0]]
(*12:*)Join[NestList[DivisorSigma[1,#]-#&,12,7],PadRight[{},18,0]]
(*28:*)Join[NestList[DivisorSigma[1,#]-#&,28,19],PadRight[{},0,0]]
(*220:*)Join[NestList[DivisorSigma[1,#]-#&,220,15],PadRight[{},0,0]]
(*276:*)Join[NestList[DivisorSigma[1,#]-#&,276,15],PadRight[{},0,0]]
(*496:*)Join[NestList[DivisorSigma[1,#]-#&,496,15],PadRight[{},0,0]]
(*562:*)Join[NestList[DivisorSigma[1,#]-#&,562,15],PadRight[{},0,0]]
(*790:*)Join[NestList[DivisorSigma[1,#]-#&,790,15],PadRight[{},0,0]]
```

The Appendix contains more Mathematica programs for calculating aliquot sequences ((0). As can be seen from the examples, there are several ways in which an aliquot sequence can end:

- prime number, followed by '1' and infinitely many '0' values (this is the 'normal' end of an aliquot sequence).

- Periodic (not 0): the cycles known hitherto have the following lengths: 1,2,4,5,6,8,9,28 (as of Jun. 2016). Numbers with cycle 1 are the perfect numbers (already discussed in 4.5). Those with cycle 2 are called ‘amicable’ numbers. Numbers in the higher cycles are called 'sociable' numbers.
- ‘Open End’ (OE). Some sequences grow to infinity without an observable 'descent'.

If we class the sequences ending in 0 along with those that become periodic, there are basically only two types of sequence: those that end periodically and those that never terminate.

The conjecture of Catalan (‘Aliquot-Catalan conjecture’) is that every aliquot sequence becomes periodic, so that no OE (non-terminating) sequences exist!

Below 1000, there are currently 5 OE sequences and a further 7 sequences that either have start values on one of these 5 sequences or end up on one of these 5 sequences (as of Jun. 2016). These are the so-called 'Lehmer Five'. Here the starting values of the 12 sequences below 1000, whose 'destiny' is uncertain:

276 (306,396, 696)
552 (888)
564 (780)
660 (828, 996)
966

As computers have increased in power in recent years, the number of OE sequences has been reduced. Some sequences invade vertiginously high number regions before they decide to 'descend' again and end up normally at a prime number. Each natural number, taken as a starting value, thus has its own private aliquot sequence. These sequences can look very different. They can consist of a single number (if a perfect number is taken as the starting value), but they can also consist of thousands of values before the sequence ends in a cycle. In these cases, the graph of the corresponding sequence looks more like a stock market price than an arithmetical function. The longest sequences calculated to date are all OE sequences with lengths of thousands of sequence members.

The longest, currently ‘calculated’ aliquot OE sequence has the starting value 933436 and has been calculated up to the term 12516 (as of June 2016)[86]. The longest sequences found so far have lengths of over 70000 sequence values. The largest values achieved by sequence members are larger than 10^{120} (same source). For OE sequences, there are 'descents' of more than 100 powers of 10 before the sequence rises again, into infinity… On the other hand, there are 'ascents' up to 120 orders of magnitude before some sequences descend again and end on a prime number. From the data empirically found so far, it can be estimated that at present about 1% of all numbers have OE sequences ('open end').

[86] http://aliquot.de/aliquote.htm#records

Note: most of the information in this chapter is taken from the following Internet pages http://www.aliquot.de, http://factordb.com (Markus Tervooren), http://christophe.clavier.free.fr/Aliquot/site/Aliquot.html

Here are a few graphs of aliquot sequences. First, the 'Lehmer Five' (open-end sequences with starting values below 1000):

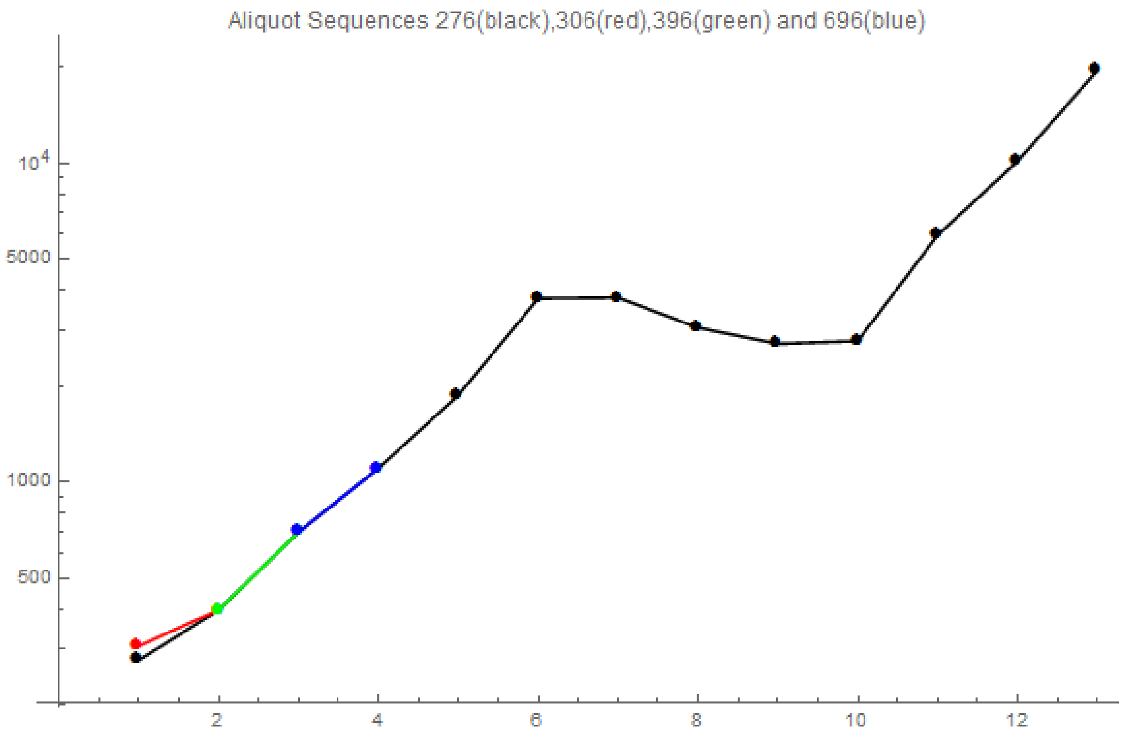


Figure 144. The first 12 values of the aliquot sequences 276,306,396,696. From the 3rd value on, the sequences are identical

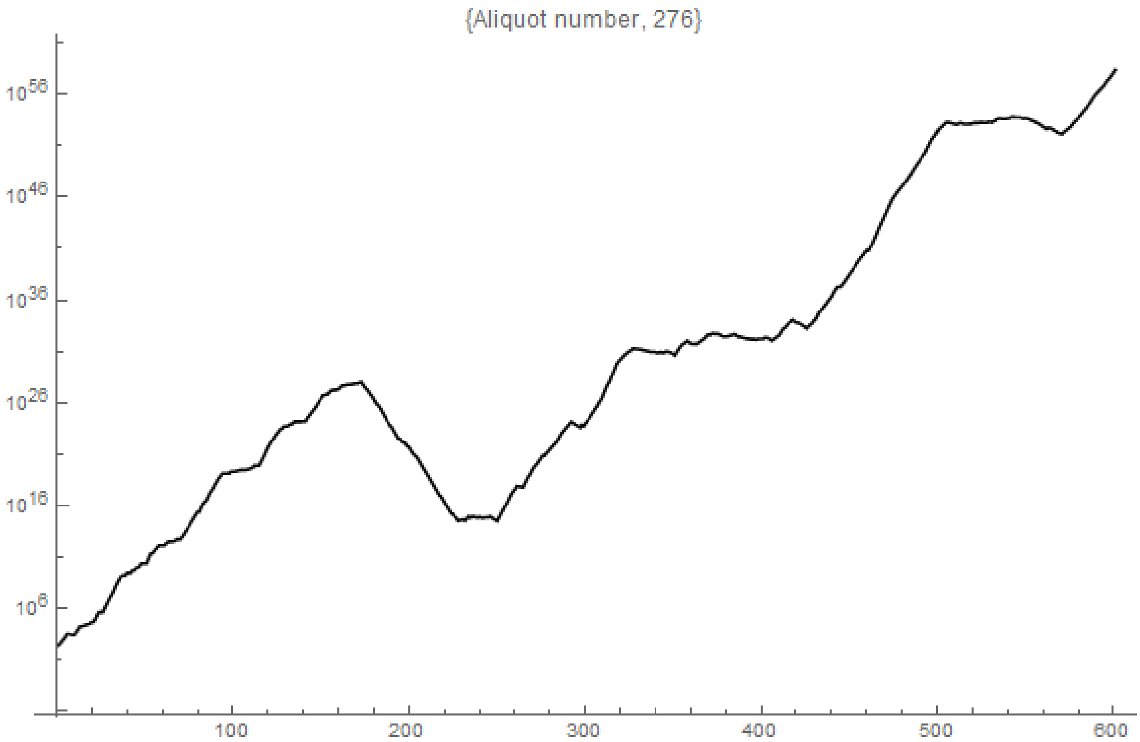


Figure 145. Aliquot sequence 276 (OE, the first 600 values)

Mathematica:

```
(*Aliquot 276 OE*)
n=276;value=n;
table=Table[value=DivisorSigma[1,value]-
value,{i,1,600}];table=Prepend[table,n];
ListLogPlot[table,PlotStyle->Black,Joined->True,ImageSize-
>Large,PlotLabel->{"Aliquot number",n}]
```

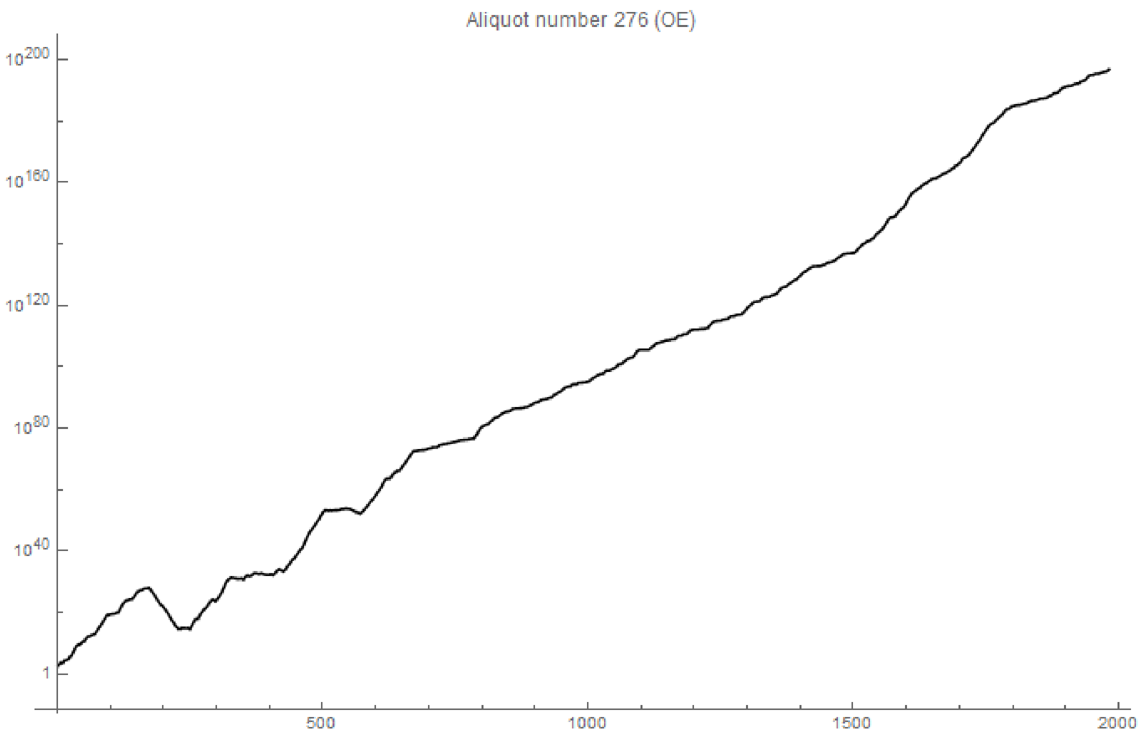


Figure 146. Aliquot sequence 276 (OE, the first 1981 values)

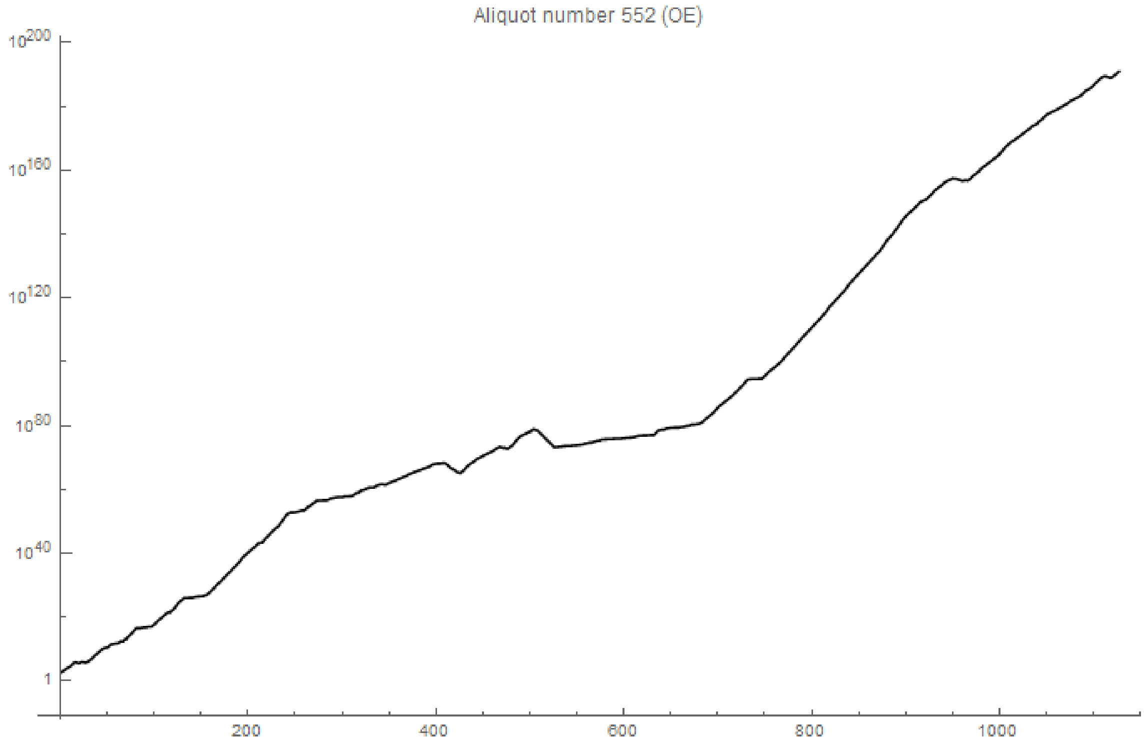


Figure 147. Aliquot sequence 552 (OE, the first 1126 values)

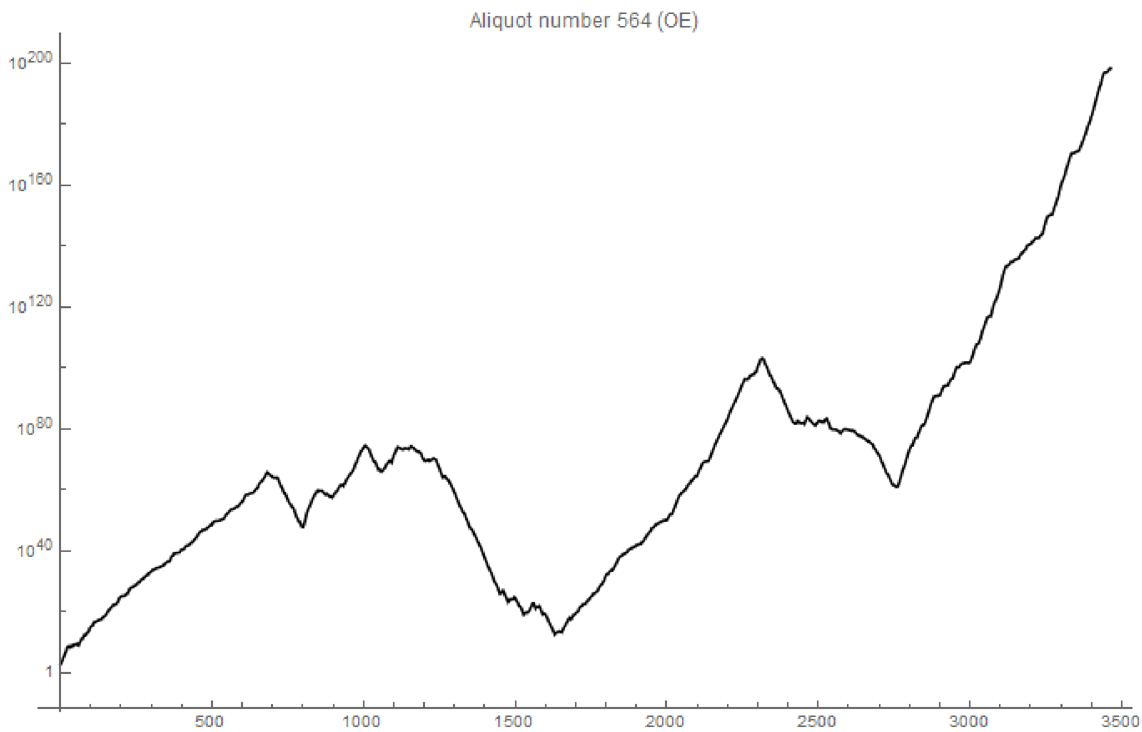


Figure 148. Aliquot sequence 564 (OE, the first 3463 values)

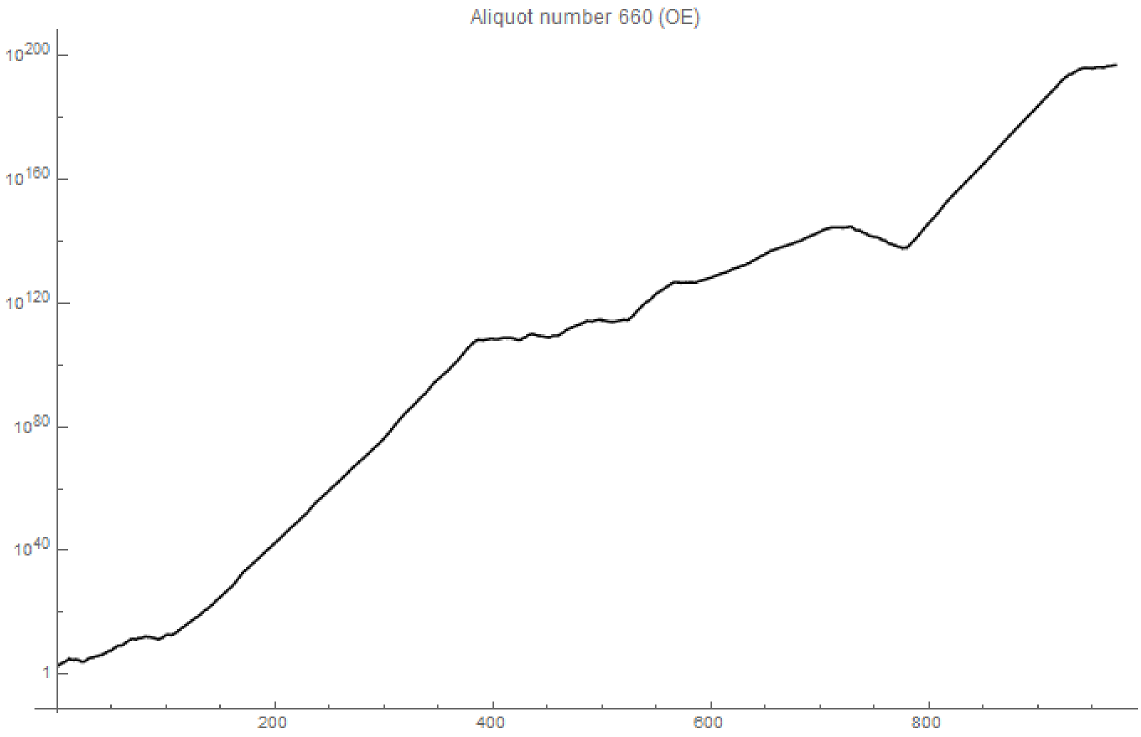


Figure 149. Aliquot sequence 660 (OE, the first 971 values)

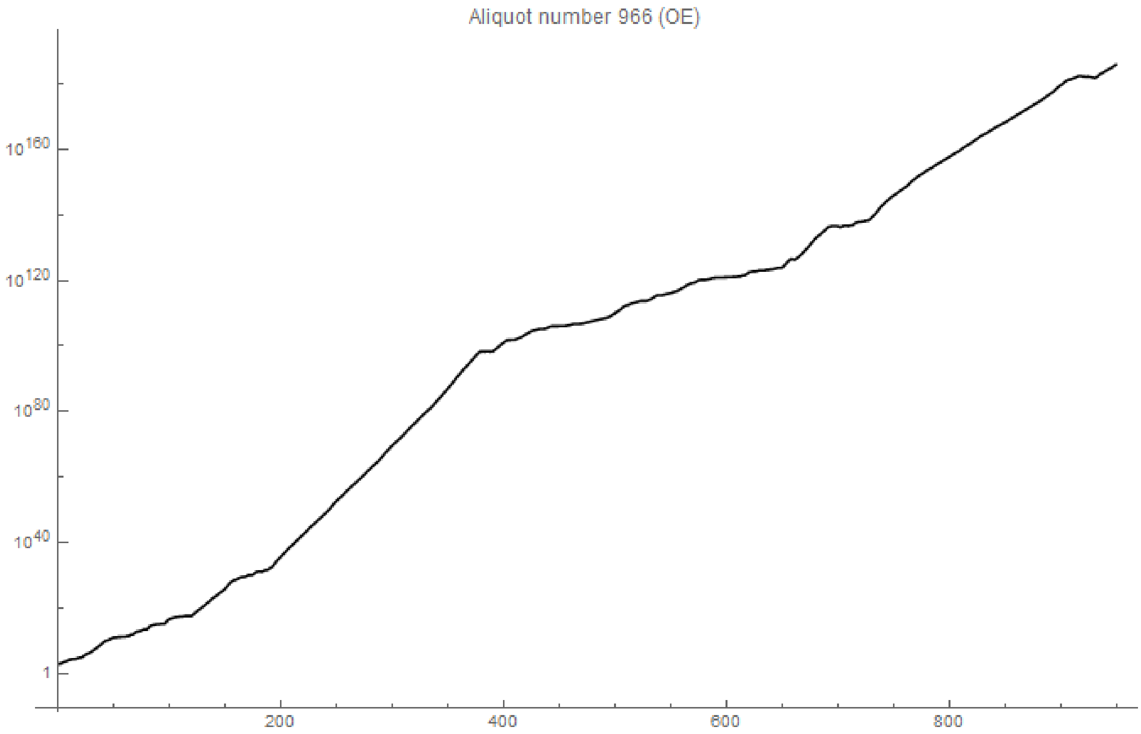


Figure 150. Aliquot sequence 966 (OE, the first 948 values)

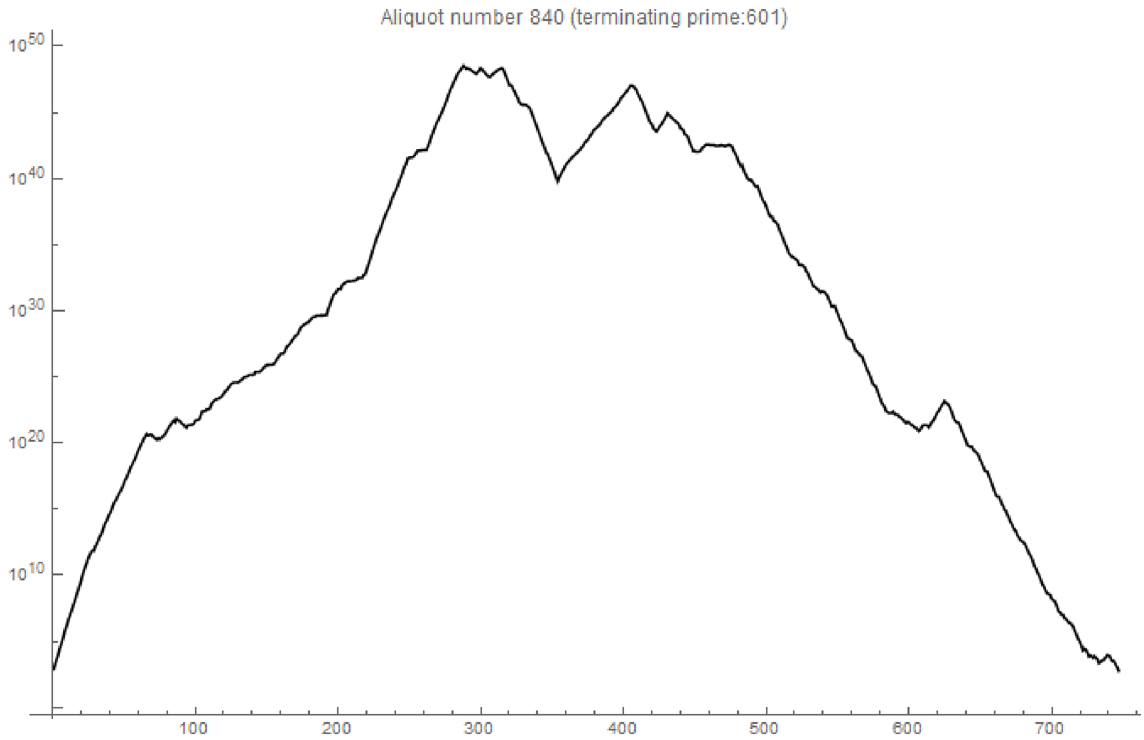


Figure 151. Aliquot sequence 840 (terminates at 601, 746 values)

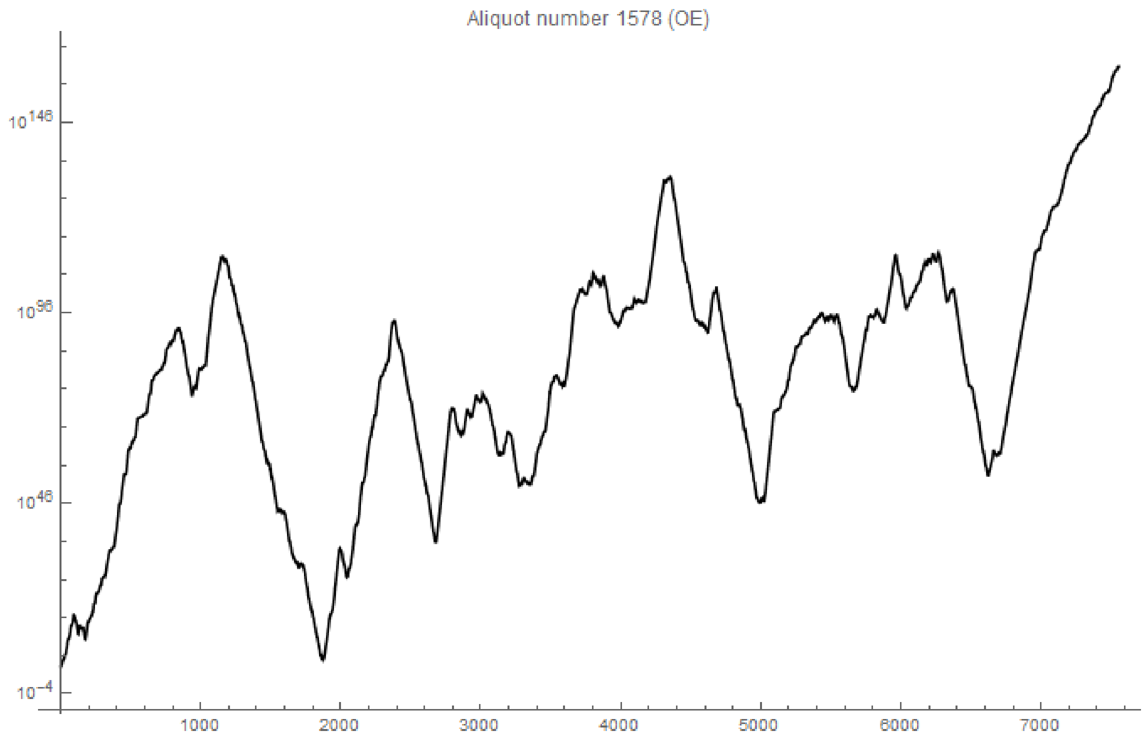


Figure 152. Aliquot sequence 1578 (OE, the first 7555 values)

And here are a few plots of terminating aliquot sequences:

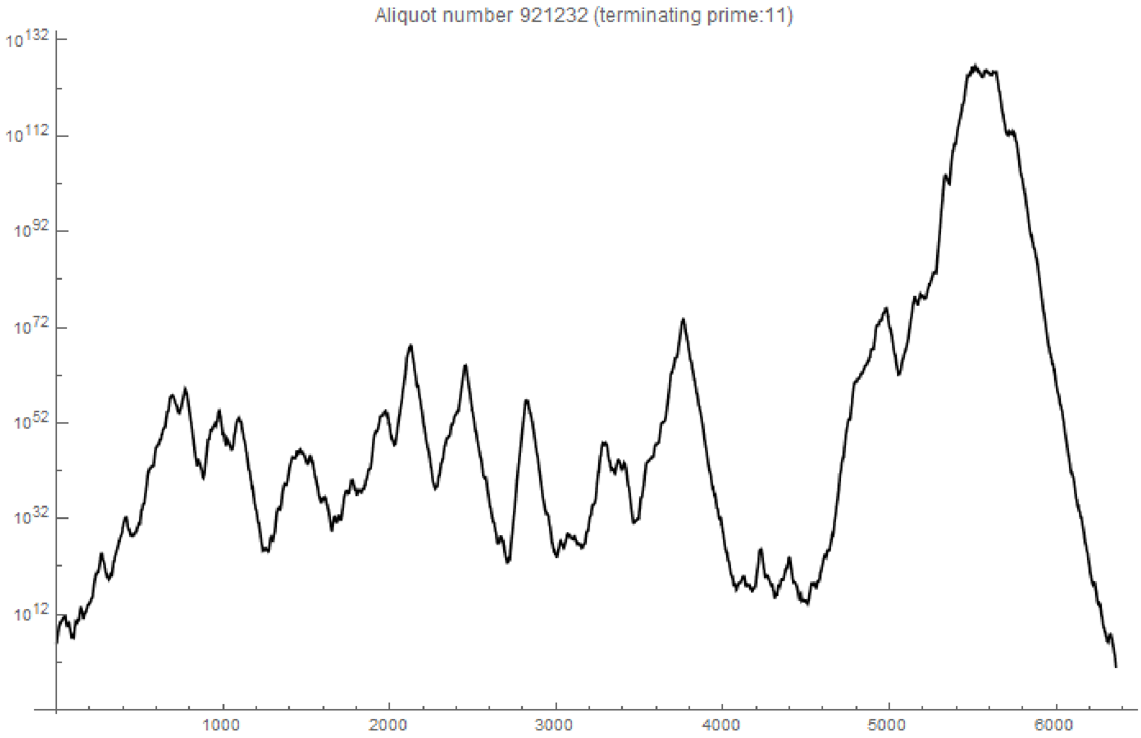


Figure 153. Aliquot sequence 921232 (terminates at 11, 6358 values)

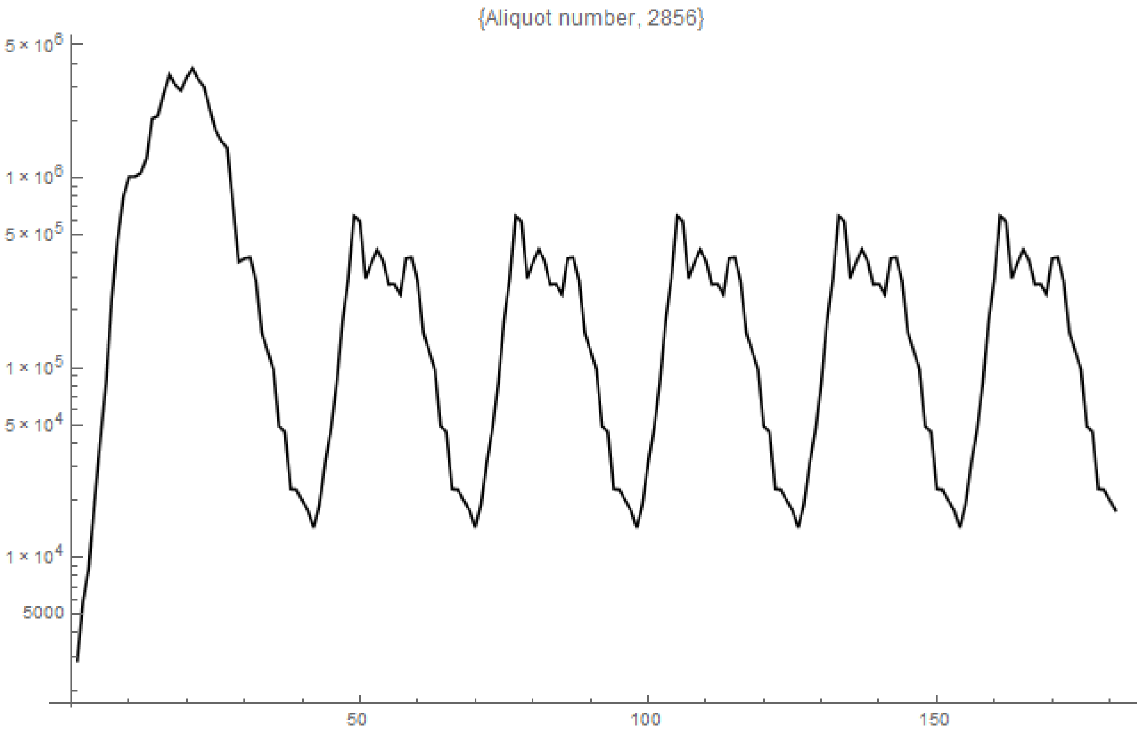


Figure 154. Aliquot sequence 2856 (terminates with a cycle of period 28)

‘Almost perfect’ numbers can also occur within an aliquot sequence, for example in the terminating sequence with starting value 840 for indices 139/140 and 140/142:

{13938528443323550460883494,13938528465780941432786**826**,13938528465780941432786**838**,23607694429544124013899**882**,23607694429544124013899**894**}

Here the successive sequence members differ only in the 26th position with a difference of 12!

20.9.2.2 FAMILIES OF ALIQUOT SEQUENCES

All aliquot sequences (belonging to different initial values) belong to the same family when they terminate in the same manner (i.e. with the same cycle, with the same prime number, or with the same OE sequence). A family of aliquot sequences can be represented very neatly by a tree structure. Here are a few examples (in which, of course, only the lower number range is represented):

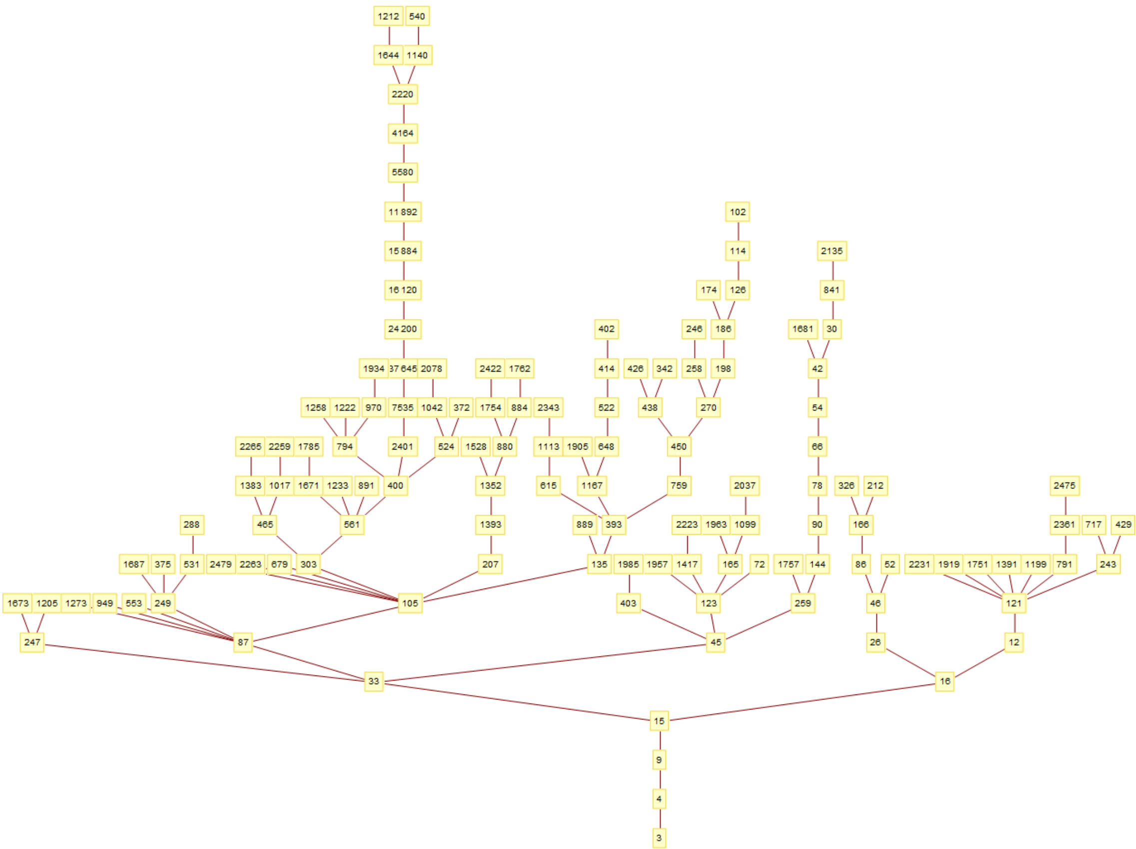


Figure 155. Family of aliquot sequences (sequences end with the prime number 3)

`Mathematica program: please contact the author.`

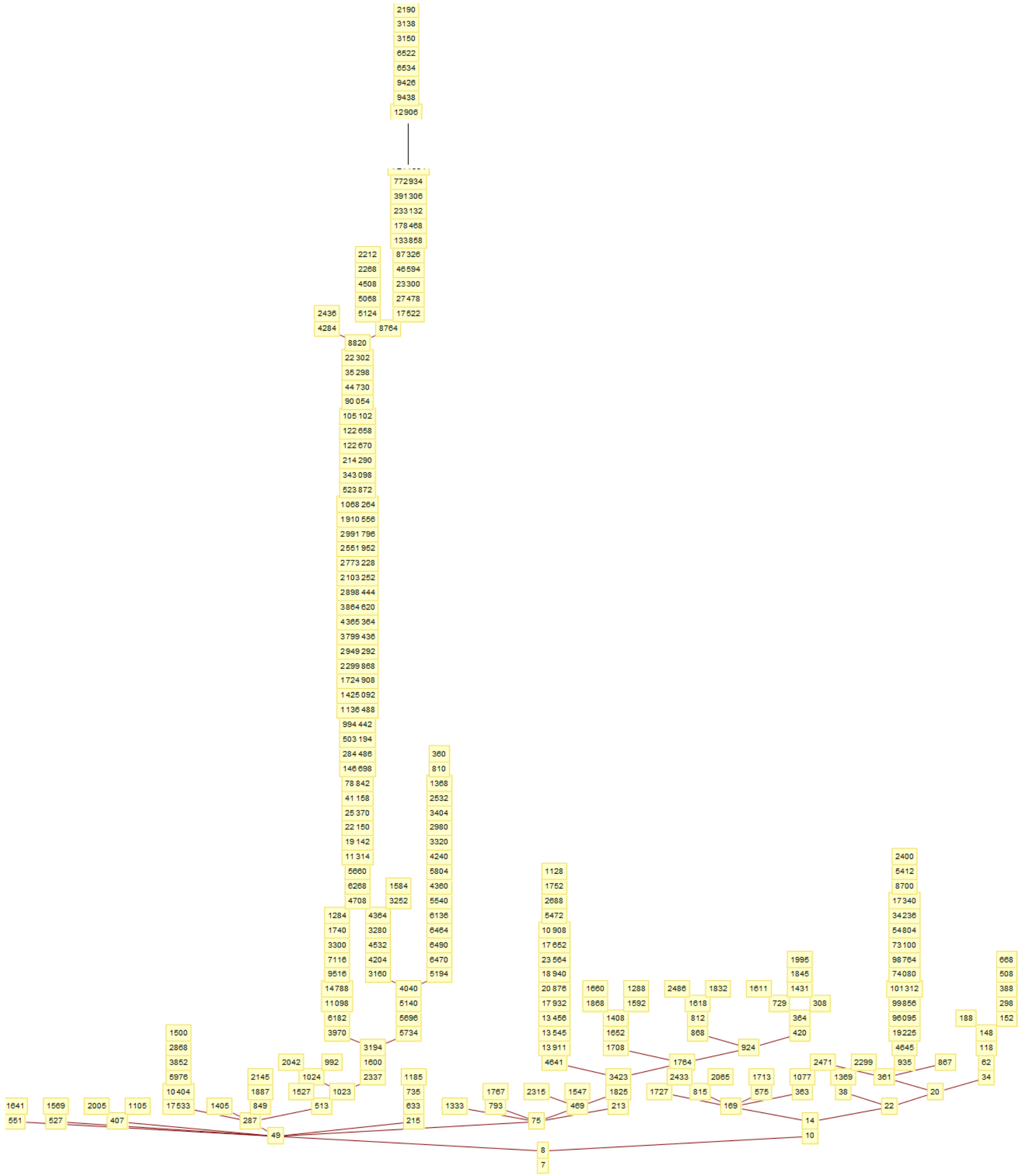


Figure 156. Family of aliquot sequences (sequences end with the prime number 7)

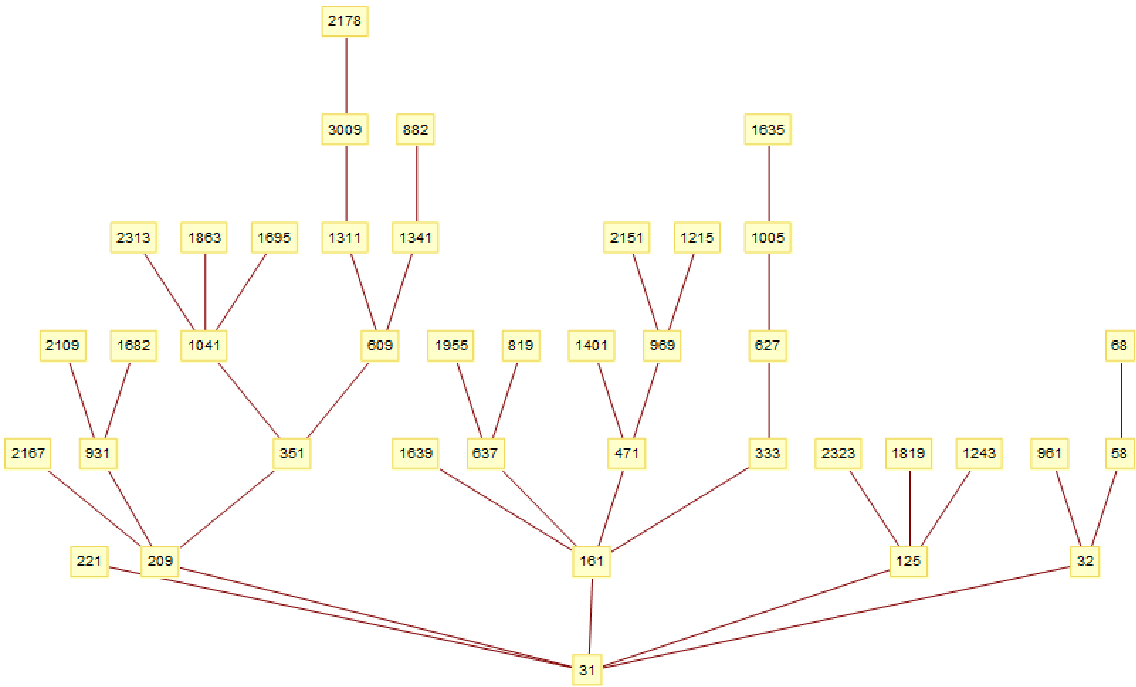


Figure 157. Family of aliquot sequences (sequences end with the prime number 31)

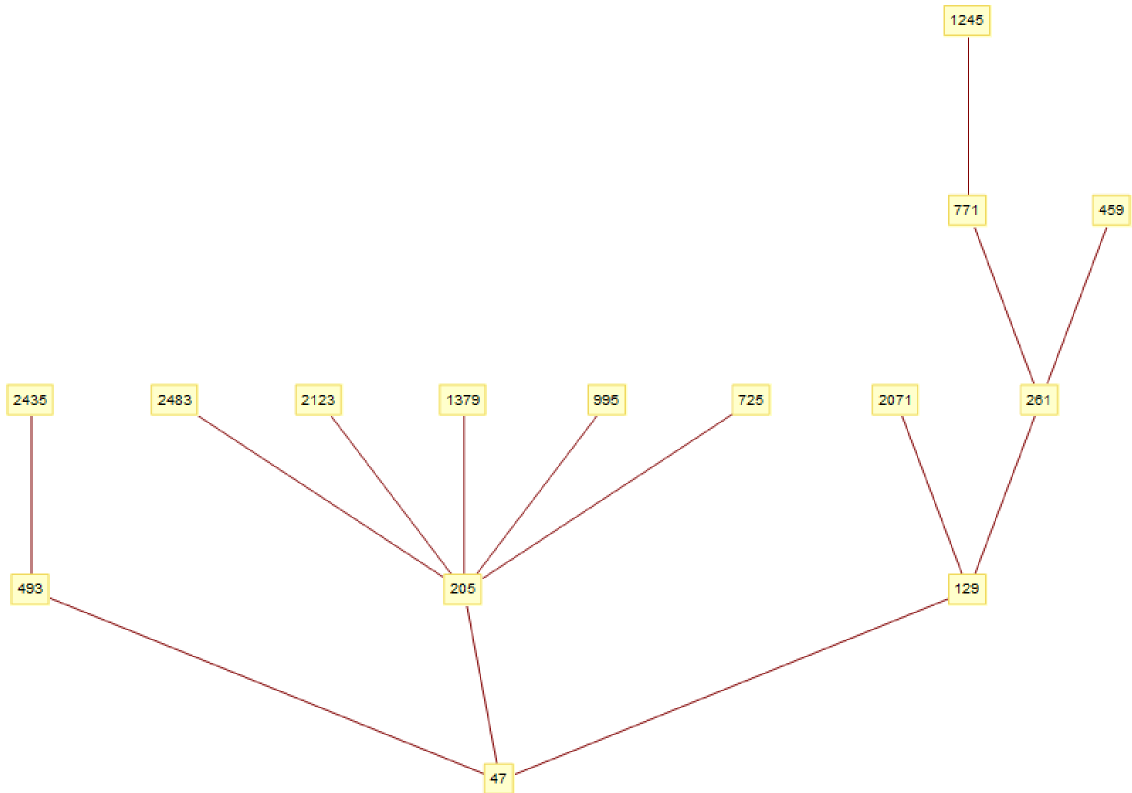


Figure 158. Family of aliquot sequences (sequences end with the prime number 47)

20.9.2.3 LENGTHS OF ALIQUOT SEQUENCES

The following convention applies to the calculation of the lengths of aliquot sequences: the sequence always starts with the initial value itself. All subsequent members are counted up to (and including) the first repeating value. Since primes have an aliquot sum of 1 and the 1 is followed by a value of 0, all primes have a sequence length of 3. Perfect numbers have a sequence length of 1. For OE sequences (admittedly somewhat arbitrarily), a sequence length of 10000 was determined.

Here is a list of the first 300 sequence lengths:

```
{2,3,3,4,3,1,3,4,5,5,3,8,3,6,6,7,3,5,3,8,4,7,3,6,2,8,4,1,3,16,3,4,7,9,
4,5,3,8,4,5,3,15,3,6,8,9,3,7,5,4,5,10,3,14,4,6,4,5,3,12,3,10,4,5,4,13,
3,6,5,7,3,10,3,6,6,6,4,12,3,8,6,7,3,7,4,10,8,8,3,11,5,7,5,5,3,10,3,4,5
,6,3,19,3,8,9,7,3,11,3,8,4,10,3,18,4,6,5,11,3,13,9,6,9,7,4,17,3,4,4,7,
3,12,5,8,10,9,3,179,3,6,6,7,3,10,5,7,7,12,3,178,3,13,7,9,4,9,3,8,5,12,
4,5,3,8,10,11,3,176,7,10,4,10,3,17,4,6,5,8,3,53,3,10,5,7,4,16,4,13,4,1
1,3,14,3,7,7,5,3,15,3,5,4,9,4,11,4,8,10,8,4,53,3,12,7,9,6,11,5,11,5,2,
4,177,3,18,9,7,3,9,3,10,8,12,3,176,4,8,4,8,3,12,3,4,10,12,4,16,8,13,9,
12,3,18,5,8,6,7,3,15,9,12,5,9,3,32,4,10,6,9,3,14,3,13,5,7,4,???,3,8,4,
17,3,18,3,2,8,12,6,11,6,13,4,8,3,17,5,8,6,14,4}
```

In the range of 80 and above, there appear to be strings for which no sequence lengths exist.

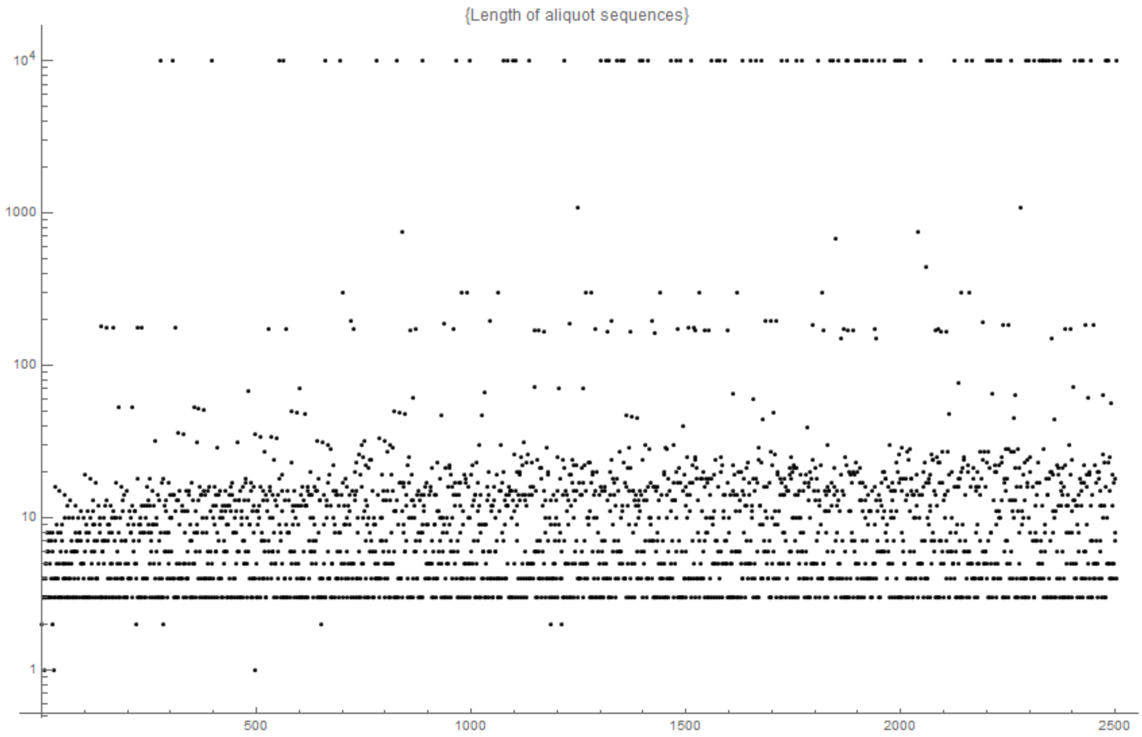


Figure 159. Aliquot sequence lengths up to n=2500, OE sequences are represented as having a length of 10000.

The Mathematica program with which the lengths were calculated can be found in the Appendix.

20.9.2.4 END VALUES OF ALIQUOT SEQUENCES

It is also interesting to consider the values with which aliquot sequences end. Since most sequences end with a 0 (with the predecessors of a prime number and a 1), such a graph would be extremely tedious if we were literally to take the last term (according to the length convention we introduced in the last chapter). So we examine the 'interesting' values and use the following convention for the final values:

In the case of sequences ending with 0, we regard the two places before the prime that appears as the end value, and for cyclical endings, we take the first element of the terminating cycle as the end value. For OE sequences, we select the value 1 (because we don't know the end…). Here is a list of the first 300 final values:

```
{1,2,3,3,5,6,7,7,3,7,11,3,13,7,3,3,17,11,19,7,11,7,23,17,6,3,13,28,29,
3,31,31,3,7,13,17,37,7,17,43,41,3,43,43,3,3,47,41,7,43,11,3,53,3,17,41
,23,31,59,43,61,7,41,41,19,3,67,31,13,43,71,3,73,43,7,41,19,3,79,41,43
,43,83,37,23,3,3,41,89,3,11,41,13,43,6,37,97,73,23,19,101,3,103,41,3,4
1,107,43,109,41,41,43,113,3,29,43,19,7,6,12161,3,41,3,19,31,3,127,127,
47,41,131,43,13,43,3,43,137,59,139,37,11,43,6,3,13,41,43,7,149,59,151,
7,43,43,37,37,157,43,23,43,31,71,163,41,3,3,167,59,7,43,89,43,173,3,73
,37,41,41,179,601,181,43,19,37,43,3,29,7,131,43,191,43,193,19,11,37,19
7,3,199,59,71,41,37,43,47,41,3,43,31,601,211,3,7,41,7,73,17,43,19,220,
31,59,223,41,41,43,227,41,229,41,43,43,233,59,53,37,83,19,239,12161,24
1,157,3,43,97,3,3,43,3,43,251,59,13,41,41,41,257,3,3,43,47,43,263,59,5
9,41,13,43,269,3,271,43,73,37,97,1,277,43,137,41,281,163,283,284,11,43
,7,3,11,43,101,43,293,163,19,37,19,7,37}
```

First of all, we notice that most endpoints consist of prime numbers. The few composite numbers belong to sequences that end cyclically. It is worth noting that the prime number 5 appears as a final value only once (namely, at position 5). In the range between 1 and 300 there is only a single OE sequence (marked by the red '1').

In the graphic representation, two lines appear that result from point accumulations. The curved line: this marks the prime numbers. The straight lines, parallel to the X-axis at the values 41, 43 and 59: here clearly an inexplicable accumulation can be seen.

The OE sequences all appear in a straight line at the value 1. On average, almost 8% of all aliquot sequences end at 43, about 5% at value 59, and 5% at 41. Other values such as 5 or 28 appear only once.

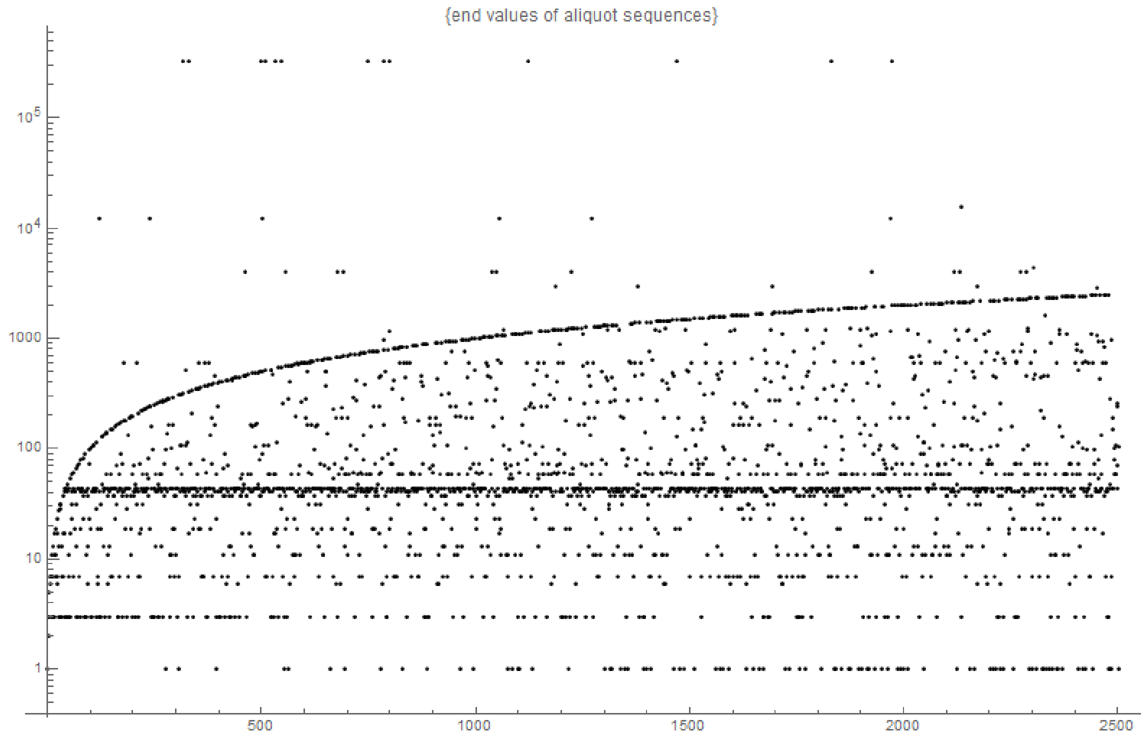


Figure 160. End values of aliquot sequences for initial values up to 2500

One may wonder which initial starting values result in cyclical end values (including the perfect numbers with cycle length 1). If these initial values are simply plotted in ascending order, then you can see that their 'density' remains constant on average because the slope is linear with a good approximation (the equation for the accompanying straight line is: $f(x) = 14.512 + 40.8404\,x$).

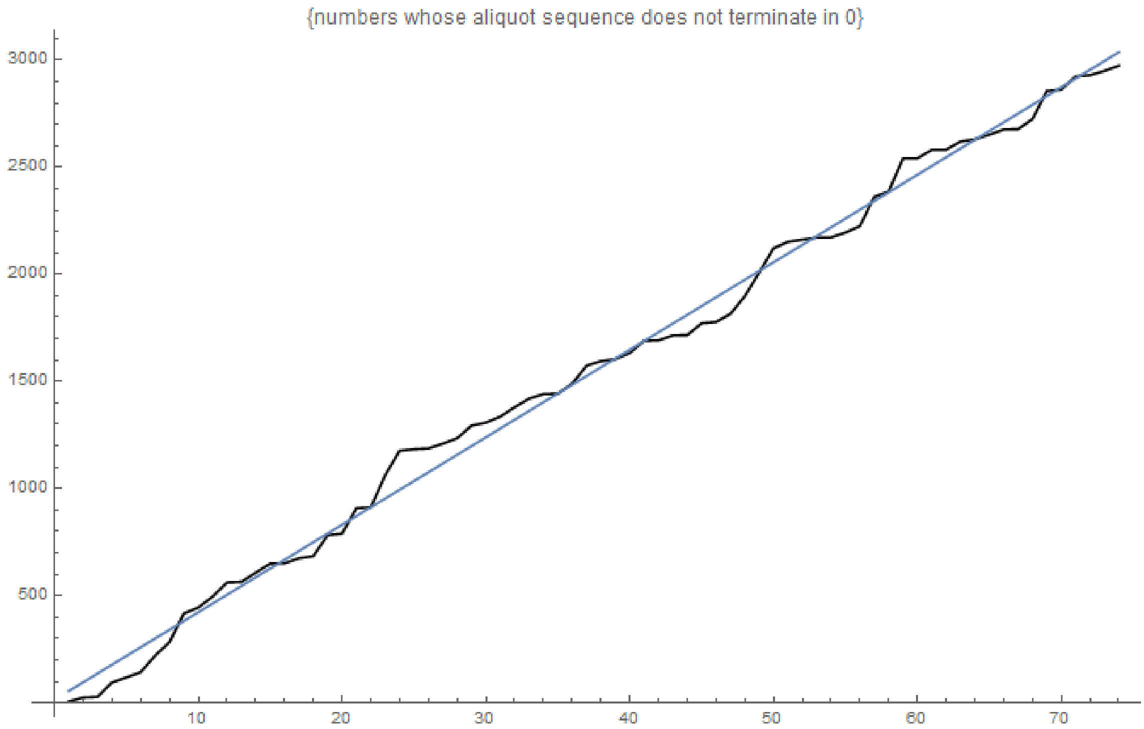


Figure 161. Initial values of aliquot sequences that terminate with a cycle

20.9.2.5 DIFFERENCES AND QUOTIENTS OF ALIQUOT SEQUENCES

The following observations were illustrated using the example of the aliquot sequence with the starting value 840. However, they generally apply in the same way for most other aliquot sequences

If we consider the differences of two successive sequence members, it is noticeable that in the majority of cases these are almost of the same order of magnitude as the sequence members themselves. An exception are the 'almost perfect' numbers (see Chapter 20.9.2.1). Moreover, a plot of the differences shows a certain 'form invariance' compared to the original aliquot sequence

This form invariance also persists in higher order differences (tested by the author to difference orders of over 20). In the plots the logarithmic values of the differences were taken. The form invariance becomes even more visible when the negative differences, "upwards folded" (i.e. the absolute values) are taken:

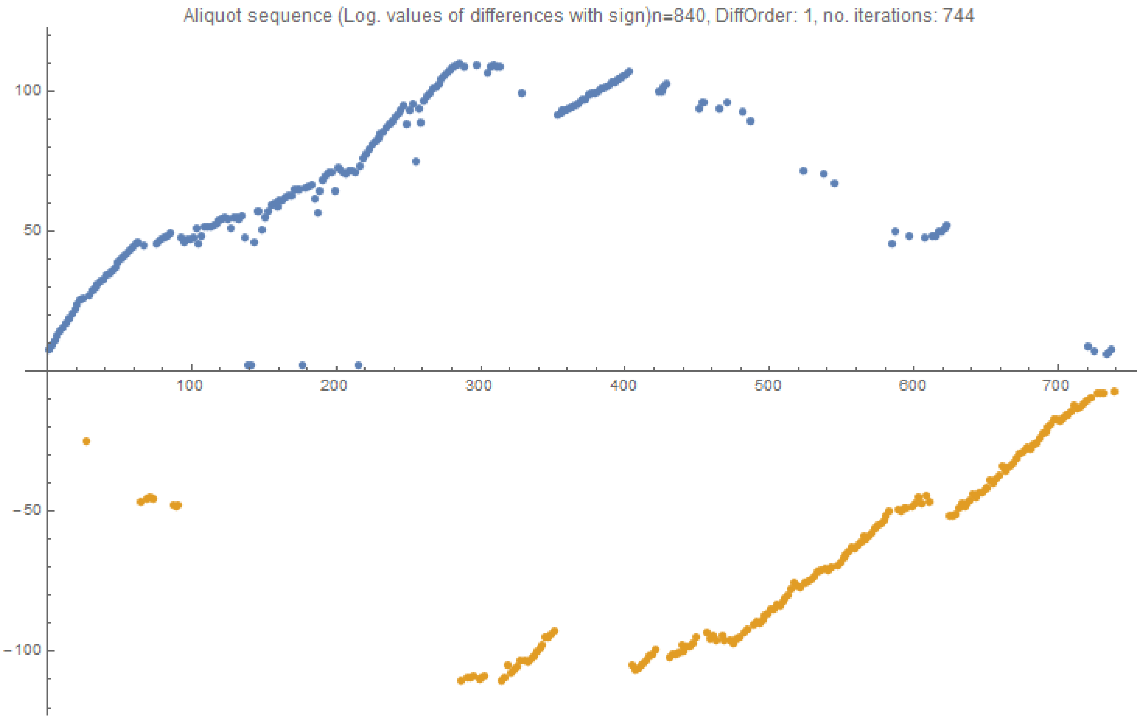


Figure 162. Aliquot sequence: log differences for initial value n= 840, preserving sign

```
Mathematica program: please contact the author.
```

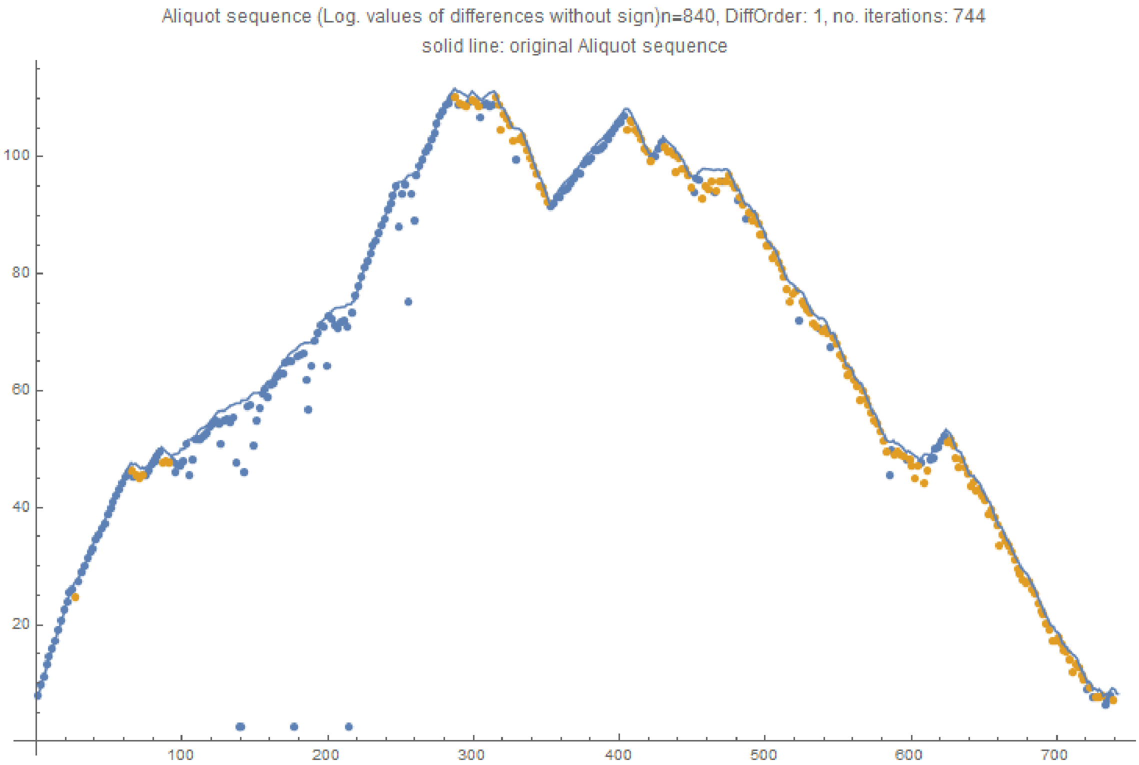


Figure 163. Aliquot sequence: log differences for initial value n= 840, absolute values

It is even more interesting if, instead of considering the logarithmic values of the differences, we look at the differences of the logarithmic values, which correspond to the quotient of two successive values. There are accumulation points which are approximately at the values $\ln\left(\frac{1}{2}\right), \ln\left(\frac{3}{4}\right), 0, \ln\left(\frac{5}{4}\right)$ and 1, which correspond to the quotient values of $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ and e. Furthermore, it can be observed that the 'ascents' within the sequences are on average steeper than the 'descents'. There are no distinct accumulation points for the gradients in the ascent, but an upper bound of 1 (or e) (apart from occasional small slip-ups in OE sequences, which are barely over 1). On descent we have a lower bound of -0.693 (or 0.5).

This means that the terms of aliquot sequences cannot grow faster than with a factor e, or become smaller by a factor of 0.5. As a matter of fact, the descent is always slower than the ascent, and yet almost all aliquot sequences redescend and come to rest on small values. Note: these are all purely empirical considerations, without claim to strict mathematical validity.

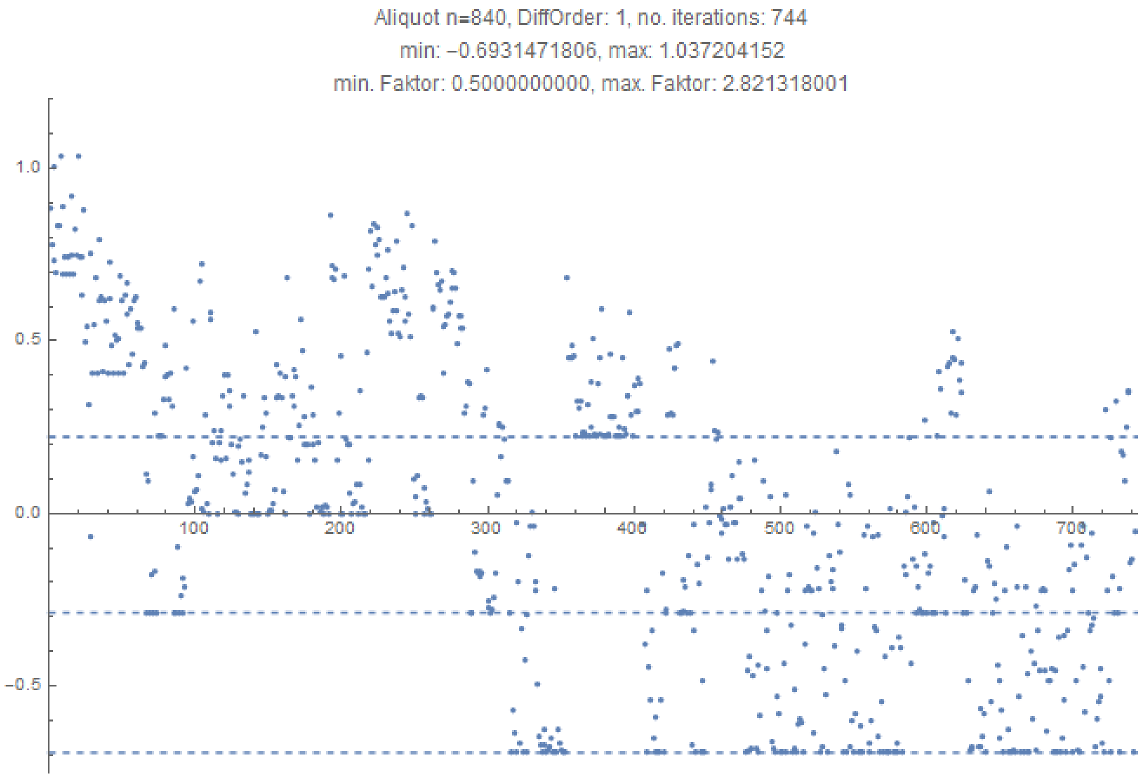


Figure 164. Aliquot sequence: differences of log values for initial value n= 840

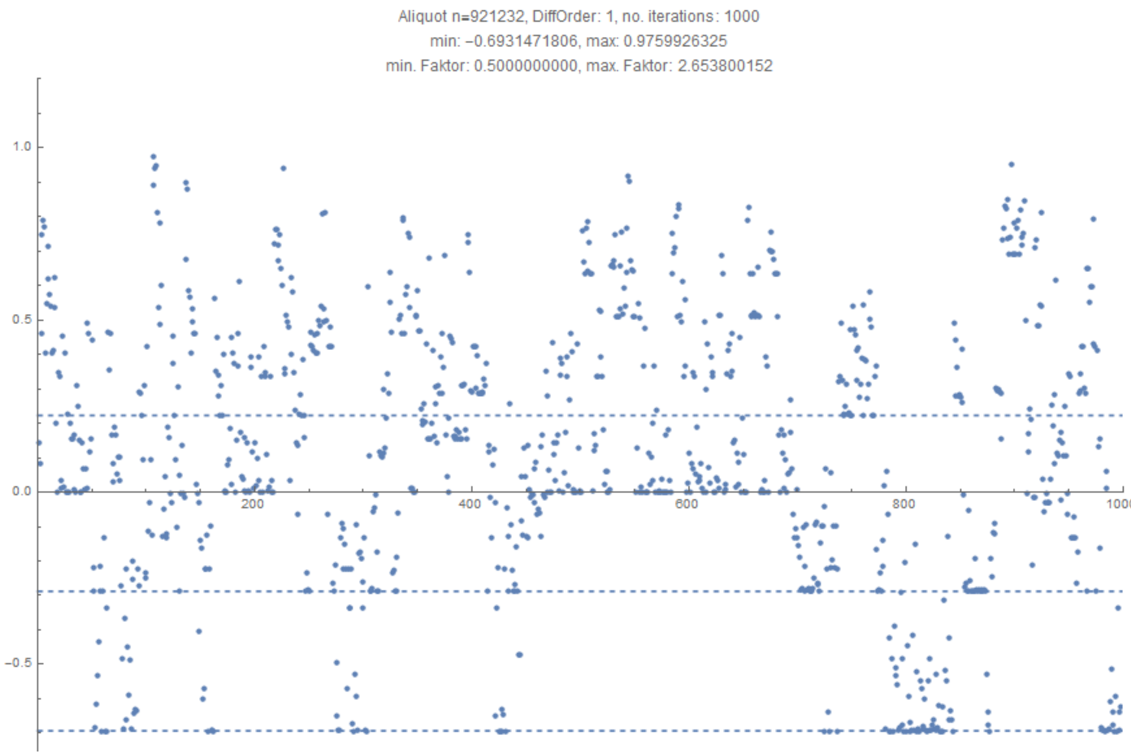


Figure 165. Aliquot sequence: differences of log values for initial value n= 921232

```
Mathematica:
(Program can be found in the Appendix)
```

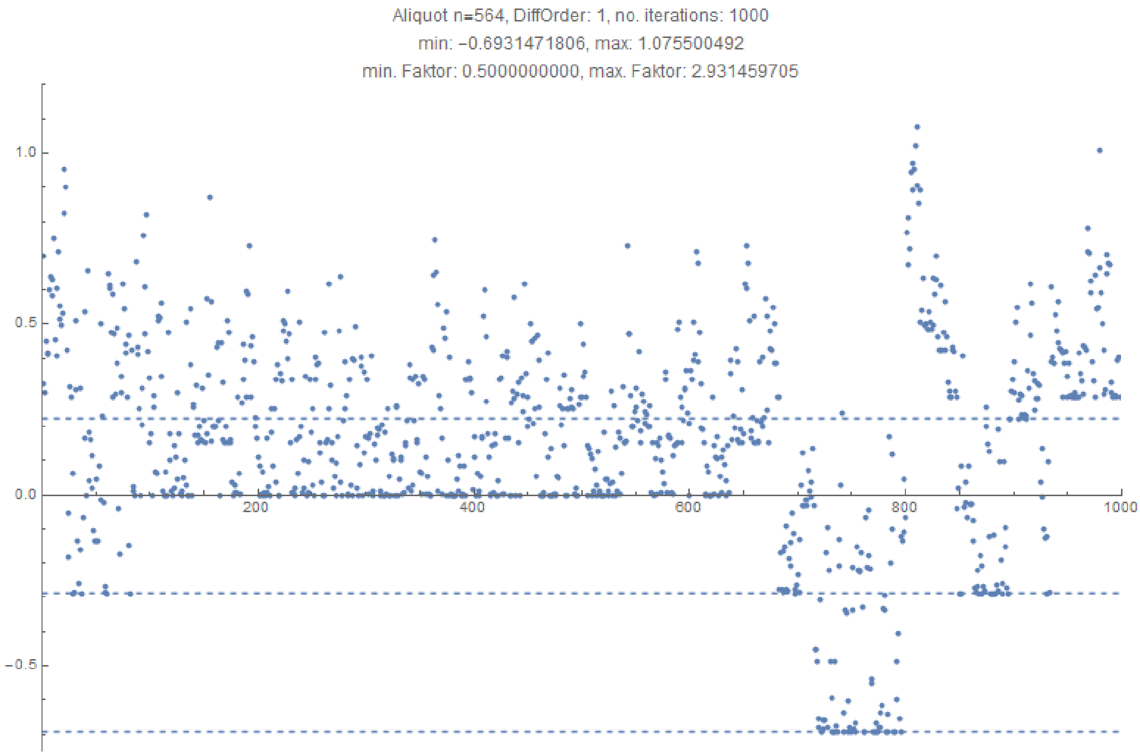


Figure 166. Aliquot sequence: differences of log values for initial value n= 564 (OE)

20.9.3 FACTORIZATION OF INTEGER NUMBERS

Mathematica can be used to factorize relatively large numbers:
For example, the following 68-digit number

```
CenterDot@@(Superscript@@@
FactorInteger[24284712165828060817808704394685584572191120513988451223045457718339])
```

returns the following prime factor after 1.5 seconds of computational time (on a 2.6 GHz Quad Core Intel PC):

$$29996224275833^2 \cdot 29996224275851^3$$

There are any number of methods using Mathematica to factorize numbers (whether or not they are very efficient is another question), for example:

```
n=1037;
Solve[x*y== n&&x>1&&x<n&&y>1&&y<n,{x,y},Integers]
```

or:

```
FindInstance[x*y== n&&x>1&&x<n&&y>1&&y<n,{x,y},Integers]
```

yields:

```
{{x->17,y->61}}
```

20.9.3.1 IMPORTANT FACTORIZATION METHODS

The most important factorization methods currently available (as of Feb. 2016) are listed here without explaining their algorithms or implementations. In practice, several methods are used depending on the number range. Several algorithms are usually combined with one another. Thus, at the beginning of a factorization process, a test with comparatively small factors (trial division) usually takes place in order to find small factors quickly before the "heavy artillery" is launched, such as the ECM method or the "number field sieve".

Factorization methods:

- Division by trial (finds out small factors)
- Fermat’s method
- Pollard ‘p-1’ method
- Pollard ‘rho’ method (searches for $x \equiv y \ (mod\ p)$ in a pseudorandom number sequence)
- Williams’s ‘p+1’ method
- Pomerance’s method
- Continued fractions methods
- ECM method using elliptic curves
- Shank’s SQUFOF method
- Quadratic sieve method
- Number field sieve

These factorization methods are documented extensively on many Internet web sites, so we will not discuss them here.

Because of its simplicity and beauty, Fermat's method is briefly described here:

Let n be the number to be factored. The algorithm works only for odd numbers n. We test the expression $\left(\lceil\sqrt{n}\rceil + i\right)^2 - n$ (by incrementing i, starting from $i = 0$) until it gives a squared number y^2:

$\left(\lceil\sqrt{n}\rceil + i\right)^2 - n = y^2$. With $x = \lceil\sqrt{n}\rceil + i$ this yields: $x^2 - n = y^2$, or

$n = (x + y)(x - y)$. Thus, we have found two factors of n.

Here is an example: n=1037. Then we have $\lceil\sqrt{1037}\rceil = 33$. We then get the following sequence:

$$(33+0)^2 - 1037 = 52$$
$$(33+1)^2 - 1037 = 119$$
$$(33+2)^2 - 1037 = 188$$
$$(33+3)^2 - 1037 = 259$$
$$(33+4)^2 - 1037 = 332$$
$$(33+5)^2 - 1037 = 407$$
$$(33+6)^2 - 1037 = 484\ (= 22 * 22)$$

Thereby x has a value of 39 and y the value 22 and thus we have both factors $p = 39 + 22 = 61$ and $q = 39 - 22 = 17. 1037 = 17 * 61$.

```
Mathematica:
n=17*61;sqN=Ceiling[Sqrt[n]];value=2;
For[i=0,i<n&&IntegerQ[Sqrt[value]]==False,i++,
Print[i,"->",value=(sqN+i)^2-n]];i--;
y=Sqrt[value]; x=sqN+i;
p=x+y; q=x-y
Print["factors: ",p,"*",q];
```

The algorithm can be accelerated by avoiding the repeated squaring and computing $(s + i)^2$ recursively from the predecessor term: $(s + 1)^2 - n = s^2 + 2s + 1 - n$. The test of whether y^2 is a square can also be accelerated by testing the last two digits of the number (there are only 22 of 100 different possibilities for the last two digits for any number of squares.

The runtime behaviour of this algorithm is good ($\sim\sqrt{n}$) if both factors are approximately of equal size. However, it becomes bad when one of the factors is very small (e.g. 3). The iteration is always finite, i.e. it will always end at a square. However, for primes, many iterations occur so that this method is unsuitable as a prime number test.

The bad run time for factors of different sizes can be circumvented by finding a suitable factor k such that the algorithm is applied to $k \cdot n$, finding two factors closer together.

Such an algorithm is much more efficient than the Fermat algorithm and is known as the 'Lehman method'[87].

20.9.3.2 OTHER FACTORIZATION METHODS

The author would like to present a few unconventional methods, regardless of their practical applicability.

The sigma phi method

Let n be the product of exactly two different prime numbers: $n = pq$.

Then:

$$\sigma(n) = (p+1)(q+1) = n + 1 + (p+q)$$

$$\varphi(n) = (p-1)(q-1) = n + 1 - (p+q)$$

p and q can be calculated thus:

$$p = \frac{(\sigma(n) - \varphi(n))}{4} - \sqrt{\left[\frac{(\sigma(n) - \varphi(n))}{4}\right]^2 - \left[\frac{(\sigma(n) + \varphi(n))}{2}\right] + 1} \tag{166}$$

$$q = \frac{(\sigma(n) - \varphi(n))}{4} + \sqrt{\left[\frac{(\sigma(n) - \varphi(n))}{4}\right]^2 - \left[\frac{(\sigma(n) + \varphi(n))}{2}\right] + 1} \tag{167}$$

Example: $n = 1037$
$\sigma(n): 1116, \varphi(n): 960, \frac{(\sigma(n)-\varphi(n))}{4}: 39$ yields $1037 = 17 * 61$

Example: $n = 519920418755535776857$
$\sigma(n): 519920418801139303860, \varphi(n): 519920418709932249856,$
$\frac{(\sigma(n)-\varphi(n))}{4}: 22801763501$ yields:
$519920418755535776857 = 22801763489 * 22801763513$

```
Mathematica:
n=519920418755535776857;
sigmaN=DivisorSigma[1,n]; eulerP=EulerPhi[n];
sum=sigmaN+eulerP; dif=sigmaN-eulerP; sqTerm=(dif/4)^2-sum/2+1;
p=dif/4-Sqrt[sqTerm]
q=dif/4+Sqrt[sqTerm]
```

Using this method, however, the problem of the factorization of n has only been ‘transformed’ to the determination $\sigma(n)$ and $\varphi(n)$, which again implies a similar complexity.

[87] Richard Crandall, Carl Pomerance: *Prime Numbers. A Computational Perspective, p. 191*

A ‘crazy’ method (analytically)

We consider the function of two variables

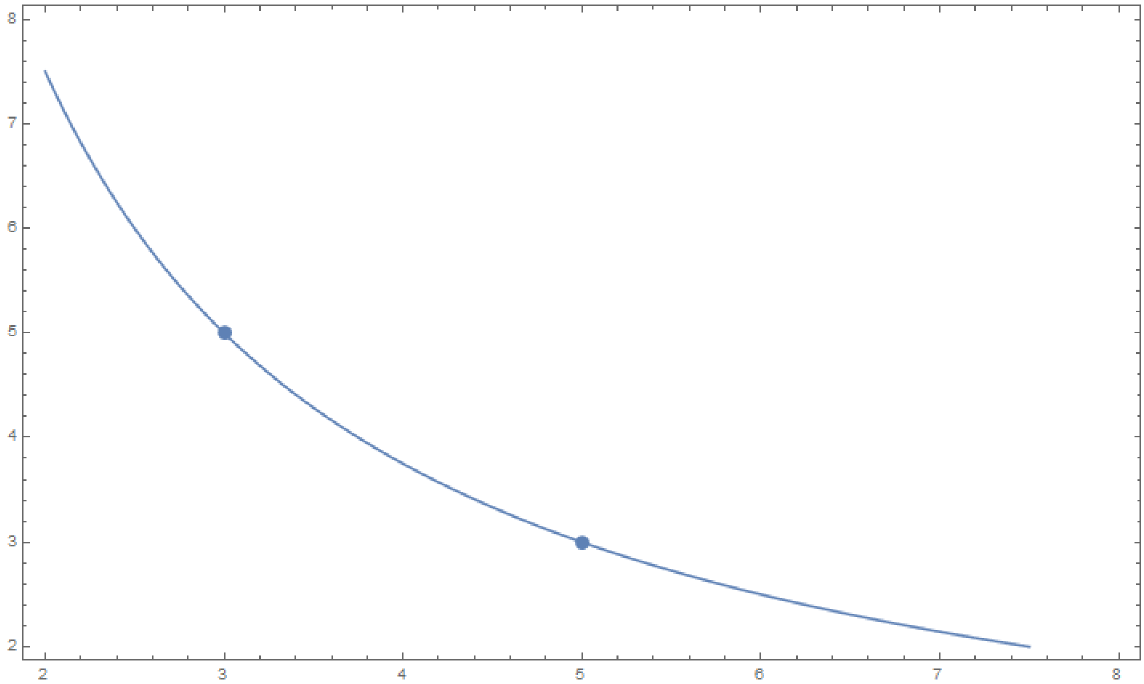
$$\text{productF}(x, y) = x * y - n$$

and examine for which values x and y this function assumes a value of 0. These values all lie on a ‘zero line’ and, so to speak, represent all 'real' factors of n (in this case, a hyperbole). If we extract the integer (x, y) −values from this ‘zero line’, then we have factorized n.

Example:

$$\text{productF}(x, y) = x * y - 15$$

The ‘zero line’ as a contour plot looks like this:



The integer values of the zero line lie, as can be seen, at the points(3,5) and (5,3).

Mathematica:

```
testF[m_]:=If[val=Abs[Round[{m}]-{m}];val[[1]][[1]]<10^(-
5)&&val[[1]][[2]]<10^(-5),True,False];
primeIndex=2; Prime[primeIndex]
Prime[primeIndex+1]
n=Prime[primeIndex]*Prime[primeIndex+1]
sqN=Round[Sqrt[n]+1];
productF[x_,y_]:=((x)*(y)-n);
(*Find Zero-Line:*)
ptsxy=ContourPlot[(productF[x,y]==0),{x,2,8},{y,2,8},MaxRecursion->4];
Show[ptsxy,ListPlot[{{3,5},{5,3}}],ImageSize->{708,425},AspectRatio->Full]
ptsxy1=Cases[Normal@ContourPlot[productF[x,y]==0,{x,2,8},{y,2,8},
MaxRecursion->4],Line[{x__}]:>x,Infinity]
```

```
Round[Select[ptsxy1,testF]]
Sort[DeleteDuplicates[Round[Select[ptsxy1,testF]]]]
```

Which yields:

```
{{3,5},{5,3}}
```

An analytic method

We start again with $x * y = n$, where $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ and search for integer solutions of x, y. This represents an equation with two unknown variables, with the boundary condition that x, y must be integer values. To solve equations with two unknowns, we need two equations. The question is now: how do we get a second equation expressing the boundary condition of being an integer? There are more possibilities for this, e.g. :

$$\sin\left(\pi\left(2x-\frac{1}{2}\right)\right)+\sin\left(\pi\left(2y-\frac{1}{2}\right)\right)+2=0 \tag{168}$$

$$x \cdot y = n \tag{169}$$

Equations (168) and (169) describe a nonlinear equation system of two equations with two unknowns. The real solution(s) of this system of equations yield the prime factors of our number n. However, the solution is difficult and not possible with simple methods. (169) can be solved for y and inserted into (168).

If we then apply a power function $(x)^{\frac{1}{3}}$ (to move the 'near-solutions' a little farther away from the X-axis), we get the following function:

$$\mathrm{fakFunc}(\mathrm{x},\mathrm{n})=\left(\sin\left(\pi\left(2x-\frac{1}{2}\right)\right)+\sin\left(\pi\left(2\frac{n}{x}-\frac{1}{2}\right)\right)+2\right)^{\frac{1}{3}} \tag{170}$$

The real zeros of fakFunc(x) give the complete list of all possible divisors of n.

Here is an example with $n = 1037$:

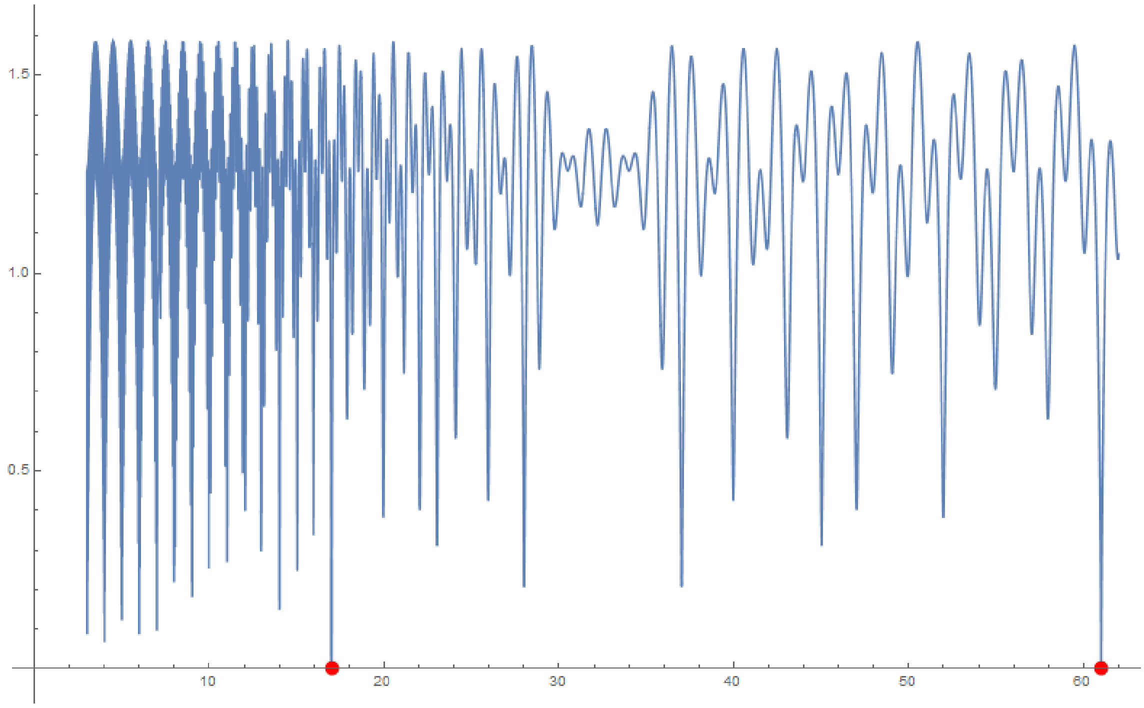


Figure 167. fakFunc(x, 1037) having zeros at the prime factors 17 and 61

```
Mathematica:
n=1037;
intFunc[x_,y_]:=(Sin[Pi*(2x-1/2)]+Sin[Pi*(2y-1/2)])+2;
Show[Plot[(intFunc[x,n/x])^(1/3),{x,3,62},MaxRecursion->15,AxesOrigin-
>{0,0}],ListPlot[{{17,0},{61,0}},PlotStyle->Red]]
```

The function intFunc(x, y), Formula (168), by the way, looks like an egg tray:

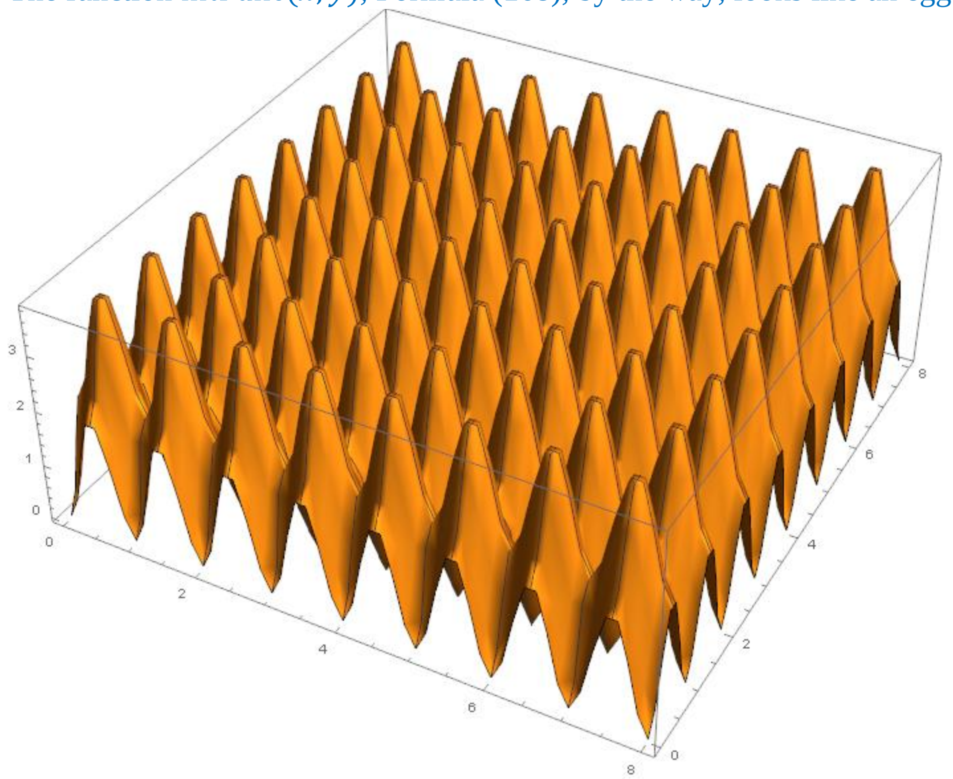


Figure 168. Function f(x,y) has zeros for each integer (x-y) point

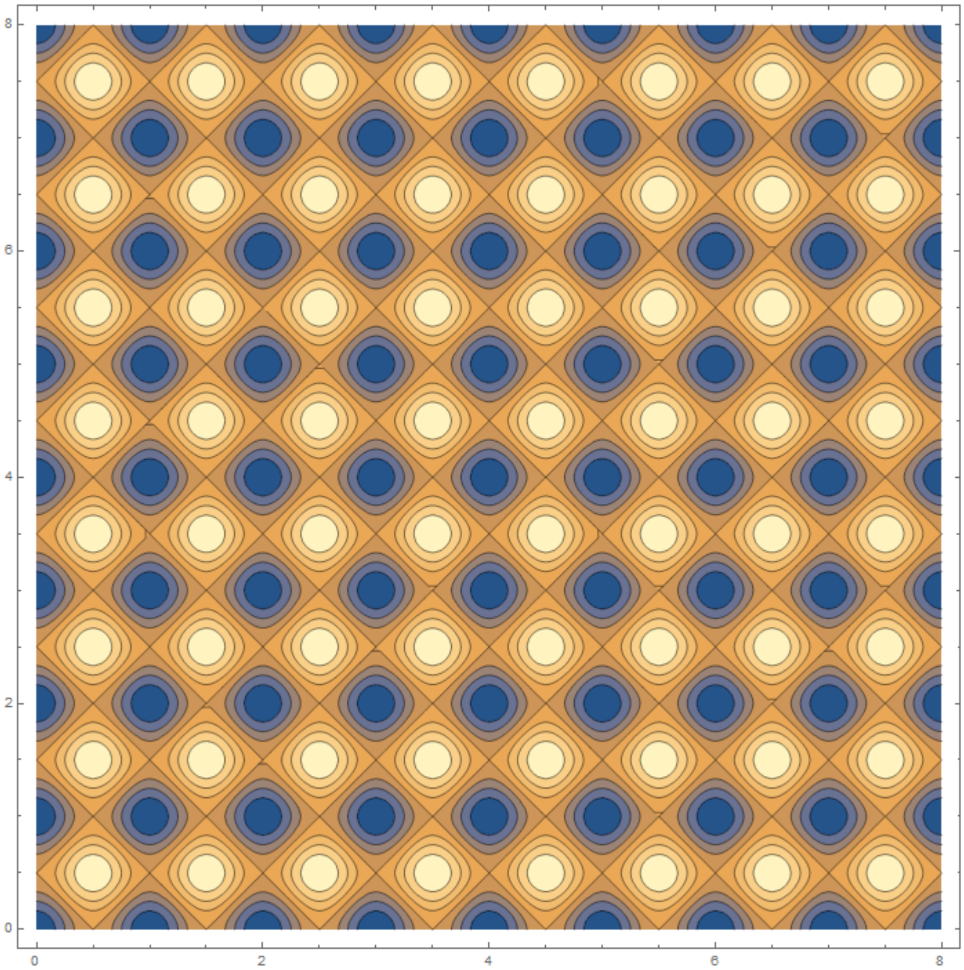


Figure 169. Same as above, but as a contour plot

```
Mathematica:

ContourPlot [intFunc[x,y],{x,0,8},{y,0,8},ImageSize-
>Large]Plot3D[intFunc[x,y],{x,0,8},{y,0,8},ImageSize->Large]
```

20.10 TABLES

20.10.1 NUMBER OF PRIMES UP TO A GIVEN LIMIT N: π(N)

Exact values of $\pi(x)$ for x to 10^{27} are available in the "Online Encyclopedia of Integer Sequences" (http://oeis.org). e.g.: A006880:

Table 29. Comparison of the exact π − function with the Riemann function (rounded)

n	$\pi(10^n)$	Riemann(10^n)	Riemann(10^n)- $\pi(10^n)$
0	0	0	0
1	4	5	1
2	25	26	1
3	168	168	0

4	1229	1227	2
5	9592	9587	-5
6	78498	78527	29
7	664579	664667	88
8	5761455	5761552	97
9	50847534	50847455	-79
10	455052511	455050683	-1828
11	4118054813	4118052495	-2318
12	37607912018	37607910542	-1476
13	346065536839	346065531066	-5773
14	3204941750802	3204941731602	-19200
15	29844570422669	29844570495887	73218
16	279238341033925	279238341360977	327052
17	2623557157654233	2623557157055978	-598255
18	24739954287740860	24739954284239494	-3501366
19	234057667276344607	234057667300228940	23884333
20	2220819602560918840	2220819602556027015	-4891825
21	21127269486018731928	21127269485932299724	-86432204
22	201467286689315906290	201467286689188773625	-127132665
23	1925320391606803968923	1925320391607837268776	1033299853
24	18435599767349200867866	18435599767347541878147	-1658989719
25	176846309399143769411680	176846309399141934626966	-1834784714
26	1699246750872437141327603	1699246750872419991992147	-17149335456
27	16352460426841680446427399	16352460426841662910939465	-17535487934
28	157589269275973410412739598		
29	1520698109714272166094258063		

Table 30. Comparison of the exact π − function with Riemann's exact formula $\pi^*(n)$ (see (132), sum over 10000 zeros, rounded

n	$\pi(10^n)$	$\pi^*(10^n)$	$\pi^*(10^n)$- $\pi(10^n)$
0	0	-	-
1	4	4	0
2	25	25	0
3	168	168	0
4	1229	1229	0
5	9592	9592	0
6	78498	78498	0
7	664579	664579	0
8	5761455	5761462	7
9	50847534	50847519	-15
10	455052511	455052528	17
11	4118054813	4118054697	-116
12	37607912018	37607911016	-1002

13	346065536839	346065537034	195
14	3204941750802	3204941747414	-3388
15	29844570422669	29844570424541	1872
16	279238341033925	279238341008610	-25315
17	2623557157654233	2623557157681368	27135
18	24739954287740860	24739954288134940	394080
19	234057667276344607	234057667277476288	1131681
20	2220819602560918840	2220819602559672832	-1246008
21	21127269486018731928	21127269486003990528	-14741400
22	201467286689315906290	201467286689365917696	50011406
23	1925320391606803968923	1925320391606731276288	-72692635
24	18435599767349200867866	18435599767349571354624	370486758
25	176846309399143769411680	176846309399143087341568	-682070112
26	1699246750872437141327603	1699246750872436043939840	-1097387763
27	16352460426841680446427399	16352460426841662628560896	-17817866503
28	157589269275973410412739598		
29	1520698109714272166094258063		

Table 31. Comparison of the exact π -function with Riemann's exact formula $\pi^*(n)$, sum over 100000 zeros, rounded

n	$\pi(10^n)$	$\pi^*(10^n)$	$\pi^*(10^n)$- $\pi(10^n)$
0	0	-	-
1	4	4	0
2	25	25	0
3	168	168	0
4	1229	1229	0
5	9592	9592	0
6	78498	78498	0
7	664579	664579	0
8	5761455	5761457	2
9	50847534	50847536	2
10	455052511	455052532	21
11	4118054813	4118054886	73
12	37607912018	37607911595	-423
13	346065536839	346065537866	1027
14	3204941750802	3204941749206	-1596
15	29844570422669	29844570413033	-9636
16	279238341033925	279238341037530	3605
17	2623557157654233	2623557157660142	5909
18	24739954287740860	24739954287711076	-29784
19	234057667276344607	234057667276885600	540993
20	2220819602560918840	2220819602559328000	-1590840
21	21127269486018731928	21127269486015279104	-3452824
22	201467286689315906290	201467286689324924928	9018638
23	1925320391606803968923	1925320391606799433728	-4535195
24	18435599767349200867866	18435599767349154021376	-46846490

25	1768463093991437694116800	176846309399143557103616	-212308064
26	16992467508724371413276003	16992467508724363123752960	-828952307
27	163524604268416804464273990	163524604268416604810772480	-19965350151
28	1575892692759734104127395980		
29	15206981097142721660942580630		

Table 32. Comparison of the exact π -function with Riemann's exact formula $\pi^*(n)$, sum over 1m zeros, rounded

n	$\pi(10^n)$	$\pi^*(10^n)$	$\pi^*(10^n)$- $\pi(10^n)$
0	0	-	-
1	4	4	0
2	25	25	0
3	168	168	0
4	1229	1229	0
5	9592	9592	0
6	78498	78498	0
7	664579	664579	0
8	5761455	5761456	1
9	50847534	50847536	2
10	455052511		
11	4118054813		
12	37607912018		
13	346065536839		
14	3204941750802		
15	29844570422669		
16	279238341033925		
17	2623557157654233		
18	24739954287740860		
19	234057667276344607		
20	2220819602560918840		
21	21127269486018731928		
22	201467286689315906290		
23	1925320391606803968923		
24	18435599767349200867866	18435599767349269364736	68496870
25	176846309399143769411680	176846309399144194637824	425226144
26	1699246750872437141327603	1699246750872437117681664	23645939
27	16352460426841680446427399	???	???
28	157589269275973410412739598		
29	1520698109714272166094258063		

The equivalence of the analytically calculated value with the exact value $\pi(10^{26})$ is remarkable: the value is exact up to 17 decimal digits! Nevertheless, the result is only three decimal places better compared to the 'normal' Riemann function (14 digits accuracy, even though the summation terms of the first 1,000,000 zeros of the zeta function were evaluated).

20.10.2 MERSENNE PRIME NUMBERS

This table contains all the exponents currently known (as of Dec. 2020).

No.	p (exponent)	Digits in M_p	Year	Discovered by
1	2	1	----	----
2	3	1	----	----
3	5	2	----	----
4	7	3	----	----
5	13	4	1456	anonymous
6	17	6	1588	Cataldi
7	19	6	1588	Cataldi
8	31	10	1772	Euler
9	61	19	1883	Pervushin
10	89	27	1911	Powers
11	107	33	1914	Powers
12	127	39	1876	Lucas
13	521	157	1952	Robinson
14	607	183	1952	Robinson
15	1279	386	1952	Robinson
16	2203	664	1952	Robinson
17	2281	687	1952	Robinson
18	3217	969	1957	Riesel
19	4253	1281	1961	Hurwitz
20	4423	1332	1961	Hurwitz
21	9689	2917	1963	Gillies
22	9941	2993	1963	Gillies
23	11213	3376	1963	Gillies
24	19937	6002	1971	Tuckerman
25	21701	6533	1978	Noll & Nickel
26	23209	6987	1979	Noll
27	44497	13395	1979	Nelson & Slowinski
28	86243	25962	1982	Slowinski
29	110503	33265	1988	Colquitt & Welsh

30	132049	39751	1983	Slowinski
31	216091	65050	1985	Slowinski
32	756839	227832	1992	Slowinski & Gage et al.
33	859433	258716	1994	Slowinski & Gage
34	1257787	378632	1996	Slowinski & Gage
35	1398269	420921	1996	Armengaud, Woltman, et al. (GIMPS)
36	2976221	895932	1997	Spence, Woltman, et al. (GIMPS)
37	3021377	909526	1998	Clarkson, Woltman, Kurowski et al. (GIMPS, PrimeNet)
38	6972593	2098960	1999	Hajratwala, Woltman, Kurowski et al. (GIMPS, PrimeNet)
39	13466917	4053946	2001	Cameron, Woltman, Kurowski et al. (GIMPS, PrimeNet)
40	20996011	6320430	2003	Shafer, Woltman, Kurowski et al. (GIMPS, PrimeNet)
41	24036583	7235733	2004	Findley, Woltman, Kurowski et al. (GIMPS, PrimeNet)
42	25964951	7816230	2005	Nowak, Woltman, Kurowski et al. (GIMPS, PrimeNet)
43	30402457	9152052	2005	Cooper, Boone, Woltman, Kurowski et al. (GIMPS, PrimeNet)
44	32582657	9808358	2006	Cooper, Boone, Woltman, Kurowski et al. (GIMPS, PrimeNet)
45	37156667	11185272	2008	Elvenich, Woltman, Kurowski et al. (GIMPS, PrimeNet)
46	42643801	12837064	2009	Strindmo, Woltman, Kurowski et al. (GIMPS, PrimeNet)
47	43112609	12978189	2008	Smith, Woltman, Kurowski et al. (GIMPS, PrimeNet)
48	57885161	17425170	2013	Cooper, Woltman, Kurowski et al. (GIMPS, PrimeNet)
??	74207281	22338618	2016	Cooper, Woltman (prime95), Kurowski & Blosser (PrimeNet), GIMPS et al
??	77232917	23249425	2017	GIMPS / Jon Pace (Prime95)
??	82589933	24862048	2018	GIMPS / Patrick Laroche (Prime95)

20.10.3 FERMAT PRIME NUMBERS

At present, only five Fermat primes are known. These are:

$$\mathbf{3, 5, 17, 257, 65537}$$

20.10.4 DEGENERATION OF TYPE 4 OCRONS AND EOCRONS

Table 33. The first 23 type 4 OCRONs, GOCRONs (GC) as well as the corresponding degenerations

n	GC(“*”=0, “P”=1, “2”=2, “^”=3), OCRON	n	GC(“*”=0, “P”=1, “2”=2, “^”=3), OCRON
2	2 2	13	609 2P2*P 657 22P*P
3	9 2P	14	2584 22*P2* 2692 222*P* 2740 222^P* 2776 22^P2*
4	40 22* 43 22^	15	2404 2PP2P* 2452 2P2PP*
5	37 2PP	16	651 22*2^ 675 222*^ 687 222^^ 699 22^2^ 2680 22P^2* 2716 222P^* 10376 22*2*2* 10400 22*22** 10412 22*22^* 10760 222**2* 10784 222*2** 10880 2222*** 10928 2222^** 10952 222^*2* 10976 222^2** 11144 22^2*2* 11168 22^22** 11180 22^22^*
6	152 2P2* 164 22P*	17	645 22*PP 693 22^PP
7	161 22*P 173 22^P	18	2488 2P2^2* 2668 22P2^* 9764 2P2*2P* 9800 2P2P*2* 9824 2P2P2** 9872 2P22P** 10532 22P*2P* 10640 22P2P**
8	167 22P^ 648 22*2* 672 222** 684 222^* 696 22^2*	19	669 22P^P 2593 22*2*P 2689 222**P 2737 222^*P 2785 22^2*P
9	155 2P2^ 612 2P2P*	20	9608 2PP2*2* 9632 2PP22** 9644 2PP22^* 10388 22*2PP* 10568 22PP*2*

			10592 22PP2** 10832 222PP** 11156 22^2PP*
10	600 2PP2* 660 22PP*	21	9860 2P22*P* 9908 2P22^P* 10340 22*P2P* 11108 22^P2P*
11	149 2PPP	22	2392 2PPP2* 2644 22PPP*
12	2440 2P2*2* 2464 2P22** 2476 2P22^* 2596 22*2P* 2632 22P*2* 2656 22P2** 2704 222P** 2788 22^2P*	23	621 2P2^P 2449 2P2P*P

Table 34. Number of degenerations for EOCRONs of type 4

n	Degener.	n	Degener.	n	Degener.	n	Degener.
1	-	26	13	51	10	76	185
2	-	27	10	52	63	77	10
3	1	28	63	53	39	78	85
4	4	29	3	54	106	79	3
5	1	30	36	55	4	80	693
6	5	31	1	56	311	81	39
7	3	32	271	57	30	82	13
8	16	33	4	58	13	83	2
9	3	34	13	59	3	84	594
10	5	35	10	60	260	85	10
11	1	36	159	61	11	86	32
12	26	37	13	62	5	87	10
13	3	38	40	63	45	88	134
14	13	39	10	64	1139	89	61
15	4	40	134	65	10	90	231
16	68	41	3	66	36	91	24
17	3	42	85	67	10	92	52
18	23	43	8	68	36	93	4
19	10	44	26	69	8	94	10
20	26	45	19	70	85	95	30
21	10	46	10	71	13	96	3508
22	5	47	2	72	997	97	2
23	2	48	693	73	5	98	111
24	134	49	15	74	55	99	19
25	3	50	23	75	19	100	159

The degeneration of EOCRONs of type 4 is significantly larger than for 'normal' OCRONs of type 4 (see Table 20).

20.10.5 ZEROS OF RAMANUJAN'S TAU L FUNCTION

Table 35. The first 128 zeros or Ramanujan' s tau L function along the critical line $\mathrm{Re}(s) = 6$

n	n-th Zero (imaginary part)
1	9.222379399921084797142611932940781116486
2	13.907549861392134005200205137953162193298
3	17.44277697823447326186396821867674589157
4	19.65651314195496013326192041859030723572
5	22.33610363720986669022749993018805980682
6	25.27463654811236537511831556912511587143
7	26.80439115835040198021488322410732507706
8	28.83168262418687532999683753587305545807
9	31.17820949836026045431935926899313926697
10	32.77487538223120822067357948981225490570
11	35.19699584121007518433543737046420574188
12	36.74146297671030936271563405171036720276
13	37.75391597562427392631434486247599124908
14	40.21903437422132299161603441461920738220
15	41.73049228930784693147870711982250213623
16	43.59174123557517077642842195928096771240
17	45.04007921377559853226557606831192970276
18	46.19731875314330693527153925970196723938
19	48.35905247802367057374794967472553253174
20	49.27605353655818021252343896776437759399
21	51.15656028143634870275491266511380672455
22	53.06671423542580612320307409390807151794
23	54.09995263156227451872837264090776443481
24	55.21778745348462535957878571934998035431
25	56.71529404472536839421081822365522384644
26	58.58016100791407154702028492465615272522
27	59.78593800331714191997889429330825805664
28	61.13672295792679989290263620205223560333
29	62.66499232630715710001823026686906814575
30	64.08664571892624906013224972411990165710
31	64.84864127982825721119297668337821960449
32	66.49476926718958225137612316757440567017
33	67.93860977475046070139796938747167587280
34	69.04339787488993351871613413095474243164
35	71.11465341424647590429231058806180953979
36	71.74750419616562169267126591876149177551
37	72.81406066758940198724303627386689186096
38	74.09582544001794701671315124258399009705
39	75.77216168976411836410989053547382354736
40	77.10183189348964560849708504974842071533
41	77.68461125026033187168650329113006591797
42	79.79293909123566663765814155340194702148
43	80.56019206809750698994321282953023910522
44	82.00757620451852858423080760985612869263
45	82.84252583957207605180883547291159629822
46	83.97564035576498042701132362708449363708
47	85.46221814858006382564781233668327331543

48	86.75597218825528500474320026114583015442
49	88.07513099425673885889409575611352920532
50	89.02289034074360074555443134158849716187
51	90.45103289616260155980853596702218055725
52	91.11271853147249544235819485038518905640
53	92.44292549472127973331225803121924400330
54	93.76912394743676770758611382916569709778
55	95.13807853977348827356763649731874465942
56	95.62492107704515831301250727847218513489
57	97.34104088984686597996187629178166389465
58	98.70980408818076057286816649138927459717
59	99.74664890030413744170800782740116119385
60	100.22461499968198950227815657854080200195
61	101.34359353371037570923363091424107551575
62	103.16663591563629154279624344781041145325
63	103.81733899744642712903441861271858215332
64	105.22181333799052538324758643284440040588
65	106.29382213420061020769935566931962966919
66	107.42670755392653347826126264408230781555
67	108.47543790163686594496539328247308731079
68	109.39169607602677558588766260072588920593
69	110.70966268400202636712492676451802253723
70	111.53473540163911081890546483919024467468
71	112.75715359897023404300853144377470016479
72	113.84343404772059216156776528805494308472
73	115.06276556053481385788472834974527359009
74	116.46348398369597987311863107606768608093
75	117.11654084727238966934237396344542503357
76	118.14687073684822848917974624782800674438
77	119.08216779664660123216890497133135795593
78	119.99454209523629799605259904637932777405
79	121.78633067852094029603904346004128456116
80	122.55731782502655846656125504523515701294
81	123.21241716312161429414118174463510513306
82	124.60624049116798062186717288568615913391
83	125.94289344930038510028680320829153060913
84	126.75939204586923381157248513773083686829
85	127.55580316015350206271250499412417411804
86	128.62383894451065202702011447399854660034
87	129.60342208412549780405242927372455596924
88	130.94859240739617689541773870587348937988
89	131.70819904811898481966636609286069869995
90	132.96854278614409849978983402252197265625
91	134.34729668877156427697627805173397064209
92	135.07869588873938937467755749821662902832
93	135.55289998752846258867066353559494018555
94	137.09033471100445922274957410991191864014
95	137.70022292031720212435175199061632156372
96	139.28400855168445104936836287379264831543
97	139.93658439005704963165044318884611129761
98	140.89653322681010649830568581819534301758
99	142.1411519890185388703685021027922630310
100	143.0835552634784448855498339980840682983
101	144.3547263694031244085635989904403686523

102	145.1653120064068502870213706046342849731
103	146.1487705718024301404511788859963417053
104	146.4097883646259958823065971955657005310
105	148.1177541226128084872470935806632041931
106	149.0412678815713718449842417612671852112
107	150.2750742969780901603371603414416313171
108	150.9064237539794532949599670246243476868
109	152.1344343784803641028702259063720703125
110	153.1151471940314081621181685477495193481
111	154.0518290966241181649820646271109580994
112	154.7953122295758987547742435708642005920
113	155.7320793911374607887410093098878860474
114	157.0957831922944762936822371557354927063
115	157.9127528865146530279162107035517692566
116	158.6608139225808713490550871938467025757
117	159.6686139103367452207749010995030403137
118	161.3063702811864743580372305586934089661
119	161.8503586051299976134032476693391799927
120	162.8714549225416021727141924202442169189
121	163.5474941087671822970150969922542572021
122	164.3389052284337310538830934092402458191
123	165.6101228957916760009538847953081130981
124	166.5807970056847295836632838472723960876
125	167.6436347091075731441378593444824218750
126	168.6591247847260888192977290600538253784
127	169.2457741065447009987110504880547523499
128	170.5979320487521135873976163566112518311

Mathematica program: please contact the author.

20.10.6 THE *ABC* CONJECTURE: FIT PARAMETER AND C3 VALUES OF THE PLANE EQUATIONS FOR DIFFERENT GÖDELIZATION METHODS

Table 36. c=30011. Fit parameter and c_3 of the plane equations for M_{abc} (type M2GOCRON4) for different assignments of Gödel symbols

C	c_3	Code table: symbols/values	Max. value	Standard error	t-statistics
30011	3.50329	1:{*,P,2,^},{0,1,2,3}	31.5607	0.00147067	2382.1
30011	3.31794	2:{*,P,2,^},{0,1,3,2}	31.8085	0.00203153	1633.22
30011	3.4018	3:{*,P,2,^},{0,2,1,3}	31.1111	0.00109664	3102.03
30011	**3.01535**	**4:{*,P,2,^},{0,2,3,1}**	**31.7433**	**0.000640746**	**4706.**
30011	3.1857	5:{*,P,2,^},{0,3,1,2}	30.9755	0.00200754	1586.86
30011	2.97014	6:{*,P,2,^},{0,3,2,1}	31.3842	0.000996648	2980.14
30011	4.00661	9:{*,P,2,^},{1,2,0,3}	30.2574	0.00451077	886.529
30011	**3.03373**	**10:{*,P,2,^},{1,2,3,0}**	**31.6698**	**0.000607757**	**4991.67**
30011	3.78849	11:{*,P,2,^},{1,3,0,2}	30.2873	0.00555072	682.522
30011	2.99422	12:{*,P,2,^},{1,3,2,0}	31.2772	0.00115591	2590.36
30011	4.50418	15:{*,P,2,^},{2,1,0,3}	30.242	0.00378901	1188.75
30011	3.35767	16:{*,P,2,^},{2,1,3,0}	31.6661	0.00190649	1761.18
30011	3.93106	17:{*,P,2,^},{2,3,0,1}	30.2885	0.00646281	608.258
30011	3.2632	18:{*,P,2,^},{2,3,1,0}	30.611	0.0024814	1315.06
30011	4.61984	21:{*,P,2,^},{3,1,0,2}	29.8592	0.00406558	1136.33

30011	3.58239	22:{*,P,2,^},{3,1,2,0}	31.2661	0.0011851	3022.85
30011	4.25748	23:{*,P,2,^},{3,2,0,1}	29.886	0.00587147	725.113
30011	3.5185	24:{*,P,2,^},{3,2,1,0}	30.6002	0.0016776	2097.34

Table 37. c=10009. Fit parameter and c_3 of the plane equations for M_{abc} (type M2GOCRON4) for different assignments of Gödel symbols

C	c_3	Code table: symbols/values	Max. value	Standard error	t-statistics
10009	3.50252	1:{*,P,2,^},{0,1,2,3}	27.4048	0.00255229	1372.31
10009	3.31693	2:{*,P,2,^},{0,1,3,2}	27.6525	0.00352484	941.016
10009	3.40264	3:{*,P,2,^},{0,2,1,3}	26.9558	0.00190137	1789.57
10009	**3.01507**	**4:{*,P,2,^},{0,2,3,1}**	**27.6497**	**0.00111371**	**2707.22**
10009	3.18746	5:{*,P,2,^},{0,3,1,2}	26.9829	0.00348145	915.555
10009	2.97092	6:{*,P,2,^},{0,3,2,1}	27.4021	0.00172728	1719.99
10009	4.00661	9:{*,P,2,^},{1,2,0,3}	26.104	0.00790261	506.999
10009	**3.0335**	**10:{*,P,2,^},{1,2,3,0}**	**27.6466**	**0.00105616**	**2872.2**
10009	3.79832	11:{*,P,2,^},{1,3,0,2}	26.278	0.0097115	391.116
10009	2.9951	12:{*,P,2,^},{1,3,2,0}	27.398	0.00200372	1494.77
10009	4.50975	15:{*,P,2,^},{2,1,0,3}	26.0831	0.00668297	674.813
10009	3.35675	16:{*,P,2,^},{2,1,3,0}	27.5776	0.00330739	1014.92
10009	3.94228	17:{*,P,2,^},{2,3,0,1}	26.277	0.0113075	348.642
10009	3.26532	18:{*,P,2,^},{2,3,1,0}	26.9845	0.00430428	758.62
10009	4.62653	21:{*,P,2,^},{3,1,0,2}	25.7004	0.00719384	643.124
10009	3.58183	22:{*,P,2,^},{3,1,2,0}	27.2467	0.00205594	1742.19
10009	4.26777	23:{*,P,2,^},{3,2,0,1}	25.9801	0.0102978	414.436
10009	3.51983	24:{*,P,2,^},{3,2,1,0}	26.8782	0.00291095	1209.17

Table 38. c=10009. Fit parameter and c_3 of the plane equations for M_{abc} (type EGOCRON4) for different assignments of Gödel symbols

C	c_3	Code table: symbols/values	Max. value	Standard error	t-statistics
10009	2.1217	1:{*,P,2,^},{0,1,2,3}	23.2525	0.00256872	825.975
10009	1.93604	2:{*,P,2,^},{0,1,3,2}	23.5014	0.00353924	547.021
10009	2.02181	3:{*,P,2,^},{0,2,1,3}	23.1332	0.00187808	1076.53
10009	**1.63421**	**4:{*,P,2,^},{0,2,3,1}**	**23.4314**	**0.00112142**	**1457.26**
10009	1.80654	5:{*,P,2,^},{0,3,1,2}	23.526	0.00345596	522.731
10009	1.59005	6:{*,P,2,^},{0,3,2,1}	23.5452	0.00169719	936.872
10009	2.61236	9:{*,P,2,^},{1,2,0,3}	23.1025	0.00785712	332.484
10009	**1.65067**	**10:{*,P,2,^},{1,2,3,0}**	**23.3575**	**0.00106003**	**1557.19**
10009	2.40797	11:{*,P,2,^},{1,3,0,2}	23.5053	0.00967604	248.859
10009	1.61167	12:{*,P,2,^},{1,3,2,0}	23.5442	0.00198794	810.723
10009	3.08406	15:{*,P,2,^},{2,1,0,3}	22.414	0.00649123	475.111
10009	1.97171	16:{*,P,2,^},{2,1,3,0}	23.3565	0.00330899	595.863
10009	2.5431	17:{*,P,2,^},{2,3,0,1}	23.5043	0.0112604	225.844
10009	1.87643	18:{*,P,2,^},{2,3,1,0}	23.5239	0.00430664	435.707
10009	3.18294	21:{*,P,2,^},{3,1,0,2}	22.4109	0.0068767	462.858
10009	2.19211	22:{*,P,2,^},{3,1,2,0}	22.9524	0.00204121	1073.92

10009	2.8499	23:{*,P,2,^},{3,2,0,1}	23.0993	0.0101915	279.636
10009	2.12547	24:{*,P,2,^},{3,2,1,0}	23.1287	0.00293149	725.048

Table 39. c=10009. Fit parameter and c_3 of the plane equations for M_{abc} (type EGOCRON4) for different assignments of Gödel symbols (order reversed)

C	c_3	Code table: symbols/values	Max. value	Standard error	t-statistics
10009	2.43885	1:{*,P,2,^},{0,1,2,3}	22.2941	0.00330491	737.948
10009	2.48199	2:{*,P,2,^},{0,1,3,2}	22.2949	0.0025768	963.203
10009	1.86469	3:{*,P,2,^},{0,2,1,3}	22.9589	0.00150375	1240.03
10009	**1.94351**	**4:{*,P,2,^},{0,2,3,1}**	**22.9598**	**0.00109807**	**1769.94**
10009	1.5486	5:{*,P,2,^},{0,3,1,2}	23.3565	0.00118561	1306.16
10009	1.58908	6:{*,P,2,^},{0,3,2,1}	23.3575	0.00199429	796.813
10009	**1.81771**	**9:{*,P,2,^},{1,2,0,3}**	**23.0656**	**0.000933021**	**1948.2**
10009	1.97304	10:{*,P,2,^},{1,2,3,0}	23.0668	0.00316066	624.249
10009	1.51979	11:{*,P,2,^},{1,3,0,2}	23.4296	0.00127364	1193.27
10009	1.63535	12:{*,P,2,^},{1,3,2,0}	23.4318	0.00446254	366.462
10009	2.27073	15:{*,P,2,^},{2,1,0,3}	22.657	0.00209251	1085.17
10009	2.46931	16:{*,P,2,^},{2,1,3,0}	22.6589	0.00254729	969.387
10009	1.54154	17:{*,P,2,^},{2,3,0,1}	23.499	0.00276984	556.546
10009	1.62921	18:{*,P,2,^},{2,3,1,0}	23.5001	0.00550417	295.996
10009	**2.25579**	**21:{*,P,2,^},{3,1,0,2}**	**22.8032**	**0.00147073**	**1533.78**
10009	2.42527	22:{*,P,2,^},{3,1,2,0}	22.8045	0.00365269	663.97
10009	1.83374	23:{*,P,2,^},{3,2,0,1}	23.2557	0.00223614	820.047
10009	1.93552	24:{*,P,2,^},{3,2,1,0}	23.2561	0.00520616	371.774

Table 40. c=10009. Fit parameter and c_3 of the plane equations for M_{abc} (type M2GOCRON4) for different assignments of Gödel symbols (order reversed)

C	c_3	Code table: symbols/values	Max. value	Standard error	t-statistics
10009	4.85748	1:{*,P,2,^},{0,1,2,3}	25.5946	0.00684048	710.109
10009	4.90055	2:{*,P,2,^},{0,1,3,2}	25.4929	0.00652405	751.152
10009	4.28317	3:{*,P,2,^},{0,2,1,3}	26.0164	0.00620476	690.305
10009	4.36192	4:{*,P,2,^},{0,2,3,1}	25.937	0.00610748	714.192
10009	3.967	5:{*,P,2,^},{0,3,1,2}	26.3287	0.00614129	645.955
10009	4.00745	6:{*,P,2,^},{0,3,2,1}	26.3326	0.00633565	632.524
10009	**3.54826**	**9:{*,P,2,^},{1,2,0,3}**	**26.8814**	**0.00192201**	**1846.12**
10009	3.6121	10:{*,P,2,^},{1,2,3,0}	26.8521	0.00222321	1624.72
10009	3.36945	11:{*,P,2,^},{1,3,0,2}	27.0258	0.00278201	1211.16
10009	3.41428	12:{*,P,2,^},{1,3,2,0}	27.0297	0.00325271	1049.67
10009	3.37628	15:{*,P,2,^},{2,1,0,3}	27.2372	0.00192779	1751.37
10009	3.42781	16:{*,P,2,^},{2,1,3,0}	27.2175	0.00241658	1418.46
10009	**3.0478**	**17:{*,P,2,^},{2,3,0,1}**	**27.4339**	**0.00142776**	**2134.67**
10009	3.06892	18:{*,P,2,^},{2,3,1,0}	27.4352	0.0023285	1317.99
10009	3.09633	21:{*,P,2,^},{3,1,0,2}	27.5812	0.00291311	1062.9
10009	3.13022	22:{*,P,2,^},{3,1,2,0}	27.5819	0.00355751	879.89

10009	**2.93734**	**23:{*,P,2,^},{3,2,0,1}**	**27.6553**	**0.00149217**	**1968.51**
10009	2.95744	24:{*,P,2,^},{3,2,1,0}	27.6556	0.00252606	1170.77

20.10.7 REED JAMESON PSEUDO PRIME NUMBERS

So far, the following Reed Jameson pseudoprimes are known:

```
      4.647.272.200.763.653
     13.145.972.926.201.741
     37.550.172.530.083.333
     91.475.036.245.333.333
    138.059.041.752.628.921
  1.017.051.023.982.373.381
  1.198.917.598.782.691.327
  2.193.915.384.965.973.241
  3.451.615.699.229.107.381
  3.512.610.370.112.161.753
  4.595.180.567.858.094.061
  6.048.451.215.682.221.781
  6.338.484.791.054.344.501
  7.928.915.800.561.771.753
  8.145.180.508.453.751.953
  8.791.425.219.802.647.241
  9.298.405.698.887.024.981
  9.538.676.189.678.282.653
 10.465.926.737.075.038.153
 10.672.259.013.245.100.833
 10.832.491.549.192.774.861
 10.877.405.928.733.495.009
 10.956.794.257.273.312.801
 11.422.820.349.626.091.841
 11.555.150.568.592.132.153
 13.383.002.224.373.603.221
 14.127.279.039.356.766.601
 17.487.206.393.334.007.501
```

Source: Peter Danzeglocke (calculated with an optimized C ++ program).
The range $n < 10^{10}$ contains no Reed Jameson pseudo primes (as of December 2020).

20.11 MATHEMATICA PROGRAMS

In this section you will find a collection of Mathematica programs, such as speed-optimized versions of the example programs described above.

Chebyshev function psi(x):

The function `myPsi[x]` can be made somewhat faster by exploiting symmetry properties and using the `Evaluate[]` and `Compile[]` functions:

```
####################################################################
```

```
myPsi[x_]:=Evaluate[-2*Sum[((x)^ZetaZero[i])/ZetaZero[i],{i,1,15}]-
0.5*Log[1-1/x^2]+x-Log[2*Pi]];
myPsic=Compile[{{x,_Complex}},myPsi[x],CompilationOptions-
>{"ExpressionOptimization"->True},
CompilationOptions->{"InlineCompiledFunctions"->Auto}]
Timing[Plot[Re[myPsic[x]],{x,1,100}]]
```

20.11.1 COMPARISON OF THE NUMBER OF TWIN, COUSIN AND SEXY PRIMES BY THE FORMULA OF HARDY-LITTLEWOOD

```
#################################################################
Mathematica-Program: please contact the author.
```

20.11.2 DIFFERENCERG SEQUENCES

RG sequences with 'Prime GOCRONs' (type 6):

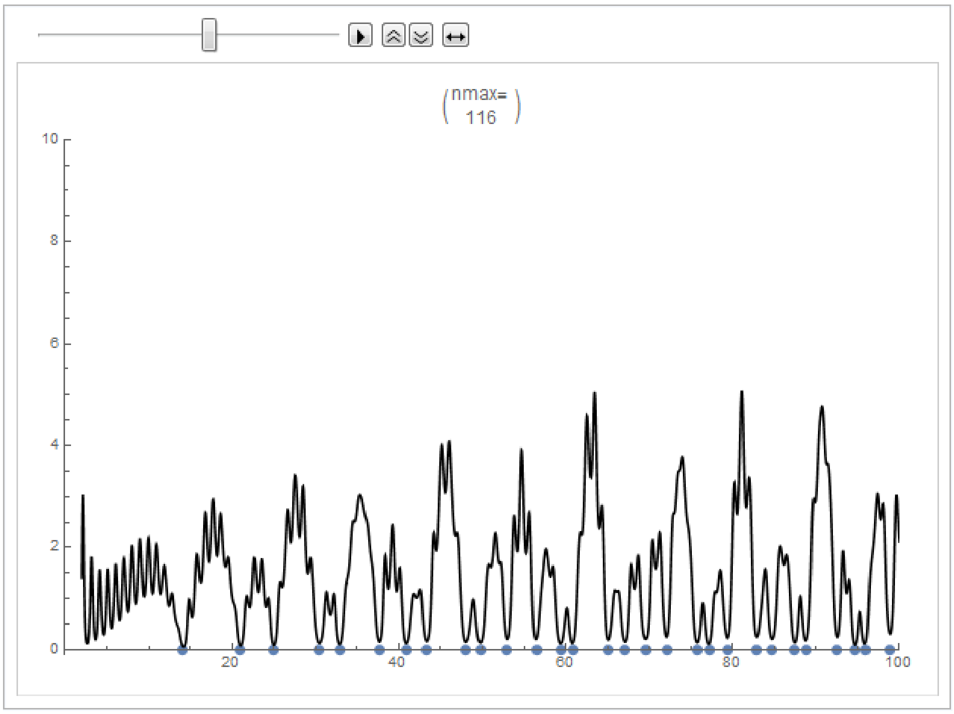
```
Mathematica program: please contact the author.
```

RG sequences with 'EGOCRONs' (type 4):

```
(*The following examples need the OCRON-library (see below*)
#################################################################
(*example:*)
Mathematica program: please contact the author.
```

20.11.3 RIEMANN'S ZETA FUNCTION

```
#################################################################
(*animation of the 'noise' of the product representation in the
complex domain:*)
Mathematica program: please contact the author.
(*(Snapshot:*)
```




```
################################################################
(*Iterative, approximat. method for the calculation of the product
representation, using prime numbers *)
(*along the critical line *)
Mathematica program: please contact the author.
```

```
################################################################
(*Iterative, approximat. method for the calculation of the product
representation, using the zeros of the zeta function*)
(*along the X-axis, zeros at prime numbers*)
Mathematica program: please contact the author.
```

```
################################################################
(*Parametric 3D-Plot of the Riemann zeta function along the crit. line
*)
Mathematica program: please contact the author.
```

20.11.4 REED JAMESON AND PERRIN SEQUENCES

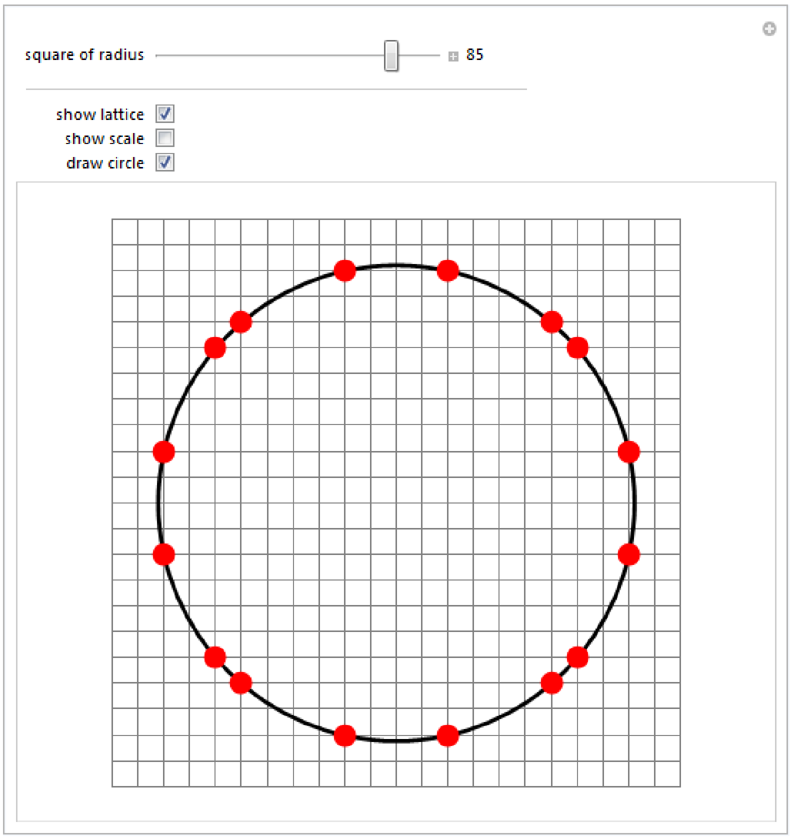
```
(*##############################################################*)
Mathematica program: please contact the author.
```

```
Mathematica program: please contact the author.
```

20.11.5 LATTICE POINTS ON *N*-SPHERES (*N*-DIMENSIONAL SPHERES)

```
################################################################
(*Interactive animation: lattice points on 1-sphere*)
```

```
m=Manipulate[Graphics[
dim=2;sqN=Sqrt[n];sqNInt=Round[sqN];
numberOfGridPoints=SquaresR[dim,n];
If[numberOfGridPoints>0,sol=FindInstance[a^2+b^2==n,{a,b},
Integers,numberOfGridPoints]];
Flatten[Table[{},{x,-sqNInt-2,sqNInt+2},
{y,-sqNInt-2,sqNInt+2}]],
Prolog->{If[ci,{{Black,Thickness[0.007],Circle[{0,0},sqN]},
If[numberOfGridPoints>0,{Red,PointSize[0.04],
Point[{a,b}]/.sol}]},{}]},
Frame->If[ft,Automatic,False],
PlotRange->{{-sqNInt-2,sqNInt+2},{-sqNInt-2,sqNInt+2}},
FrameTicks->If[ft,Automatic,None],
ImageSize->{480,400},ImageMargins->10,
GridLines->If[lattice,{Range[-sqNInt-2,sqNInt+2],
Range[-sqNInt-2,sqNInt+2]}]],
{{n,10,"square of radius"},2,100,1,
Appearance->"Labeled"},Delimiter,{{lattice,True,"show
lattice"},{True,False}},
{{ft,False,"show scale"},{True,False}},{{ci,True,"draw
circle"},{True,False}},
AutorunSequencing->Automatic]
(*snaphot:*)
```



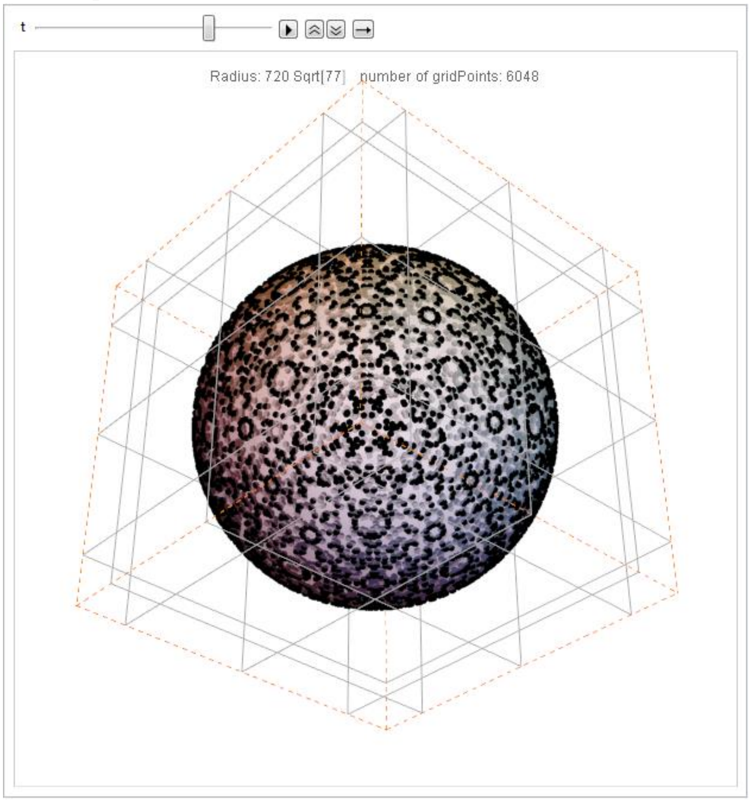

```
Export["C:\\animations\\latticePointsOnN-
spheresInNDimensions\\latticePointsOn1-spheresIn2Dimensions_RQ2-
100.mov",m]
#################################################################
```

(*Integer Grid-Points, touching the surface of a sphere for a given square of radius*)
Mathematica program: please contact the author.

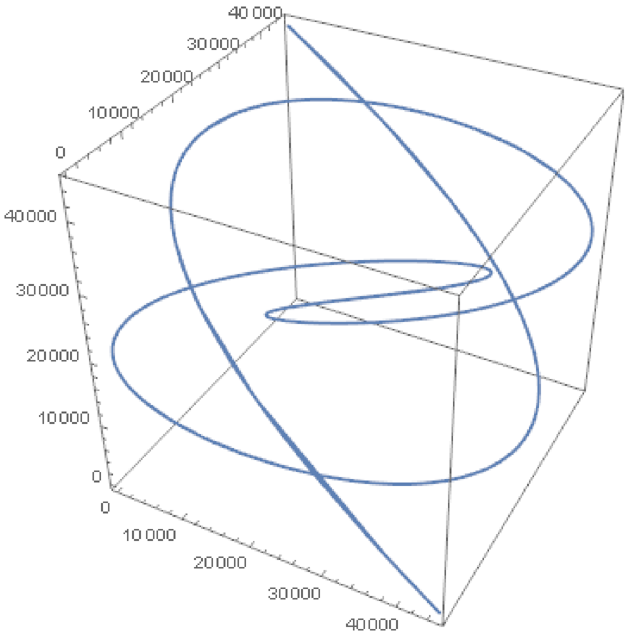
##
(* Lattice points on the surface of a 3 dimensional sphere*)
(*angles of spherical coordinates interpreted as 2-dimensional Cartesisian coordinates *)
Mathematica program: please contact the author.

##
(*Integer Grid-Points, touching the surface of a sphere for a given square of radius*)
Mathematica program: please contact the author.

(*Snapshot:*)



(*used viewvector:*)



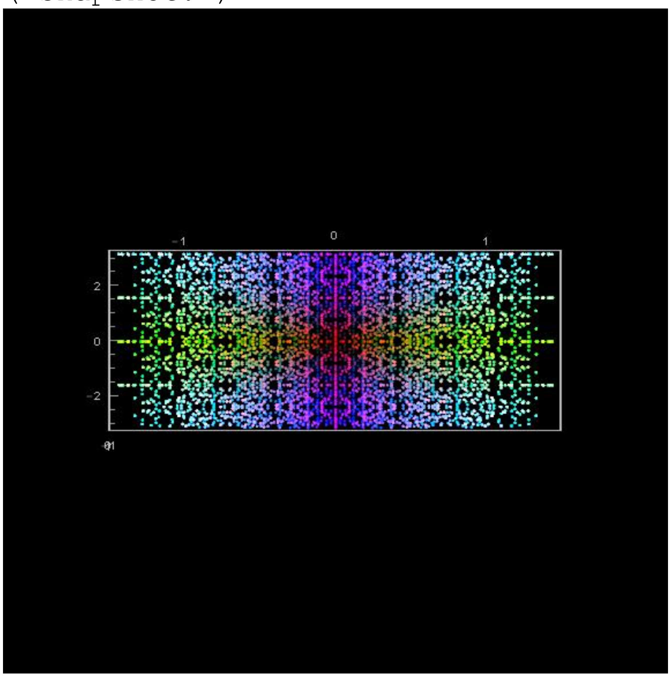
(*###*)
(*Figure 86*)
(*3D Plots of glomes, interpreting phi, theta and psi as Cartesian Coordinates *)

Mathematica program: please contact the author.

```
################################################################
(*Journey through the surface of a 4-dim Sphere *)
(*Animation: 3D Plot of Glome, interpreting phi, theta and psi as
Cartesian coordinates *)
```

Mathematica program: please contact the author.

(*Snapshot:*)



20.11.6 EVALUATION AND STATISTICS OF DIFFERENCES OF THE PRIME SEQUENCE

```
(*Statistics with prime number differences of higher order*)
Mathematica program: please contact the author.
```

20.11.7 THE *ABC* CONJECTURE

```
(*abc-conjecture: calculates log Gödel-GOCRON4 codes of abc-points. Representation *)
(*as 3D-Plot together with a 'fitted' plane from different views *)
(* execution needs the OCRON Mathematica-library! *)
```

Mathematica program: please contact the author.

20.11.8 OTHER MATHEMATICA PROGRAMS

```
(*Polynomial with 26 variables of degree 25, whose positive values are identical to the set pf primes *)
(*This program searches for positive solutions *)
c0=w z+h+j-q;
c1=(g k+2g+k+1)*(h+j)+h-z;
c2=2n+p+q+z-e;
c3=16(k+1)^3 *(k+2)*(n+1)^2+1-f^2;
c4=e^3*(e+2)*(a+1)^2+1-o^2;
c5=(a^2-1)*y^2+1-x^2;
c6=16r^2 y^4*(a^2-1)+1-u^2;
c7=((a+u^2*(u^2-a))^2-1)*(n+4d y)^2+1-(x+c u)^2;
c8=n+l+v-y;
c9=(a^2-1)*l^2+1-m^2;
c10=a i+k+1-l-i;
c11=p+l*(a-n-1)+b*(2a n+2a-n^2-2n-2)-m;
c12=q+y*(a-p-1)+s*(2a p+2a-p^2-2p-2)-x;
c13=z+p l*(a-p)+t*(2a p-p^2-1)-p m;
k=0;
FindInstance[Element[k+2,Primes]&&c0==0&&c1==0&&c2==0&&c3==0&&c4==0&&c5==0&&c6==0&&c7==0&&c8==0&&c9==0&&c10==0&&c11==0&&c12==0&&c13==0&&a>=0&&b>=0&&c>=0&&d>=0&&e>=0&&f>=0&&g>=0&&h>=0&&i>=0&&j>=0&&k>=0&&l>=0&&m>=0&&n>=0&&o>=0&&p>=0&&q>=0&&r>=0&&s>=0&&t>=0&&u>=0&&v>=0&&w>=0&&x>=0&&y>=0&&z>=0,{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z},Integers]
```

```
(*Calculation of the sigma1 function*)
myDivisorSigma[k_,n_]:=
Sum[m^(k-1)Sum[Cos[(2 pi j n)/m],{j,1,m}],{m,1,n}]
(*Example: n= 31*)
myDivisorSigma[1,31]
```

This expression is not seen as identical to 32

$$
\begin{aligned}
-2\Big(-19 &+ 2\sin\left(\frac{\pi}{14}\right) - 2\sin\left(\frac{3\pi}{14}\right) - 2\sin\left(\frac{\pi}{18}\right) + \sin\left(\frac{\pi}{22}\right) - \sin\left(\frac{3\pi}{22}\right) + \sin\left(\frac{5\pi}{22}\right) - \sin\left(\frac{\pi}{26}\right) + \sin\left(\frac{3\pi}{26}\right) \\
&- \sin\left(\frac{5\pi}{26}\right) + \sin\left(\frac{\pi}{30}\right) - \sin\left(\frac{7\pi}{30}\right) - \sin\left(\frac{\pi}{34}\right) + \sin\left(\frac{3\pi}{34}\right) - \sin\left(\frac{5\pi}{34}\right) + \sin\left(\frac{7\pi}{34}\right) \\
&+ \sin\left(\frac{\pi}{38}\right) - \sin\left(\frac{3\pi}{38}\right) + \sin\left(\frac{5\pi}{38}\right) - \sin\left(\frac{7\pi}{38}\right) + \sin\left(\frac{9\pi}{38}\right) - \sin\left(\frac{\pi}{42}\right) - \sin\left(\frac{5\pi}{42}\right) \\
&+ \sin\left(\frac{\pi}{46}\right) - \sin\left(\frac{3\pi}{46}\right) + \sin\left(\frac{5\pi}{46}\right) - \sin\left(\frac{7\pi}{46}\right) + \sin\left(\frac{9\pi}{46}\right) - \sin\left(\frac{11\pi}{46}\right) - \sin\left(\frac{\pi}{50}\right) \\
&+ \sin\left(\frac{3\pi}{50}\right) + \sin\left(\frac{7\pi}{50}\right) - \sin\left(\frac{9\pi}{50}\right) + \sin\left(\frac{11\pi}{50}\right) + \sin\left(\frac{\pi}{54}\right) + \sin\left(\frac{5\pi}{54}\right) - \sin\left(\frac{7\pi}{54}\right) \\
&- \sin\left(\frac{11\pi}{54}\right) + \sin\left(\frac{13\pi}{54}\right) - \sin\left(\frac{\pi}{58}\right) + \sin\left(\frac{3\pi}{58}\right) - \sin\left(\frac{5\pi}{58}\right) + \sin\left(\frac{7\pi}{58}\right) - \sin\left(\frac{9\pi}{58}\right) \\
&+ \sin\left(\frac{11\pi}{58}\right) - \sin\left(\frac{13\pi}{58}\right) + 2\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{\pi}{9}\right) - 2\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{\pi}{11}\right) - \cos\left(\frac{2\pi}{11}\right) \\
&+ \cos\left(\frac{\pi}{13}\right) - \cos\left(\frac{2\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{\pi}{15}\right) - \cos\left(\frac{2\pi}{15}\right) + \cos\left(\frac{\pi}{17}\right) - \cos\left(\frac{2\pi}{17}\right) \\
&+ \cos\left(\frac{3\pi}{17}\right) - \cos\left(\frac{4\pi}{17}\right) + \cos\left(\frac{\pi}{19}\right) - \cos\left(\frac{2\pi}{19}\right) + \cos\left(\frac{3\pi}{19}\right) - \cos\left(\frac{4\pi}{19}\right) + \cos\left(\frac{\pi}{21}\right) \\
&- \cos\left(\frac{2\pi}{21}\right) - \cos\left(\frac{4\pi}{21}\right) + \cos\left(\frac{5\pi}{21}\right) + \cos\left(\frac{\pi}{23}\right) - \cos\left(\frac{2\pi}{23}\right) + \cos\left(\frac{3\pi}{23}\right) - \cos\left(\frac{4\pi}{23}\right) \\
&+ \cos\left(\frac{5\pi}{23}\right) + \cos\left(\frac{\pi}{25}\right) - \cos\left(\frac{2\pi}{25}\right) + \cos\left(\frac{3\pi}{25}\right) - \cos\left(\frac{4\pi}{25}\right) - \cos\left(\frac{6\pi}{25}\right) + \cos\left(\frac{\pi}{27}\right) \\
&- \cos\left(\frac{2\pi}{27}\right) - \cos\left(\frac{4\pi}{27}\right) + \cos\left(\frac{5\pi}{27}\right) + \cos\left(\frac{\pi}{29}\right) - \cos\left(\frac{2\pi}{29}\right) + \cos\left(\frac{3\pi}{29}\right) - \cos\left(\frac{4\pi}{29}\right) \\
&+ \cos\left(\frac{5\pi}{29}\right) - \cos\left(\frac{6\pi}{29}\right) + \cos\left(\frac{7\pi}{29}\right)\Big)
\end{aligned}
$$

Therefore:

```
FullSimplify[myDivisorSigma[1,31]]
32
```

or, even better:

```
N[myDivisorSigma[1,31]]
32.
```

20.11.9 OCRONS AND THE *ABC* CONJECTURE: PROGRAM LIBRARY

20.11.9.1 OCRON FUNCTIONS

```
(* ###############################################################################*)
(* Library: GOCRON-Routines actual version Sept.2016*)
(* ###############################################################################*)
Mathematica Program: please contact the author.
(* OCRON-Functionlist:
nToGoedelSymbolList[n_],goedelSymbolListToN[symbolList_]
nToGoedelSymbolListForPrimeOCRONS[n_],goedelSymbolListToNForPrimeOCRONS[symbolList_]
nToGoedelSymbolListForVirtualOCRONs[n_],goedelSymbolListToNForVirtualOCRONS[symbolList_]
##################### OCRON4, GOCRON4 #################
nToOCRON4[n_], oCRON4ToN[symbolList_], oCRON4ToNMaxVal[symbolList_,maxVal_]
nToGOCRON4[n_], GOCRON4ToN[n_], GOCRON4ToNMaxVal[symbolList_,maxVal_]
checkOCRON4[n_]
##################### M2OCRON4, M2GOCRON4 without leading 2 #################
nToM2OCRON4[n_], m2OCRON4ToN[symbolList_]
nToM2GOCRON4[n_], msGOCRON4ToN[n_]
##################### EOCRON4, EGOCRON4 #################
nToEOCRON4[n_],eOCRON4ToN[symbolList_], eOCRON4ToNMaxVal[symbolList_,maxVal_]
nToEGOCRON4[n_],eGOCRON4ToN[n_], eGOCRON4ToNMaxVal[symbolList_,maxVal_]
##################### PrimeOCRON, PrimeGOCRON (type 6)#################
nToPrimeOCRON[n_], primeOCRONToN[n_]
nToPrimeGOCRON[n_],primeGOCRONToN[n_]
##################### Miscellaneous ########################
createAscendingEOcron4List[n_]
createAscendingEVirtualOcron4List[n_]
createAscendingVirtualOcron4List[n_]
createAllValuesListFromAscendingVirtualOcron4s[n_]
createIntValuesListFromAscendingVirtualOcron4s[n_]
createAscendingIntList[n_]
createAscendinGOCRONListFromNaturalNumbers[n_]
createAscendingGOCRONListFromNaturalNumbers[n_]
createAscendingEOCRONListFromNaturalNumbers[n_]
createAscendingEGOCRONListFromNaturalNumbers[n_]
createAscendingM2OCRONListFromNaturalNumbers[n_]
createAscendingM2GOCRONListFromNaturalNumbers[n_]
resetGloc4Codes[]
setGLoc4CodeSymbols[symbols_]
setGLoc4CodeValues[values_]
######################### Virtual OCRONs ######################
checkVirtualOCRON4[n_], virtualOCRON4ToOCRON4[symbolList_]
virtualOCRON4ToN[symbolList_]
#################################################################################
Evaluating OCRONS by converting the Polish RPN-representation used in OCRONS to 'normal'
Mathematica expressions before numerical evaluation ######################
#################################################################################
oCRON4ToExpression[symbolList_]
loGOCRON4ToExpression[symbolList_]
loGOCRON4ToExpressionSimplify[symbolList_]
oCRON4ToExpressionPowerExpand[symbolList_]
loGOCRON4ToExpressionPowerExpand[symbolList_]
logLoGOCRON4ToExpressionPowerExpand[symbolList_]
convertOcronToTraditionalForm[symbolList_]
convertLogOcronToTraditionalForm[symbolList_]
convertLogLogOcronNToTraditionalForm[symbolList_]
*)
```

20.11.9.2 THE *ABC* CONJECTURE

```
(*################################################################################*)
(*################################################################################*)

(*radicals, quality, isPossibleABC, radABC() computes radABC from c, but does
not always get the smallest one!!*)
```

```
(*minRadABC takes always the smallest*)
```

20.11.9.3 THE DEGENERATION OF OCRONS

```
(*###########################################################################*)
(*Degeneration-values of OCRONs type 4 *)
(*Needs GOCRON4-Library *)
(*###########################################################################*)
gloc4Codes={{"*","P","2","^"},{0,1,2,3}}; (*actual code-Table,*)
(*Note: maxEGocrons should be at least 13 symbols long (e.g. 22*2*2*2*2*2* =
2^7=128) *)
(* Because of
goedelSymbolListToN[{"^","^","^","^","^","^","^","^","^","^","^","^","^"}]=671
08863 *)
(* maxEOcrons should be at least 67.108.863 to get all degeneration values up
to maxValue = 128 (=2^7 *)
(* maxEOcrons should be at least 1073741823 to get all degeneration values up
to maxValue = 256 (=2^8 *)
maxValue=128; maxEOcrons=67200000;
ocron4DegenList=Table[{},{i,1,maxValue}];
For[i=1,i<maxEOcrons,i++,
eOcron=nToGoedelSymbolList[i];
iValue=oCRON4ToNMaxVal[eOcron,maxValue];
If[iValue>0&& iValue<= maxValue,AppendTo[ocron4DegenList[[iValue]],eOcron];
];
If[Mod[i,100000]==0,Print[N[i/67200000]]];
]
For[i=1,i<maxValue,i++,
Print[i,": ",Length[ocron4DegenList[[i]]],"->",ocron4DegenList[[i]]];
]
```

20.11.10 SOUND ROUTINES

```
(*############Prime Sound-Library################################*)
(*Generate a sorted list of the combined (sieve number, prime number)-
pairs by: *)
Mathematica program: please contact the author.
```

20.11.11 RSA ENCRYPTION AND DECRYPTION

```
(*Example 1: encode/Decode a number (1115) *)
(*Very simple example of RSA encryption*)
(*Without Encoding/Decoding Functions from Mathematica…*)
(*############## implement coding mechanism ####################*)
(*choose two different prime numbers:*)
Mathematica program: please contact the author.

(*Example 2: same as Example 1: encode a number (1115) *)
(*using Mathematica built-in functions*)
(*publicKey[], privateKey[], Encrypt[], Decrypt[]*)
(*used padding-mode: ‚none‘ *)
(*IMPORTANT: in this Mathematica-version (10.3) Encoding with
PublicKey-Objects and padding: "None" only works for up to 16-bit
Modules *)
Mathematica program: please contact the author.
(*############## Encode and decode messages:######################*)
(*this is our message to be encoded:*)
```

Mathematica program: please contact the author.

PrivateKey[
cipher: RSA
private exponent length: 8 bits
public modulus length: 12 bits
padding: None
public exponent: 17
private exponent: 157
public modulus: 2773
]

PublicKey[
cipher: RSA
public modulus length: 12 bits
padding: None
public exponent: 17
public modulus: 2773
]

(*Example 3: rSA-Encoding/Decoding a small String ("OK") using PKCS1 padding*)
(*let Mathematica choose p, q and the modulus using Mathematica built-in functions*)
Mathematica program: please contact the author.

⟨|"PrivateKey" → PrivateKey[
cipher: RSA
public modulus length: 97 bits
private exponent length: 93 bits
padding: PKCS1
public exponent: 65537
public modulus: 12202433704 ≪7≫ 277596949541
private exponent: 76245427803 ≪5≫ 502629985493
], "

PublicKey" → PublicKey[
cipher: RSA
public modulus length: 97 bits
padding: PKCS1
public exponent: 65537
public modulus: 12202433704 ≪7≫ 277596949541
]|⟩

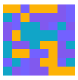
(*Example 4: hacking a private Key from a public key using PKCS1 padding with key length 192*)
(*We use Mathematica built-in functions PrivateKey[], Decrypt[] and FactorInteger[]*)
(*##*)
Mathematica program: please contact the author.
(*###### decoding can be done different ways::#########*)

(* Decrypted data by using Decrypt[] will not contain padded data…*)

```
bCryptArray=ByteArray[IntegerDigits[mCrypt,256]];
decryptedByteArray=Normal[Decrypt[privKey,bCryptArray]]
decryptedString=FromCharacterCode[decryptedByteArray];
Print["Decryption-result (original String: ",decryptedString];
(**** program - output:****)
prime p from RSA:module: 683570719408201946116823965l3
prime q from RSA:module: 785536274840425653125330065 67
private Exponent:
4844991859660492495555967871982611572207133532958607342401
Private Modulus:
5369695965139088101081485235420567443013865529391511497792
Hacked private key:
cipher: rSA
private exponent length: 192 bits
public modulus length: 192 bits
padding: pKCS1
```

```
public exponent: 65537
```

PrivateKey[



cipher: RSA
private exponent length: 192 bits
public modulus length: 192 bits
padding: PKCS1
public exponent: 65537
private exponent: 484499185966 ≪34≫ 958607342401
public modulus: 536969596513 ≪34≫ 315726900871

]

```
Original Text as Bytearray including Bytes padded by PKCS1 algorithm:
{2,11,165,77,224,174,48,231,225,235,0,69,108,118,105,115,32,108,105,11
8,101,115,33}
\.02\.0b¥Mà®0çáë\.00Elvis lives!
Decryption-result (original Byte array:
{69,108,118,105,115,32,108,105,118,101,115,33}
Decryption-result (original String):
Elvis lives!
```

```
(*Example 5: hacking a private Key from a public key using PKCS1
padding with key length 2048*)
(*We use Mathematica built-in functions PrivateKey[], Decrypt[] and
FactorInteger[]*)[88]
(*#################################################################*)
Mathematica program: please contact the author.
(* Decrypted data by using Decrypt[] will not contain padded data…*)
bCryptArray=ByteArray[IntegerDigits[mCrypt,256]];
decryptedByteArray=Normal[Decrypt[privKey,bCryptArray]]
decryptedString=FromCharacterCode[decryptedByteArray];
Print["Decryption-result (original String: ",decryptedString];

(**** program - output:****)
prime p from RSA-module:
5042275217484184784387456407481025964634418009557323862771843210015347
0233709220326765100448150841802101002465172565326870447505988642493935
6768330261183984688981022399271959231632244880124488202703458535772508
3261691330915873078509567583024603043325764678776810906881522663421805
761981426998378611173580023640569
prime q from RSA-module:
6230506107037277994848859713460167565407435407248343949280547643268996
2794782739027185058685887929516793028417110464017369917383411955467390
4850630650913159085619714144991810427714356257581581356417704599361604
8753993877513046170835382583128576250458442756910532302852222491700224
633245295180992749493823
private Exponent:
1030819360403968961043390501763010666063077815038257405347287921525051
2535022756352365421194768891684069779277718177933402369048702835912585
4131450497268501685537802072878657793084753037172733458691935491519525
1853460378496829887538632390637136150965931733628074554699429623111223
3269880545420815346132763817866426056363791352182879224677368503022618
4983798138489051261011671669601896311386511911972803990381100552366494
4073403141189139015889364169952790178286921216796288440782997538376575
```

[88] The program runs with Mathematica Version 15.5 (2026) or higher

```
7085206627441700960078133155365855116355242551141828940174105853199096
690164133765434208900272472536995205015922393069952712705
Private Modulus:
3141592653589793238462643383279502884197169399375105820974944592307816
4062862089986280348253421170679821480865132823066470938446095505822317
2535940812848111745028410270193852110555964462294895493038196442881097
5665933446128475648233786783165271201909145648566923460348610454326648
2133936072602491412737302005743673942332300681176030308206877770767919
8534374004936614234231860407863629025266826226514213872656537709283991
0702130843755236406451881492103609092001021009355392277882966409625448
3914303698969808213385445154539250686410599473315757219688912541904259
662541240447603317926765114985912144304711024675664570896
Hacked private key:
cipher: rSA
private exponent length: 2047 bits
public modulus length: 2048 bits
padding: pKCS1
public exponent: 65537
```

PrivateKey[cipher: RSA
private exponent length: 2047 bits
public modulus length: 2048 bits
padding: PKCS1
public exponent: 65537
private exponent: 10308193604 ≪594≫ 069952712705
public modulus: 31415926535 ≪594≫ 248437705287]

```
Original Text as Bytearray including Bytes padded by PKCS1 algorithm:
{2,169,246,29,163,145,193,96,236,157,15,189,194,238,0,73,102,32,68,111
,110,97,108,100,32,84,114,117,109,112,32,115,104,111,117,108,100,32,11
9,105,110,32,116,104,101,32,112,114,101,115,105,100,101,110,116,105,97
,108,32,101,108,101,99,116,105,111,110,115,44,32,116,104,105,115,32,11
9,111,117,108,100,32,98,101,32,97,32,100,105,115,97,115,116,101,114,32
,102,111,114,32,116,104,101,32,85,110,105,116,101,100,32,83,116,97,116
,101,115,32,111,102,32,65,109,101,114,105,99,97,46,10,39,116,119,97,11
5,32,98,114,105,108,108,105,103,44,32,97,110,100,32,116,104,101,32,115
,108,105,116,104,121,32,116,111,118,101,115,10,100,105,100,32,103,121,
114,101,32,97,110,100,32,103,105,109,98,108,101,32,105,110,32,116,104,
101,32,119,97,98,101,58,10,65,108,108,32,109,105,109,115,121,32,119,10
1,114,101,32,116,104,101,32,98,111,114,111,103,111,118,101,115,44,10,9
7,110,100,32,116,104,101,32,109,111,109,101,32,114,97,116,104,115,32,1
11,117,116,103,114,97,98,101}
\.02©ö\.1d£‘Á`ì \.0f½Âî\.00If Donald Trump should win the presidential
elections, this would be a disaster for the United States of America.
'twas brillig, and the slithy toves
did gyre and gimble in the wabe:
All mimsy were the borogoves,
and the mome raths outgrabe
Decryption-result (original String: if Donald Trump should win the
presidential elections, this would be a disaster for the United States
of America.
'twas brillig, and the slithy toves
did gyre and gimble in the wabe:
All mimsy were the borogoves,
and the mome raths outgrabe
```

20.11.12 ALIQUOT SEQUENCES

(* computes aliquot sequences for a few interesting initial values*)

```
Mathematica program: please contact the author.
Output:

{1,Terminating,{1,0}}
{2,Terminating,{2,1,0}}
{3,Terminating,{3,1,0}}
{4,Terminating,{4,3,1,0}}
{5,Terminating,{5,1,0}}
{6,Perfect,{{6}}}
{7,Terminating,{7,1,0}}
{8,Terminating,{8,7,1,0}}
{9,Terminating,{9,4,3,1,0}}
{10,Terminating,{10,8,7,1,0}}
{11,Terminating,{11,1,0}}
{12,Terminating,{12,16,15,9,4,3,1,0}}
{28,Perfect,{{28}}}
{496,Perfect,{{496}}}
{220,Amicable,{{220,284}}}
{1184,Amicable,{{1184,1210}}}
{12496,Sociable,{{12496,14288,15472,14536,14264}}}
{1264460,Sociable,{{1264460,1547860,1727636,1305184}}}
{790,Aspiring,{790,650,652,{496}}}
{909,Aspiring,{909,417,143,25,{6}}}
{562,Cyclic,{562,{284,220}}}
{1064,Cyclic,{1064,1336,{1184,1210}}}
{1488,Non-
terminating,{1488,2480,3472,4464,8432,9424,10416,21328,22320,55056,957
28,96720,236592,459792,881392,882384,1474608}}


(*Aliquot 276 (306,396,696)OE:*)
Mathematica program: please contact the author.


(*##############################################################*)
(*Plot Differences of Log of aliquot sequences, using ListPlot*)
n=921232;noIterate=1000; diffOrder=1;
Mathematica program: please contact the author.
```

20.11.13 THE ARECIBO MESSAGE

(*Arecibo-Message*)

```
n=23;
t=Table[BitShiftRight[BitAnd[27886402056107263551714831669687744330301
5886191896083753494207226153602508928851994608485761550978983329982259
3335259720410959738432212343758921014182008038517667278025253709464080
0567916516636264434941344165234644984933485655114374616110243082450500
4833981684141550381731028954290673308020242293291528914499592811145845
8595397126461136347103419178098716188118662826517986311913829406689871
7096729057657705911386899499333419586667745206851413286366090402386362
1169622066629371322105035882727404788841080,2^i],i],{i,1679,0,-
1}];ArrayPlot[Partition[t,n],Mesh->All,
ColorRules->{1->RGBColor[.0,.1,.9],0->RGBColor[.9,.5,.2]},
ImageSize->Medium,PlotLabel->{"Arecibo-Message"},
PlotLegends->Automatic]
```

20.11.14 CORRELATIONS OF THE LAST DIGITS IN THE PRIME NUMBER SEQUENCE

```
(*statistical properties and correlations, concerning the last digits
in the prime sequence*)
(* one predecessor:*)
Mathematica program: please contact the author.
```

20.11.15 PRIME N-TUPLETS AND MAXIMAL PRIME NUMBER DENSITY

```
 (* Construction of a maximal prime number density *)
(* the generated Sequence of possible prime positions is identical with OEIS A020498 *)
(* 1,3,7,9,13,19,21,27,31,33,37,43,49,51,57,63,69,73,,, *)
(*The patterns resulting from the differences of the p-positions (generated by sieving)
repeat after the following cycles: *)
(*sieving up to 2: {2 } length of period: 1,    primorial(1)=2 *)
(*RotateLeft[Differences[Select[Range[2+1],GCD[#,2]\[Equal]1&]],0]; *)
(*sieving up to 3: {2,4} length of period: 2,  primorial(2)=6*)
(*RotateLeft[Differences[Select[Range[2*3+1],GCD[#,2*3]\[Equal]1&]],1]; *)
(*sieving up to 5: {2,4,2,4,6,2,6,4} length of period: 8,    primorial(3)=30  *)
(*RotateLeft[Differences[Select[Range[2*3*5+1],GCD[#,2*3*5]\[Equal]1&]],2]; *)
(*sieving up to 7: {2,4,2,4,6,2,6,4,2,4,6,6,2,6,4,2,6,4,6,8,...2,10,2,10 } length of
period: 48,     primorial(4)=210 *)
(*RotateLeft[Differences[Select[Range[2*3*5*7+1],GCD[#,2*3*5*7]\[Equal]1&]],1];*)
(*sieving up to 11:
{2,4,2,4,6,2,6,4,2,4,6,6,2,6,6,6,4,6,8,4,2,4,2,4,8,6,4,8,4,6,2,6,6,4,2,4,6,8,4,2,4,2,10,
...,2,10,2,10 } length of period: 480,     primorial(5)=2310*)
(*RotateLeft[Differences[Select[Range[2*3*5*7*11+1],GCD[#,2*3*5*7*11]\[Equal]1&]],262]*)
(*sieving up to 13:
{2,4,2,4,6,2,6,4,2,4,6,6,2,6,6,6,4,6,8,4,6,2,4,8,6,4,8,4,6,2,...2,10,2,10 } length of
period: 5760,     primorial(6)=30030    *)
(*RotateLeft[Differences[Select[Range[2*3*5*7*11*13+1],GCD[#,2*3*5*7*11*13]\[Equal]1&]],
2899]*)
(*sieving up to 17: {... } length of period: 92160 ,     primorial(7)=510510    *)
(*RotateLeft[Differences[Select[Range[primorial[7]+1],GCD[#,primorial[7]]\[Equal]1&]],89
465]*)
(*sieving up to 19: {... } length of period: 1658880,     primorial(8)=9699690 *)
(*RotateLeft[Differences[Select[Range[2*3*5*7*11*13*17*19+1],GCD[#,2*3*5*7*11*13*17*19]\
[Equal]1&]],???]*)
(* The period length can be easily calculated by the formula: a(0)=1;for
n>0,a(n)=(prime(n)-1)*a(n-1) *)
(* Mathematica: RecurrenceTable[{a[0]\[Equal]1,a[n]\[Equal](Prime[n]-1)a[n-1]},a,{n,10}]
*)
(* Arguments of Range[] and GCD[]: primorial[n]:
2,6,30,210,2310,30030,510510,9699690,... Mathematica:
primorial[n_]:=Product[Prime[i],{i,n}];*)
(* The Sequence {0,1,2,1,262,2899,89465...} of the arguments for the RotateFeft function
is unknown *)
(* The values for the corresponding RotateRight operatons read:
{0,1,6,47,218,2861,2695,... *)
(* functions: *)
```

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LIST OF ILLUSTRATIONS

LIST OF TABLES

SEARCH INDEX

THE ENCLOSED CD CONTAINING COMPUTER PROGRAMS

The printed version of this book contains a CD which is enclosed.

This CD contains animations (MOV or Flash format), Mathematica notebooks, graphics (PDF or PGN format) and sound files (MP3, MIDI or Melodyne format), as well as the free, shortened online version this book in pdf format (in German and English language).

ANIMATIONS

directory: lattice-Points_On_N-spheres_In_N_Dimensions:

```
latticePointsOn1-spheresIn2Dimensions_RQ2-100.mov
latticePointsOn2-spheresIn3Dimensions_RQ11Fakultät_rotateWithViewVector_1Minute.mov
latticePointsOn2-spheresIn3Dimensions_RQ1001_rotateWithViewVector.mov
latticePointsOn2-spheresIn3Dimensions_RQ1001_rotateWithViewVector_1Minute.mov
```

directory: Zeta_Function:

```
ZetaProductOverPrimeTerms_n2_200_1_x_2_100.mov
ZetaProductOverPrimeTerms_n10_100_1_x_0_71.mov
ZetaProductOverPrimeTerms_n10_100_1_x_0_71.swf
```

MATHEMATICA NOTEBOOKS

The following directories contain over 120 Mathematica notebooks which have been used to generate the numerous tables, graphical representations and animations:

```
1_F_Noise_PrimePi-Signals
ABC-Conjecture
Aliquot sequences
DNA sequences
Factorization
Fibonacci
Functions_Generating_Primes
Functions_Having_Zeros_Or_Minima_At_Primes
Gradus_Suavitatis_Music_General
Last_Digits_In_Prime_Sequence
Lattice-Points_In_4_Dimensions
Matrix
MersennePrimes
Moebius_Mertens
OCRONS
Plots_Of_Zetafunction_Using_Product-Representation
Prime_NTuples_MaxPrimeDensity
Prime-Polynom_With_26_Variables
Primes_And_Star_Constellations
Ramanujan-Sums
```

```
Ramanujan-Tau
RG_Numbers
Riemann_Exact_Explicit_Formula
RSA
Sigma_Function_Tests
Special_Types_Of_Primes_And_Other_Numbers
Tests_With_Recursive_Sequences_(Perrin_Reed_Jameson)
Twin_Triple_Sexy_Primes
Using_Zeta_Zeros_To_Compute_Numbertheoretic_Functions
Wieferich_And_Similar_Primes
Zeta-Function
```

SOUNDS

Directory: Sounds

```
primeNumberSong46Sec.mid
primeNumberSong46Sek.mp3
Eratosthenes.mpd (Melodyne file)
```

GRAPHICS

The directory `Images` contains numerous graphics in vector and raster formats.

Note: these graphics may not be distributed, reproduced or displayed on the Internet without permission of the author.

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https://www.primes.utm.edu (Chris Caldwell)

https://www.oeis.org (Neil J. A. Sloane)

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http://www.Mersenne.org (George Woltman and Scott Kurowski)

http://www.primzahlen.de (Hans-Michael Elvenich)

http://www.seti.org (Bill Diamond)

http://www.aliquot.de (Dr.rer.nat Wolfgang Creyaufmüller)

http://www.mathpages.com (Kevin Brown)

http://www.wikipedia.org

and, of course*: http://www.wolfram.com/mathematica*

Finally, my thanks go to the many programmers of the Mathematica software, without which I could not have written this book.

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