

# Prime Numbers – things long-known and things new-found

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## 5.7 \_THE RIEMANN SPECTRUM

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We use the term “Riemann spectrum” or “Riemann transformation” if we mean the transformation of the prime numbers (and their powers) into the set of the imaginary parts of the zeros of the Zeta function.

On the other side we use the term “Inverse Riemann spectrum” or “Inverse Riemann transformation” if we mean the transformation of the imaginary parts of the Zeta-zeros into the set of prime numbers (and their powers).

**The following formula for the Inverse Riemann transformation shows very impressive peaks at the positions of primes and their powers.** The formula displays the ‘Inverse Riemann spectrum’ of the zeros of the Zeta function. We go from the “Zeta domain” to the “Prime number domain”:

$$f(x) = \text{nFact} * \sum_{n=1}^{nMax} \cos(\Im(\rho_n) \cdot \ln(x))$$

$\Im(\rho_n)$ : Imaginary part of the n-th zero of the Zeta function;

nFact: normalization factor (e. g.  $\frac{\ln(nMax)}{nMax}$ ).

The sum goes over the imaginary parts of the non trivial zeros of Riemanns Zeta function, taking  $nMax$  zeros. The more zeros are taken for summation, the more sharp peaks appear at positions of primes and prime powers. The length of the peaks belonging to a prime power  $p^n$  appear to have a length of  $\frac{1}{n}$  of the original prime peaks.

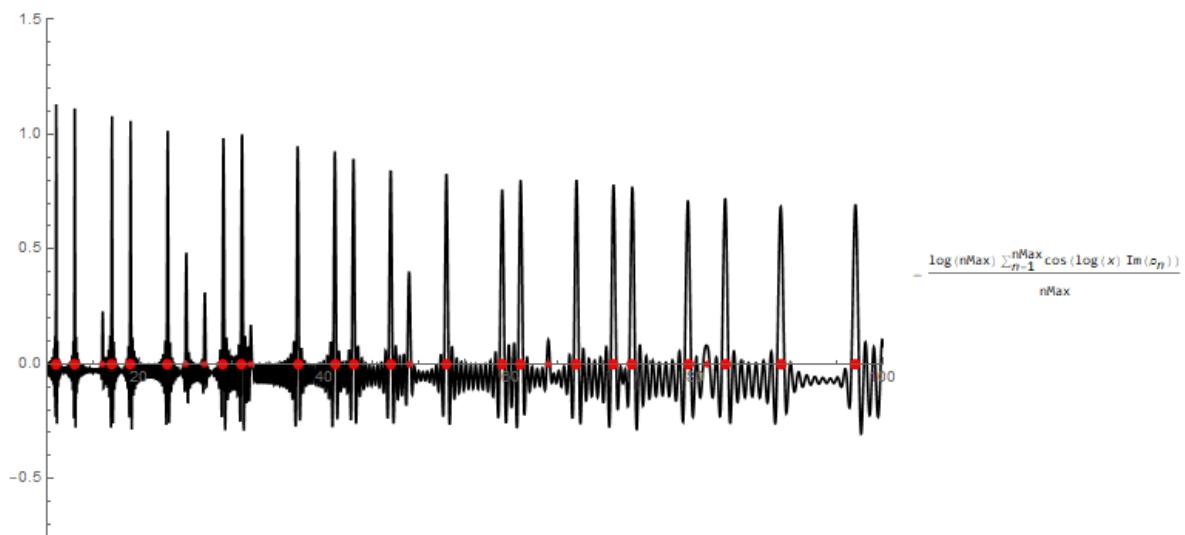
For  $n$  going to  $\infty$  the function is similar to a kind of delta function (known from physics). If a gifted mathematician succeeds in normalizing and building the integral over  $f(x)$  one could get a formula for  $\pi(x)$  (strictly speaking for the function  $J(x)$ , but  $\pi(x)$  can be easily derived from  $J(x)$  by using the Moebius inversion formula).

Here is an example computed with the first 400 zeros of the Zeta function, plotting  $f(x)$  in the range 10 – 100. Primes and prime powers marked by big and small red circles:

Mathematica:

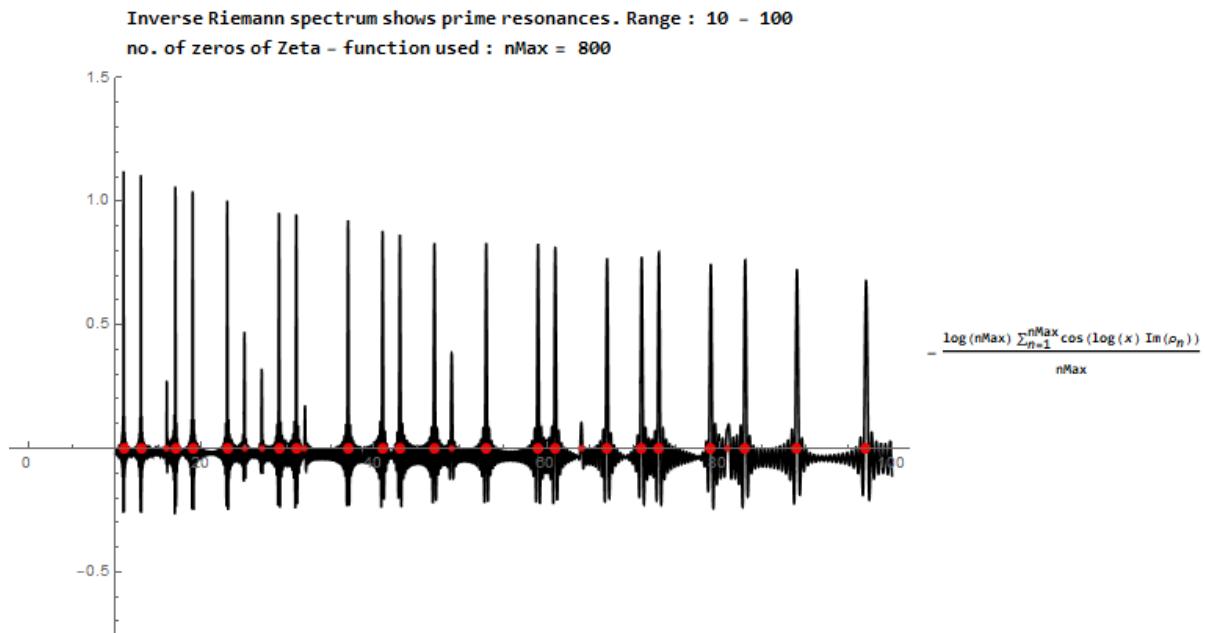
```
ClearAll[myFunc];
nMax=400;
o[k_]:=Im[ZetaZero[k]];
myFunc[p_]:=-1*Log[nMax]/nMax*Sum[Cos[o[n]*Log[p]],{n,1,nMax}]
xmin=10;xmax=100;ymin=-0.75;ymax=1.5;
grPlot=Plot[myFunc[x],{x,xmin,xmax},PlotRange->{{xmin,xmax},{ymin,ymax}},PlotStyle->Black,ImageSize->Large];//AbsoluteTiming
grPrimeMarkers=ListPlot[Table[{Prime[i],0},{i,PrimePi[xmin]+1,PrimePi[xmax]}],PlotStyle->{Red,PointSize[Large]},ImageSize->Large];
primePowers=Select[Range[xmin,xmax],PrimePowerQ[#]==True&&PrimeQ[#]==False]&;
grPrimePowerMarkers=ListPlot[Table[{primePowers[[i]],0},{i,1,Length[primePowers]}],PlotStyle->{Red,PointSize[Medium]},ImageSize->Large,PlotLegends->Placed["Inverse Riemann spectrum shows prime resonances. Range: 10 - 100\nno. of zeros of Zeta-function used: nMax= "<>ToString[nMax],Above]];
ClearAll[nMax];
grFormula=ListPlot[{},ImageSize->Large,PlotLegends->TraditionalForm[myFunc[x]],ImageSize->Large];
Show[grPlot,grPrimeMarkers,grPrimePowerMarkers,grFormula]
```

Inverse Riemann spectrum shows prime resonances. Range: 10 – 100  
 no. of zeros of Zeta-function used: nMax= 400



Here are still a few more examples:

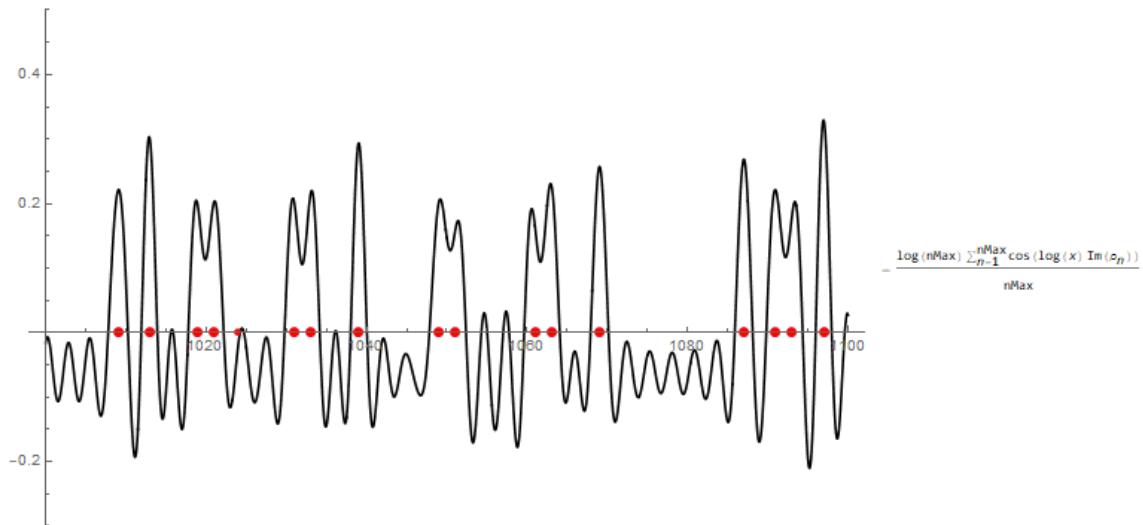
Using 800 zeros of the Zeta function,  $f(x)$  is plotted in the range 10 – 100. Primes and prime powers are marked by big and small red circles:



If we go to higher number ranges, we need to take more zeros.

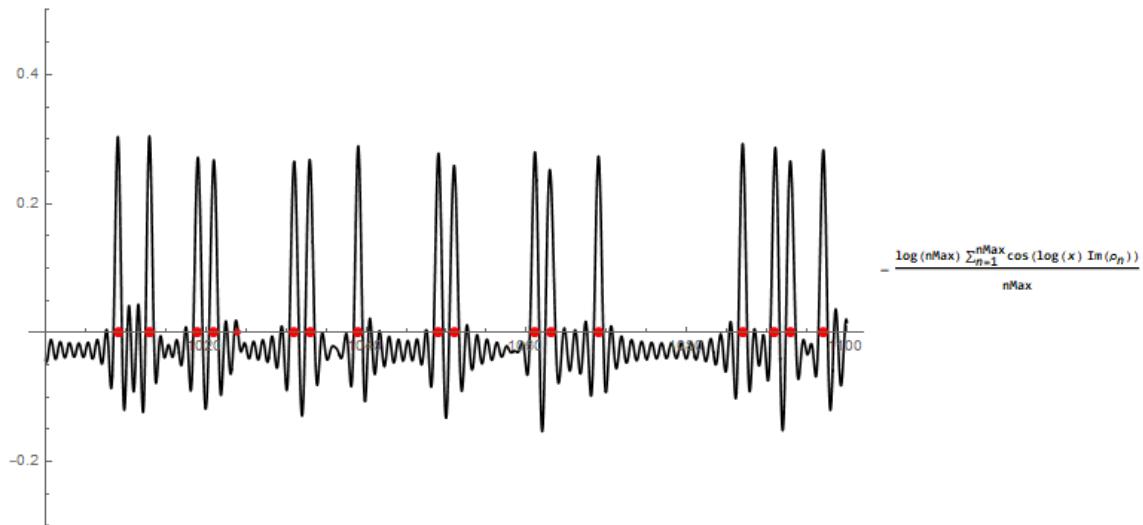
Using 2000 zeros of the Zeta function, plotting  $f(x)$  in the range 1000 – 1100:

Inverse Riemann spectrum shows prime resonances. Range: 1000 - 1100  
 no. of zeros of Zeta-function used: nMax= 2000



Using 5000 zeros of the Zeta function, plotting  $f(x)$  in the range 1000 - 1100:

Inverse Riemann spectrum shows prime resonances. Range : 1000 - 1100  
 no. of zeros of Zeta - function used : nMax = 5000



**The original formula for the Riemann transformation reads:**

$$f(x) = -\frac{1}{\pi} \sum_{p^n} \frac{\ln(p)}{p^2} \cos(x \cdot \ln(p^n))$$

Note:  $p^n$  means: running over all primes and powers of primes with values up to a given limit,  $p$  runs over all primes (no prime powers) up to that limit.

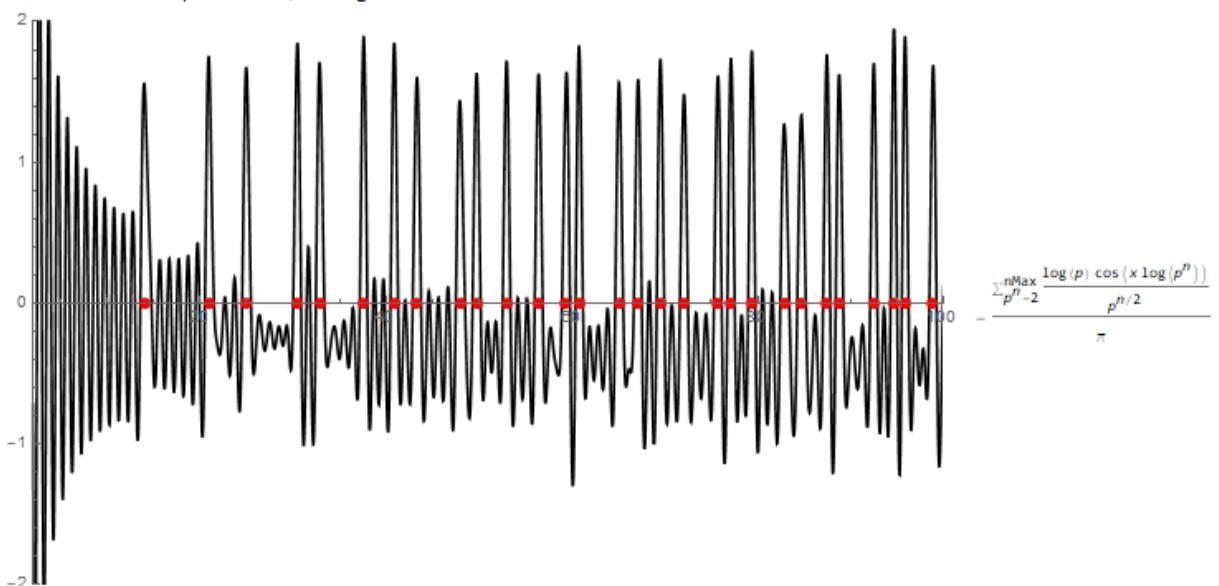
Again, the function graphs for  $f(x)$  show peaks. The peaks are located at the positions of the imaginary parts of the non trivial zeros of the Zeta function. The more primes and prime powers are taken for summation, the more sharp peaks appear at the positions of the zeros. There is a fast oscillating ‘noise’ which tends to have negative values.

Here is an example, which evaluates the first 100 prime numbers. The plot range is from 2 up to 100, the positions of the zeros are marked with red circles:

Mathematica:

```
ClearAll[myFunc]; Clear[nMax];
Clear[grPlot]; Clear[grZeroMarkers]; Clear[p]; Clear[n];
pmax=100; primeRange=Prime[pmax];
pP=Select[Range[primeRange], PrimePowerQ]; nmax=Length[pP];
myFunc[s_]:=-1/Pi*Sum[Log[FactorInteger[pP[[n]]][[1,1]]]*pP[[n]]^(-1/2)*Cos[s*Log[pP[[n]]]],{n,1,nmax}];
myFuncLegend[s_]:=-1/Pi*Sum[Log[p]/p^(n/2)*Cos[s*Log[p^n]],{p^n,2,nMax}];
xmin=2;xmax=100;ymin=-2; ymax=2;
firstZetaIndexInst=FindInstance[N[Im[ZetaZero[n]]]>=xmin,n]; firstZetaIndex=n/.firstZetaIndexInst[[1]];
lastZetaIndexInst=FindInstance[N[Im[ZetaZero[n]]]>xmax,n]; lastZetaIndex=n/.lastZetaIndexInst[[1]];
lastZetaIndex--; numberofZetaZeros=lastZetaIndex-firstZetaIndex+1;
grPlot=Plot[myFunc[x],{x,xmin,xmax},PlotRange->{{xmin,xmax},{ymin,ymax}},PlotStyle->Black,ImageSize->Large];//AbsoluteTiming
grZeroMarkers=ListPlot[Table[{Im[ZetaZero[i]],0},{i,firstZetaIndex,lastZetaIndex}],PlotStyle->{Red,PointSize[Large]},ImageSize->Large,
PlotLegends->Placed["Riemann spectrum shows resonances in the Zeta domain
nClear peaks at " <>ToString[numberofZetaZeros] <> " zero-positions\nno.
of primes and prime powers used: nmax= " <>ToString[nmax] <> "\nnpmax:
" <>ToString[pmax] <> ", xRange: " <>ToString[xmin] <> "-" <>ToString[xmax],Above]];
grFormula =ListPlot[{}],ImageSize->Large,PlotLegends->TraditionalForm[myFuncLegend[x]],ImageSize->Large];
Show[grPlot,grZeroMarkers,grFormula]
```

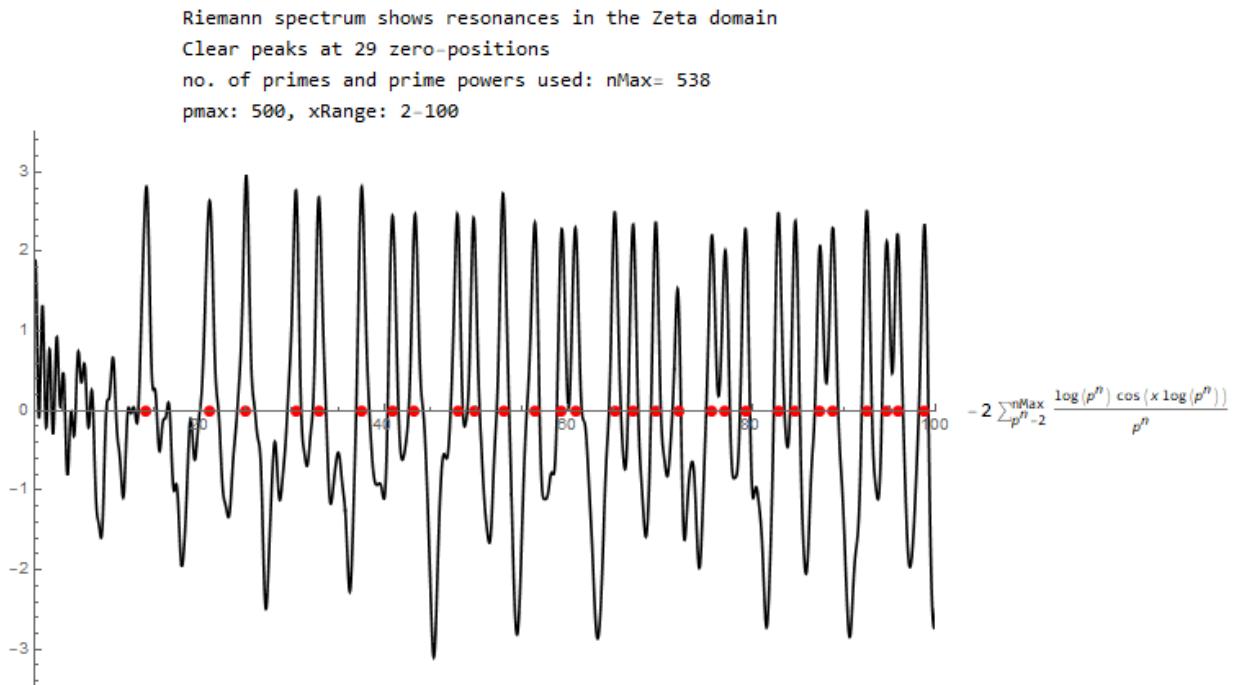
```
Riemann spectrum shows resonances in the Zeta domain
Clear peaks at 29 zero-positions
no. of primes and prime powers used: nmax= 121
pmax: 100, xRange: 2-100
```



The oscillating ‘noise’ can be suppressed by modifying the formula for the Riemann transformation: We replace the term  $\frac{\ln(p)}{p^2}$  by  $\frac{\ln(p^n)}{p^n}$  and the ‘normalization’ factor  $\frac{1}{\pi}$  by the factor 2 and get the slightly modified formula:

$$f(x) = -2 \sum_{p^n} \frac{\ln(p^n)}{p^n} \cos(x \cdot \ln(p^n))$$

The summation goes now over all primes and prime powers. The result is a smoother graph. The drawback is, that zeros with small distances will not be ‘caught’. Here is an example for the resulting function graph, which shows very sharp peaks for the first 29 zeros when prime powers for the first 500 prime numbers are used (locations of the zeros are marked by red circles):



## 8.9\_SYSTEMS OF EQUATIONS HAVING INTEGER SEQUENCES AS SOLUTIONS (VANDERMONDE MATRICES)

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### 8.9.1\_SOLUTION VECTOR: THE SEQUENCE OF NATURAL NUMBERS

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We consider equation systems of the following kind:

$$1: x_1 + x_2 + x_3 + \cdots + x_n = a_1 = 1$$

$$2: x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = a_2$$

$$3: x_1^3 + x_2^3 + x_3^3 + \cdots + x_n^3 = a_3$$

⋮

$$n: x_1^n + x_2^n + x_3^n + \cdots + x_n^n = a_n$$

This matrix is commonly referred to as the Vandermonde matrix. The solutions of this system of equations have the form:

$a_n = r_1^n + r_2^n + r_3^n + \cdots + r_n^n$ , in which  $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  is the solution vector and  $r_n$  are the  $n$  complex roots of a polynomial  $p_n$  with a degree of  $n$ :

$$p_n(y) = c_0 + c_1y + c_2y^2 + \cdots + c_ny^n$$

The polynomial  $p_n(y)$  depends on the sequence  $\mathbf{intF}_n = \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n$  in a typical way.

Let the constants  $c_{0,n}, c_{1,n}, c_{2,n}, \dots, c_{n,n}$  be the lowest line in a triangular matrix. At the top of this triangular matrix is the line consisting of the single value 1. Then the following holds: From the knowledge of the first diagonal row  $c_{0k}$  (from above to the lower left corner) of the triangular matrix we can calculate **all** values of the triangular matrix (and thus the values also of the lowest line). E.g. the concrete calculation of the line with index  $i$  from the preceding line with index  $i - 1$  reads:

We assume that  $c_{0,i}$  is given, then  $\mathbf{c}_{k,i} = \mathbf{c}_{k-1,i-1} \cdot (i - 1)$ , for  $k = 1, 2, \dots, i - 1$

The values of the first diagonal row (in descending order) beginning from the top value '1' we call **Coeff<sub>0,k</sub>** =  $\mathbf{c}_{0,1}, \mathbf{c}_{0,2}, \mathbf{c}_{0,3}, \dots, \mathbf{c}_{0,n}$  (with  $k \leq n$ ), because they represent each the constant term of a polynomial  $p_k(y) = c_{0k} + c_{1k}y + c_{2k}y^2 + \cdots + c_{nk}y^n$ . We don't want to consider here the polynomials  $p_n(y)$  and their zeros but the relation between **Coeff<sub>0,k</sub>** and **a<sub>k</sub>**. Interesting is also, that the sequence **a<sub>k</sub>** can be extended beyond the limit  $a_n$  (although only the values **a<sub>1</sub>** up to  $a_n$  were given). The values of the sequence members beyond  $a_n$  ( $a_{n+1}, a_{n+2}, \dots$ ) turn out to be rational numbers.

This connection between **intF<sub>n</sub>** and **Coeff<sub>0,k</sub>** represents a transformation, with which integer and rational sequences of numbers can be transformed back and forth, in both directions. In the following, 'PEQS' is the abbreviation for the term 'Polynomial EQuation System'. Thus, by 'PEQS transformation' we mean the transformation with which the sequences **intF<sub>n</sub>** and **Coeff<sub>0,k</sub>** can be transformed into each other. Here is an example for a PEQS equation system of order  $n$ , whose solutions are identical with the set of positive natural numbers going from 1 up to  $n$ :

$$1: x_1 + x_2 + x_3 + \cdots + x_n = 1$$

$$2: x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 2$$

$$3: x_1^3 + x_2^3 + x_3^3 + \cdots + x_n^3 = 3$$

⋮

$$n: x_1^n + x_2^n + x_3^n + \cdots + x_n^n = n$$

A concrete example:  $n = 5$ ,  $\text{intF}_k = a_k = \{1, 2, 3, 4, 5\}$ . From this we get a triangular matrix, whose line values are each the coefficients of polynomials of order 0 up to 5:

			1		
			-1	1	
			-1	-2	2
{ PEQS triangle based on natural numbers up to 5,			-1	-3	-6
			1	-4	-12
			19	5	-20
			-4	-12	-24
			-6	-120	24
			24	120	

The first diagonal row reads:  $\text{Coeff}_{0,k} = \{1, -1, -1, -1, 1, 19\}$

$p_5(y) = 19 + 5y - 20y^2 - 60y^3 - 120y^4 + 120y^5$  having the 5 complex zeros:

roots

$$= \{-0.577053079542094, 0.5579610065070687, 1.3884072929663431, -0.18465760996565891 \\ - 0.5657660375057798i, -0.18465760996565891 + 0.5657660375057798i\}$$

Now, the 5 solutions of the equation system read:

$$\text{seqFunc}[n] = (-0.577\dots)^n + (0.558\dots)^n + (1.39\dots)^n + (-0.185\dots - 0.566\dots i)^n + (-0.185\dots + 0.566\dots i)^n$$

In which we use the values von 1 bis 5 for  $n$ . The first 8 values of  $\text{seqFunc}[n]$  read:

$\{1, 2, 3, 4, 5, \frac{871}{120}, \frac{599}{60}, \frac{3313}{240}, \dots\}$  It can be seen, that  $\text{seqFunc}[n]$  is identical with the first 5 values of  $\text{intF}_n$  (respectively  $a_n$ , using the terms above), however, can be extended beyond  $a_n$ .

In this case ( $a_n$  being the sequence of natural numbers) the formula for  $\text{Coeff}_{0,n}$  is known:

$\text{Coeff}_{0,n} = -n! {}_1F_1(1-n; 2; 1)$ , in which  ${}_1F_1$  is the Kummer confluent hypergeometric function (A293226 in <https://oeis.org>).

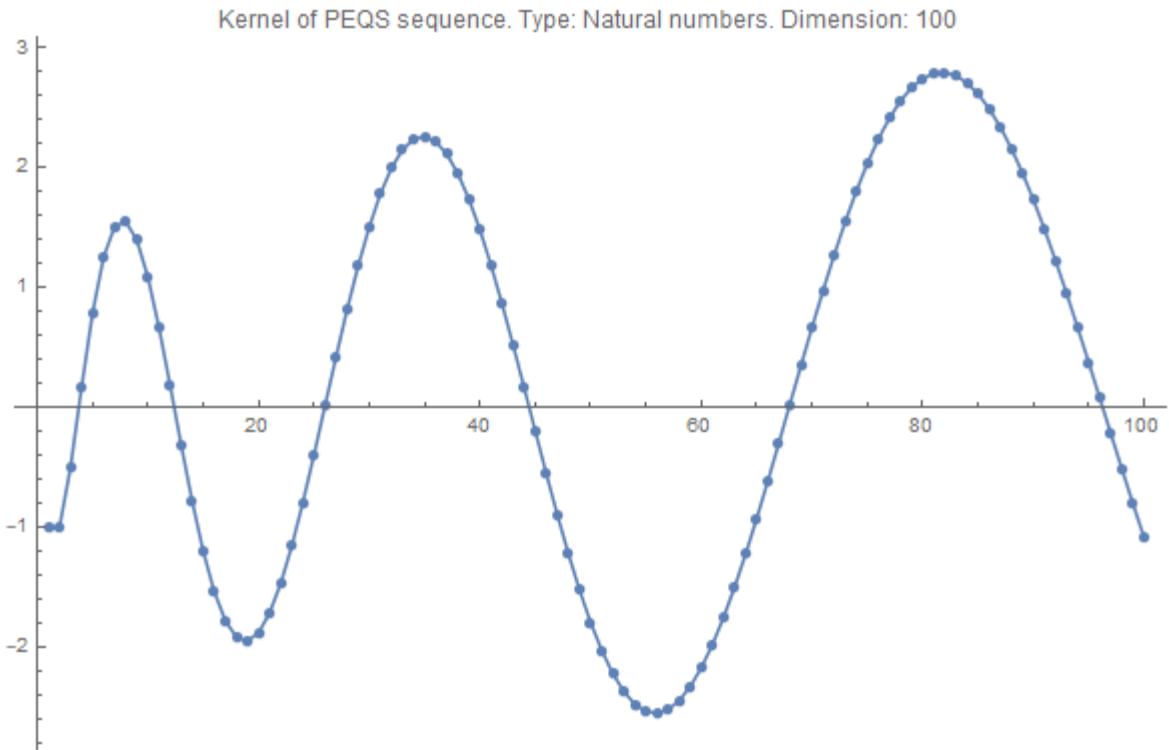
The sequence  $\text{seqFunc}[n]$  also belongs to the group of the linear recursive functions with the kernel:

$\{1, \frac{1}{2}, \frac{1}{6}, -\frac{1}{24}, -\frac{19}{120}\}$  and it is easy to calculate in this way.

Note: If we turn this principle of construction for the triangular matrix around (construction from the first diagonal row  $\text{Coeff}_{0,n}$ ) then the triangular matrix can be extended to a square matrix (in direction to the upper left corner). In the first (and second) line we have now the negative kernel of the recursive relation in reversed order:  $\{\frac{19}{120}, \frac{1}{24}, -\frac{1}{6}, -\frac{1}{2}, -1\}$ :

$$\begin{pmatrix} \frac{19}{120} & \frac{1}{24} & -\frac{1}{6} & -\frac{1}{2} & -1 & 1 \\ \frac{19}{120} & \frac{1}{24} & -\frac{1}{6} & -\frac{1}{2} & -1 & 1 \\ \frac{19}{120} & \frac{1}{24} & -\frac{1}{6} & -\frac{1}{2} & -1 & 1 \\ \frac{19}{60} & \frac{1}{12} & -\frac{1}{3} & -1 & -2 & 2 \\ \frac{19}{20} & \frac{1}{4} & -1 & -3 & -6 & 6 \\ \frac{19}{5} & 1 & -4 & -12 & -24 & 24 \\ 19 & 5 & -20 & -60 & -120 & 120 \end{pmatrix}$$

The rational values of the kernel can also be calculated from the quotients of the first and the last coefficients  $c_0$  and  $c_n$  of the  $n$ -th polynomial (respectively marked in red and blue color). The graph of the first 100 kernel values look like this:



The real problem with these calculations is determining  $\mathbf{Coeff}_{0,n}$  from a given solution vector  $\mathbf{intF}_k$ . For the case of natural numbers we have an explicit method (see above) by means of the Hypergeometric function. In this case, however, we can make it even more easy:

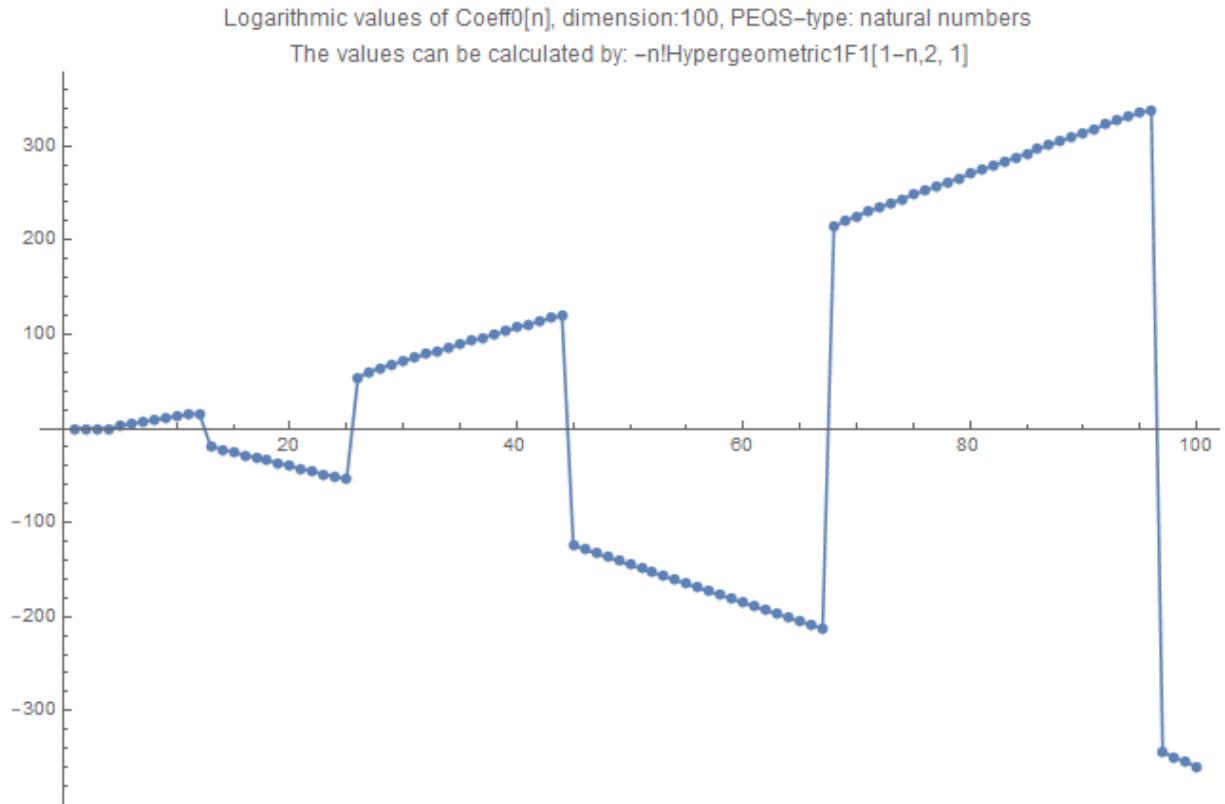
Let  $\mathbf{mat}_n$  be the sequence:

$$(1), \left( \begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix} \right), \left( \begin{matrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{matrix} \right), \left( \begin{matrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{matrix} \right), \left( \begin{matrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 4 & 1 \end{matrix} \right), \left( \begin{matrix} 1 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \end{matrix} \right), \dots$$

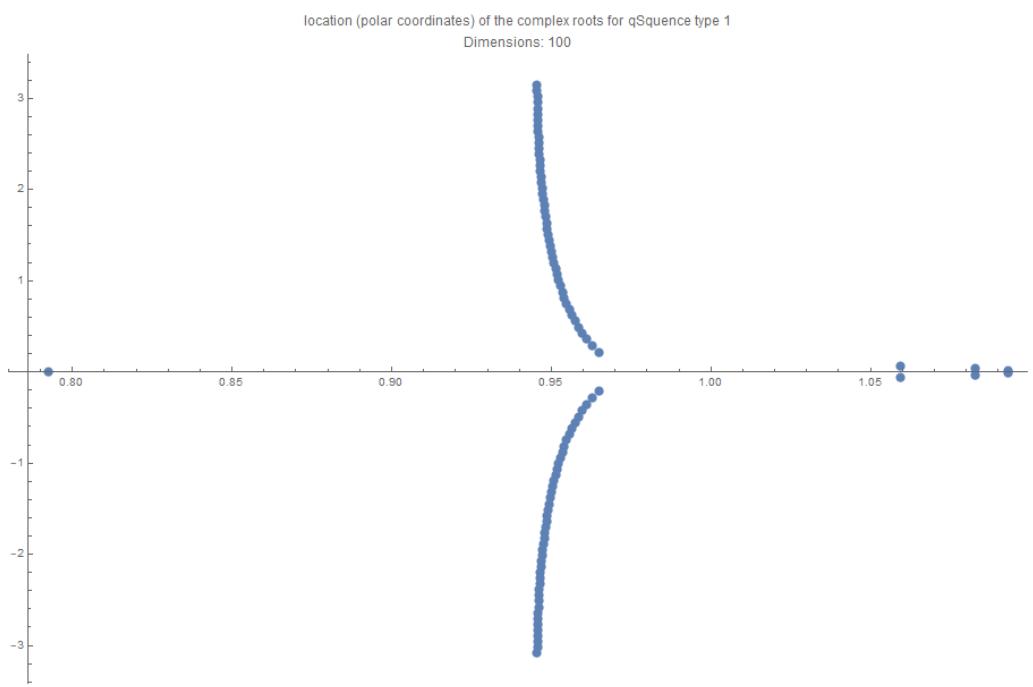
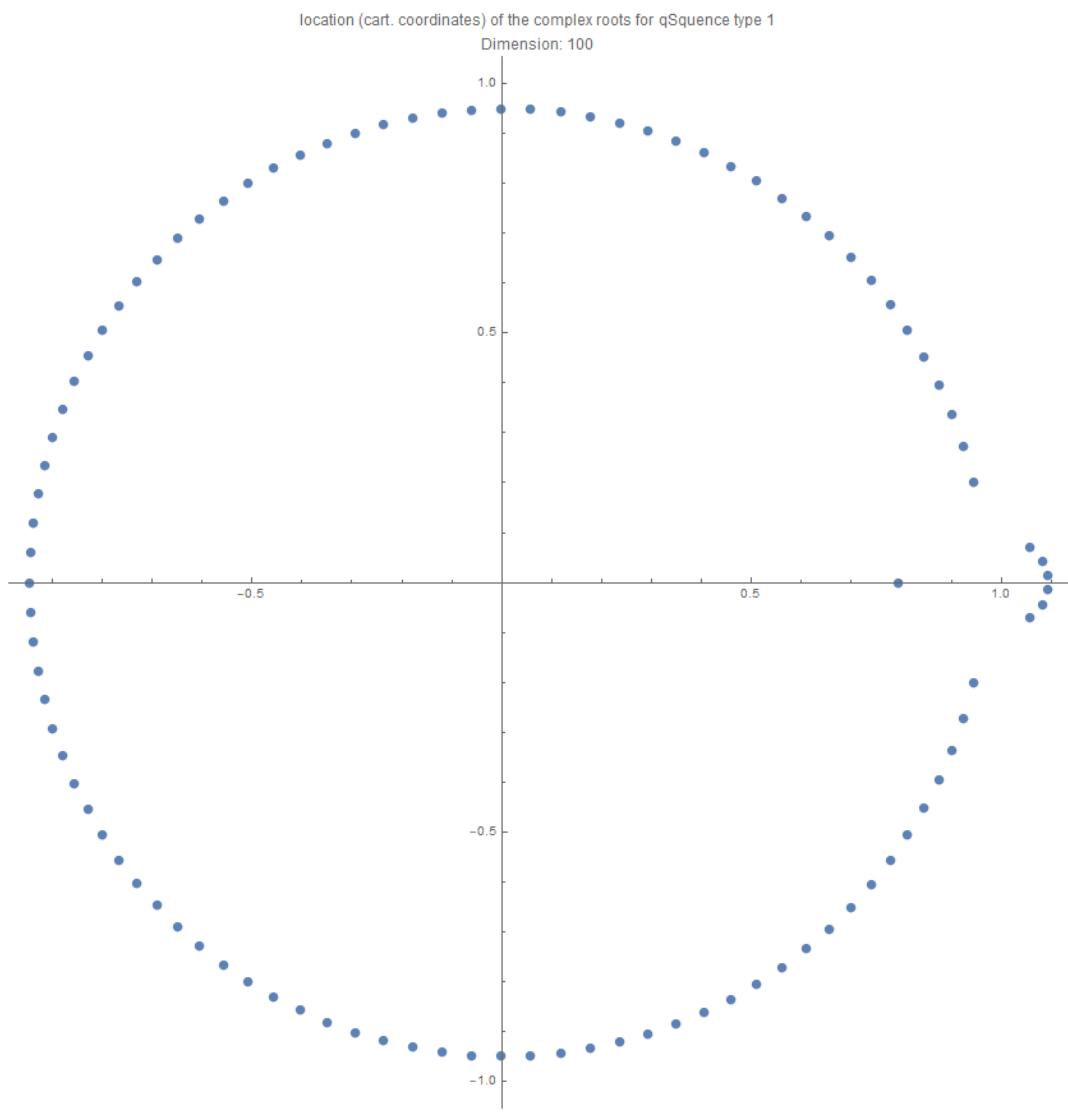
Following the construction rule:  $\text{mat}[i,j] = \begin{cases} i, & \text{if } i == j \\ 1, & \text{if } i > j \\ j - i, & \text{else} \end{cases}$

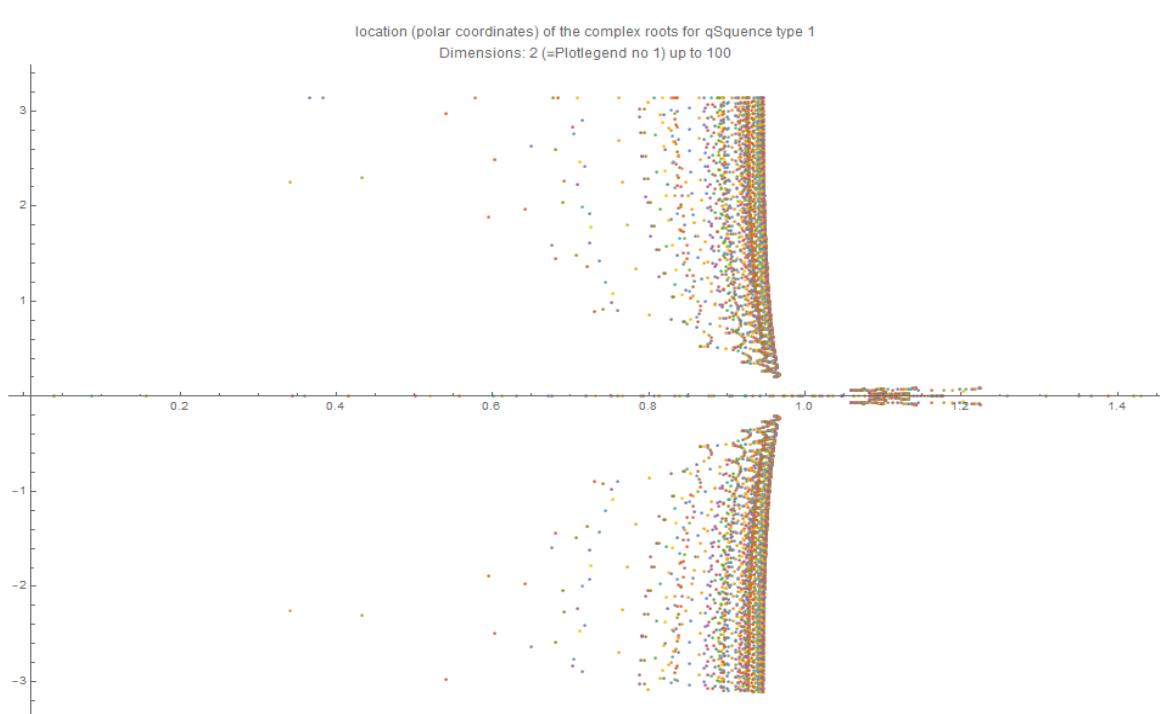
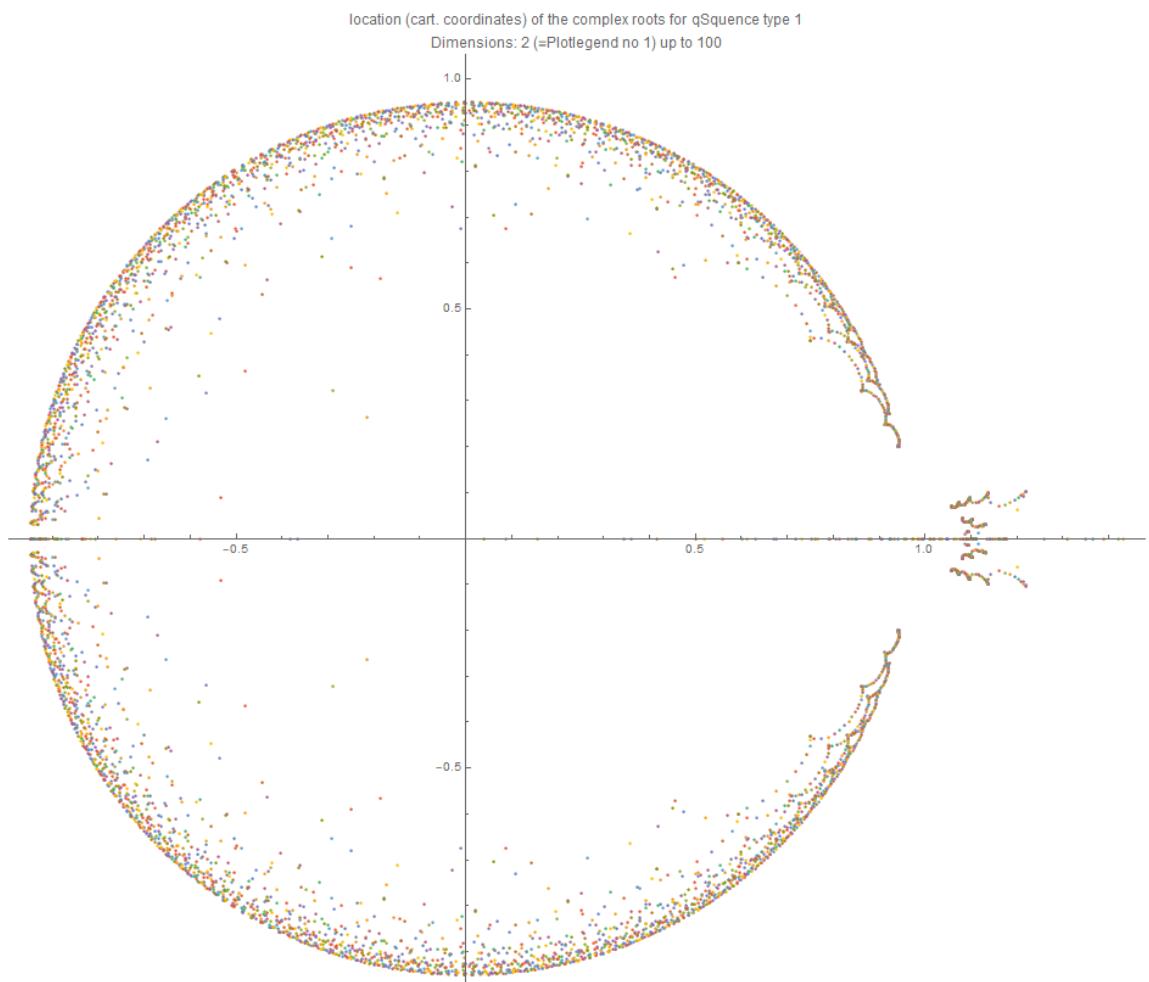
then it holds:  $\text{Det}(\text{mat}_n) = \text{Coeff}_{0,n}$

The graph of the first 100 Coeff0 values looks like this:



The things that have been explained using this example for the specific case of dimension 5 can be extended for any higher dimension. Here is another example for  $n = 100$  with a graphical representation of the complex zeros (in cartesian and spherical coordinates):





And here still another look at the prime factor decompositions of the first 50 values of  $\text{Coeff}_{0,n}$ , which show no abnormalities:

```

1      {{1,1}}
2      {{1,1}}
3      {{1,1}}
4      {{1,1}}
5      {{19,1}}
6      {{151,1}}
7      {{1091,1}}
8      {{7841,1}}
9      {{56519,1}}
10     {{223,1},{1777,1}}
11     {{23,1},{103,1},{1031,1}}
12     {{7701409,1}}
13     {{23,1},{6316067,1}}
14     {{5867,1},{823787,1}}
15     {{59,1},{1763664719,1}}
16     {{2002667085119,1}}
17     {{3362633,1},{11035753,1}}
18     {{4289,1},{158516607841,1}}
19     {{1801,1},{6907445473321,1}}
20     {{103,1},{49393,1},{44771341561,1}}
21     {{2731,1},{4993,1},{54583,1},{5583649,1}}
22     {{548079347,1},{136339117843,1}}
23     {{67891,1},{19047624174793631,1}}
24     {{19,1},{70195849,1},{15281774905949,1}}
25     {{2273,1},{132632251,1},{809726832587,1}}
26     {{59,1},{1031,1},{152791,1},{28786290689309,1}} )
27     {{29,1},{137,1},{17837,1},{2431572221454948151,1}}
28     {{1091,1},{13687,1},{1702690571,1},{351811724743,1}}
29     {{361702062324149751903132843499,1}}
30     {{2857,1},{4674028378481509129971784783,1}}
31     {{61,1},{16067,1},{984103174279,1},{490600467281563,1}}
32     {{457,1},{941,1},{7829,1},{4884665371786076435244233,1}}
33     {{79,1},{7349,1},{976046331680316442389698582029,1}}
34     {{23,1},{67,1},{7331,1},{1723166171858963399165217371761,1}}
35     {{195593,1},{52855733933,1},{242222367593,1},{266946775787,1}}
36     {{23,1},{998229455402129726075516494398995399207,1}}
37     {{10397987,1},{394418522203,1},{192099812273276678223331,1}}
38     {{239,1},{43403,1},{2596080359706103738851141696073679723,1}}
39     {{149,1},{670844597,1},{9124587114735743732707412704918043,1}}
40     {{228828199674511,1},{132492806214159332040949011205711,1}}
41     {{6566801183,1},{148066316705471672725042558703660116793,1}}
42     {{67,1},{267193810720792588447,1},{1621966420250627774099864251,1}}
43     {{19,1},{38720220357651297635515977184282535937987689769821,1}}
44     {{997,1},{30632132437,1},{5724052011253,1},{57737875048226218431440213,1}}
45     {{10712407481724409,1},{47962155441999128904276109046869785709,1}}
46     {{65712222112317165183589964642284924435961435874518608889,1}}
47     {{38501,1},{51461,1},{39710001171251,1},{62486369789949521475437558024641039,1}}
48     {{148279,1},{36911387,1},{910785131,1},{63219590349635839362267416075137730033,1}}
49     {{199,1},{146375891,1},{10795417408492229,1},{59936493382835752308549010189780921,1}}
50     {{53,1},{137,1},{3467,1},{5903,1},{91997,1},{6439003,1},{7639455163,1},{2004907113617,1},{806219781329822269,1}}

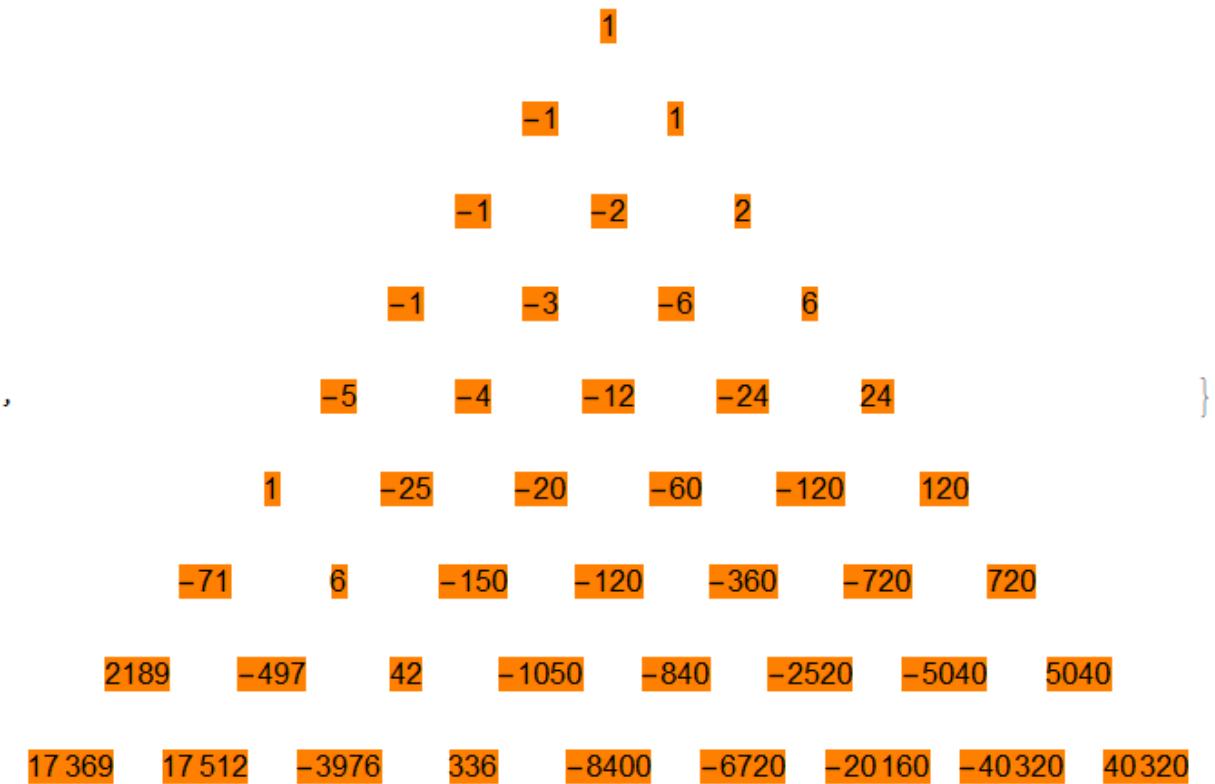
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### 8.9.2 SOLUTION VECTOR: SEQUENCE OF PRIME NUMBERS (INCLUDING THE '1')

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A concrete example:  $n = 8$ ,  $\text{intF}_k = a_k = \{1, 2, 3, 5, 7, 11, 13, 17\}$ . From this we get a triangular matrix, whose line values are each the coefficients of polynomials of order 0 up to 8:

PEQS triangle based on Primenumbers up to dimension 8



The first diagonal row reads:  $\text{Coeff}_{0,k} = \{1, -1, -1, -1, -5, 1, -71, 2189, 17369\}$

$p_8(y) = 17369 + 17512y - 3976y^2 + 336y^3 - 8400y^4 - 6720y^5 - 20160y^6 - 40320y^7 + 40320y^8$  having the following 8 complex zeros:

$$0.970\ldots, 1.43\ldots, -0.714\ldots - 0.172\ldots i, -0.714\ldots + 0.172\ldots i, -0.363\ldots - 0.761\ldots i, -0.363\ldots + 0.761\ldots i, 0.377\ldots - 0.818\ldots i, 0.377\ldots + 0.818\ldots i$$

We have the following 8 solutions of the equation system:

$$+ \textcolor{blue}{\text{(F)}} 0.970... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} 1.43... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} -0.714... - 0.172... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} -0.714... + 0.172... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} -0.363... - 0.761... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} -0.363... + 0.761... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} 0.377... - 0.818... \textcolor{brown}{\text{i}} + \textcolor{blue}{\text{(F)}} 0.377... + 0.818... \textcolor{brown}{\text{i}}$$

In which we use the values from 1 up to 8 for  $n$ . The first 12 values of  $\text{seqFunc}[n]$  read:

$\{1, 2, 3, 5, 7, 11, 13, 17, \frac{207619}{8064}, \frac{496943}{13440}, \frac{470619}{8960}, \frac{18053729}{241920}, \dots\}$  It can be seen, that  $seqFunc[n]$  is for the first 8 values identical with  $intF_n$  (respectively  $a_n$ , using the terms above), however, can be extended beyond  $a_n$ .

In this case ( $a_n$  is the sequence of prime numbers including ‘1’) the underlying formula for  $\text{Coeff}_{0,n}$  is not (yet) known. The sequence  $\text{seqFunc}[n]$  also belongs to the group of linear recursive functions with the kernel:

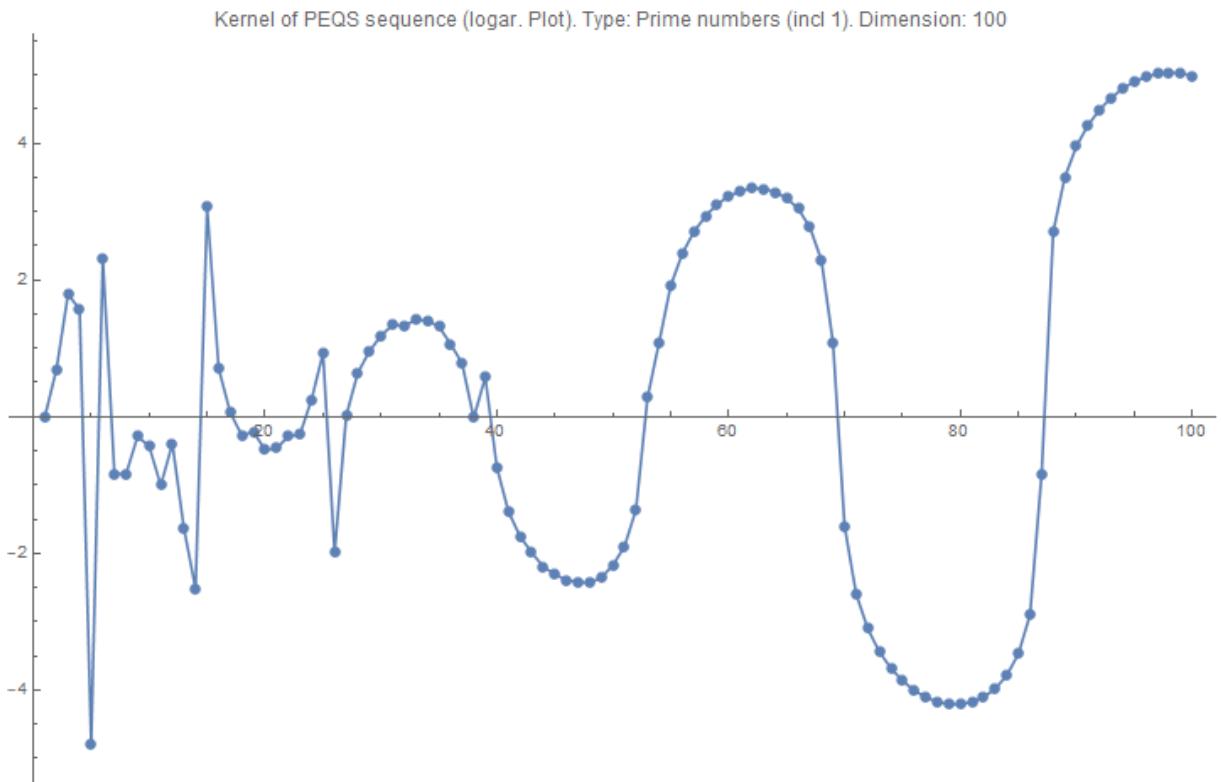
$\{1, \frac{1}{2}, \frac{1}{6}, \frac{5}{24}, -\frac{1}{120}, \frac{71}{720}, -\frac{2189}{5040}, -\frac{17369}{40320}\}$  and it is easy to calculate in this way.

Note: If we turn this principle of construction for the triangular matrix around (construction from the first diagonal row  $\text{Coeff}_{0,n}$ ) then the triangular matrix can be extended to a square matrix (in direction to the upper left corner):

$$\left( \begin{array}{ccccccccc} 17369 & 2189 & -\frac{71}{120} & \frac{1}{24} & -\frac{5}{6} & -\frac{1}{2} & -1 & \textcolor{red}{-1} \\ \frac{5040}{17369} & \frac{720}{2189} & -\frac{71}{120} & \frac{1}{24} & -\frac{5}{6} & -\frac{1}{2} & \textcolor{red}{-1} & \textcolor{blue}{-1} \\ \frac{5040}{17369} & \frac{720}{2189} & -\frac{71}{120} & \frac{1}{24} & -\frac{5}{6} & -\frac{1}{2} & \textcolor{red}{-1} & \textcolor{blue}{-1} \\ \frac{2520}{17369} & \frac{360}{2189} & -\frac{71}{60} & \frac{1}{12} & -\frac{5}{3} & \textcolor{red}{-1} & -2 & -2 \\ \frac{17369}{840} & \frac{2189}{120} & -\frac{71}{20} & \frac{1}{4} & \textcolor{red}{-5} & -3 & -6 & -6 \\ \frac{17369}{210} & \frac{2189}{30} & -\frac{71}{5} & \textcolor{red}{1} & -20 & -12 & -24 & -24 \\ \frac{17369}{42} & \frac{2189}{6} & \textcolor{red}{-71} & 5 & -100 & -60 & -120 & -120 \\ \frac{17369}{7} & \textcolor{red}{2189} & -426 & 30 & -600 & -360 & -720 & -720 \\ \textcolor{red}{17369} & 15323 & -2982 & 210 & -4200 & -2520 & -5040 & -5040 \end{array} \right)$$

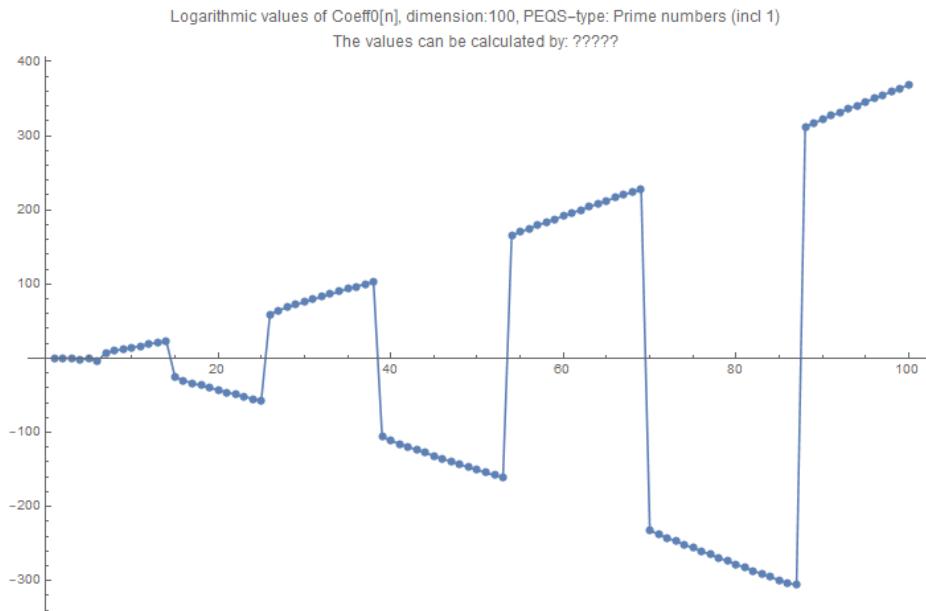
The rational values of the kernel can also be calculated from the quotients of the first coefficients  $c_0$  (of the  $n$ th line) and the last coefficient  $c_n$  (of the  $(n+1)$ -th line (each marked in red and blue color). The kernel sequence can simple also be written as:  $\frac{\text{Coef } c_0[n]}{n!}$

The graphic representation of the first 100 kernel values look like this:



After initial irregularities up to the 40-th prime number, the graph appears to follow a smooth function.

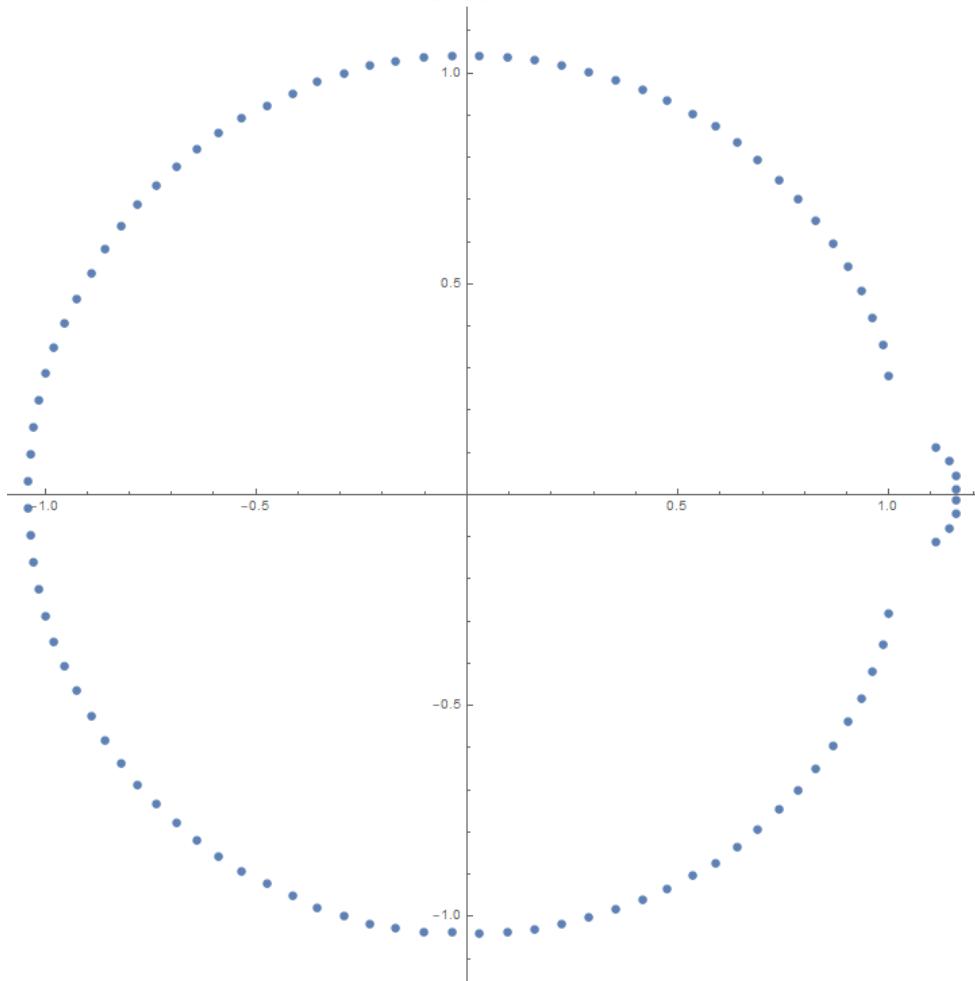
The graph of the first 100 Coeff0 values:



Here another example for  $n = 100$  with a graphical representation of the complex zeros based on the sequence of the prime numbers (including '1'), in cartesian and spherical coordinates:

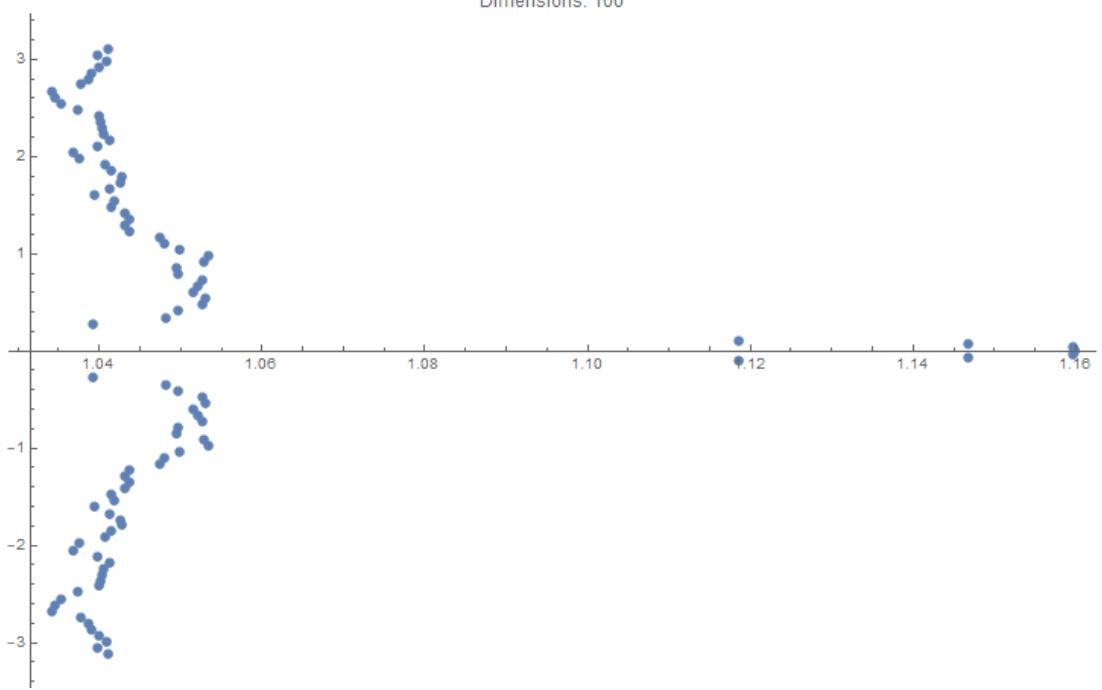
location (cart. coordinates) of the complex roots for qSquence type P1

Dimension: 100

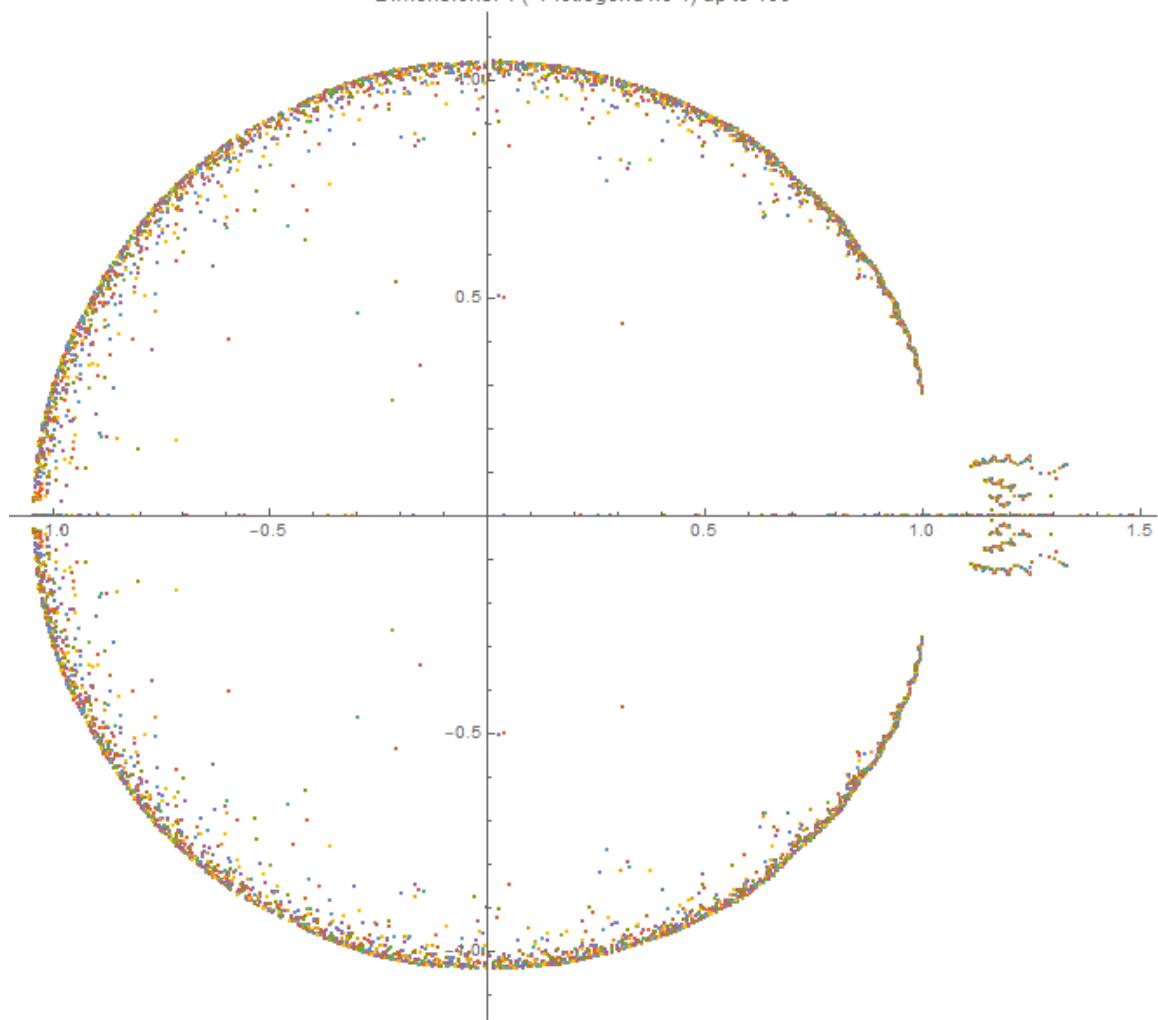


location (polar coordinates) of the complex roots for qSquence type P1

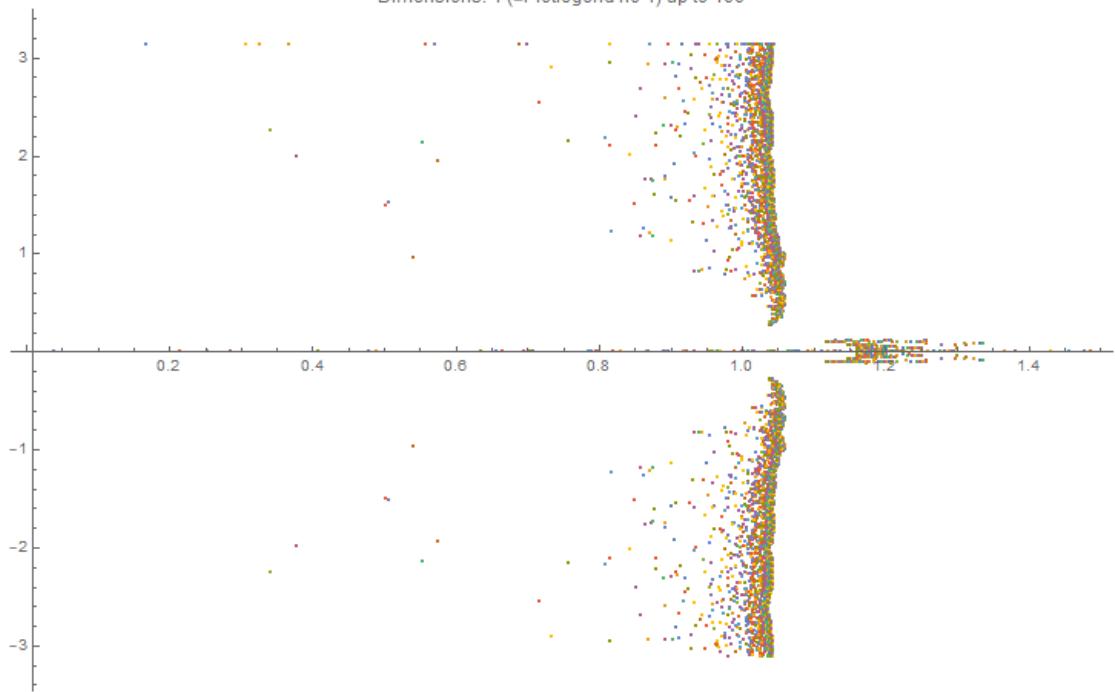
Dimensions: 100



location (cart. coordinates) of the complex roots for qSequence type P1  
Dimensions: 1 (=Plotlegend no 1) up to 100



location (polar coordinates) of the complex roots for qSequence type P1  
Dimensions: 1 (=Plotlegend no 1) up to 100



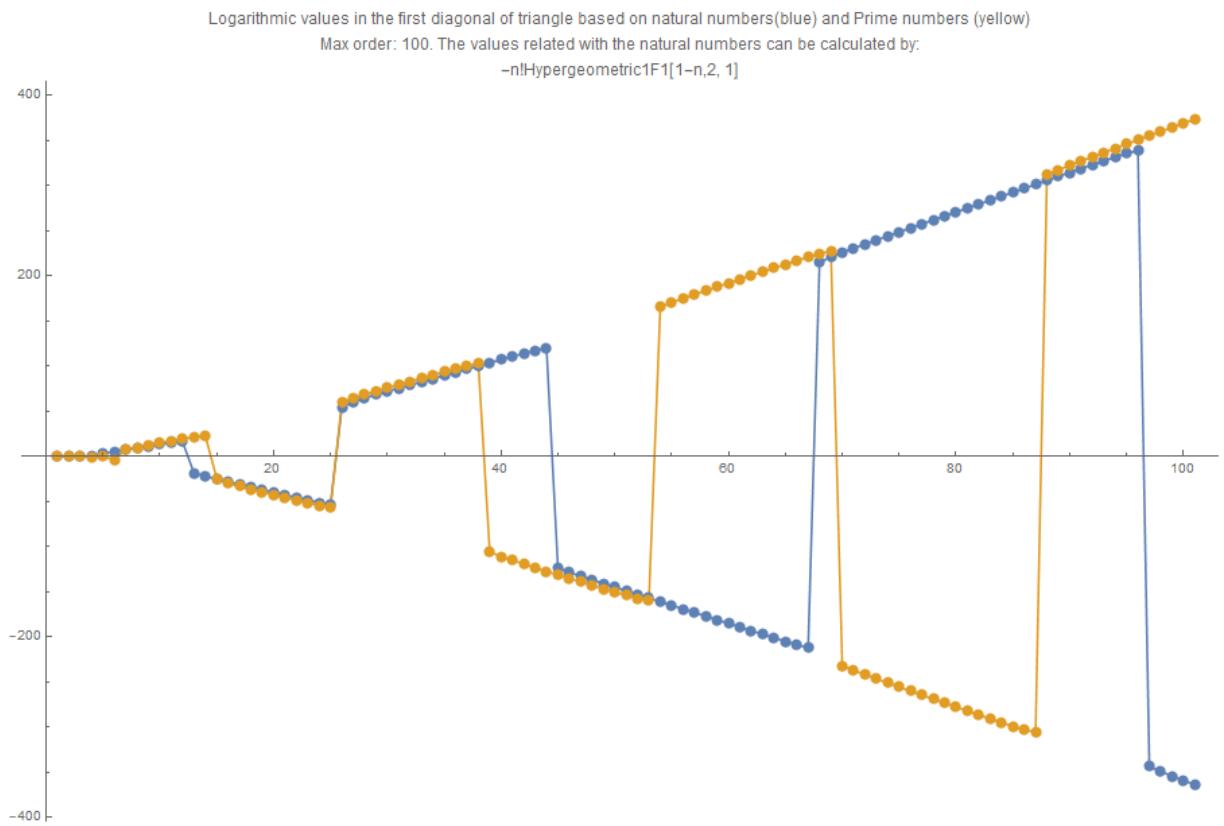
And here is another look at the prime factor decompositions of the first 50 values of  $\text{Coeff}_{0,n}$  (showing no abnormalities):

```

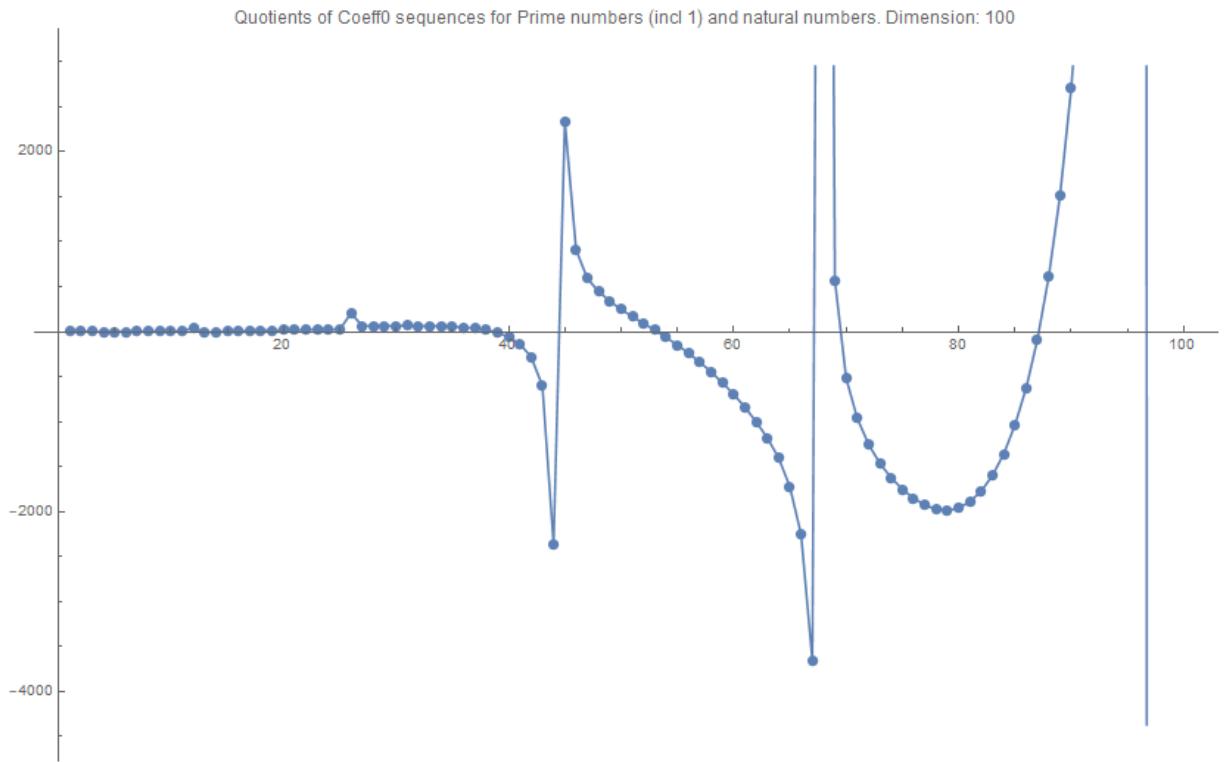
1          {{1,1}}
2          {{1,1}}
3          {{1,1}}
4          {{5,1}}
5          {{1,1}}
6          {{71,1}}
7          {{11,1},{199,1}}
8          {{11,1},{1579,1}}
9          {{5,1},{54403,1}}
10         {{31,1},{149,1},{509,1}}
11         {{14621239,1}}
12         {{281,1},{1147739,1}}
13         {{1217307569,1}}
14         {{5,1},{10529,1},{133733,1}}
15         {{4951,1},{12278989,1}}
16         {{1120907,1},{9248653,1}}
17         {{31,1},{167,1},{64949345833,1}}
18         {{11,1},{953,1},{21961,1},{36566627,1}}
19         {{5,1},{11,1},{17679667,1},{155914477,1}}
20         {{31,1},{30588403,1},{4178880553,1}}
21         {{29,1},{420803,1},{565463,1},{11764271,1}}
22         {{13537,1},{110848891214900903,1}}
23         {{32609,1},{25427321,1},{40639606619,1}}
24         {{5,3},{5550251869,1},{698493033631,1}}
25         {{643,1},{38261,1},{248881304175496463,1}}
26         {{1973,1},{27904679166759164551363,1}}
27         {{197,1},{55897240950785955716993467,1}}
28         {{8863,1},{77263,1},{844586701020730907971,1}}
29         {{5,1},{11,1},{17579,1},{3407271193,1},{6979935756062957,1}}
30         {{11,1},{137,1},{9043,1},{11777,1},{28364647,1},{190533140945879,1}}
31         {{67,1},{1245191,1},{1373495499781,1},{274557304919197,1}}
32         {{657469,1},{1506125486212660074232764836981,1}}
33         {{88681,1},{303781,1},{2357807,1},{9520206263,1},{59202318419,1}}
34         {{5,1},{281,1},{96849439788919813,1},{8841427248979156591,1}}
35         {{199,1},{911,1},{1355731193,1},{193793756917,1},{812267729749219,1}}
36         {{61,1},{1498843,1},{11802277715851056385913568043367893,1}}
37         {{11827,1},{98993,1},{25598033811869283428758498123282219,1}}
38         {{61,1},{13297,1},{97665349,1},{9301825277573,1},{696736579259231941,1}}
39         {{5,1},{1483,1},{1272989,1},{12514917277,1},{96886196973294584331021277,1}}
40         {{11,2},{41,1},{2417,1},{202291,1},{707907639285813015424527889937058317,1}}
41         {{11,1},{31,1},{196357319,1},{9563926637,1},{21193516919,1},{9879495844107365633,1}}
42         {{2861,1},{8905199908952780767189,1},{317931904100938067261450929,1}}
43         {{71,1},{7540793471249,1},{2220057913322783,1},{370405541768408417718593,1}}
44         {{5,1},{1297,1},{15299,1},{55987,1},{1270445153450608341769,1},{3383979802038295048681,1}}
45         {{83,1},{1009,1},{8939407,1},{38074871,1},{42192534451089560702168725307868912781,1}}
46         {{34537,1},{926707,1},{3696593,1},{30537990645505529089,1},{16599250962806839019557,1}}
47         {{383,1},{96858953530575753108751,1},{78406389023851848767152440248791867,1}}
48         {{31,1},{277,1},{2173307,1},{346102703,1},{21842292327057143973217122168009604136019833,1}}
49         {{5,2},{257,1},{2339,1},{123863,1},{7633829256046840371293,1},{451394577456034889716827336793,1}}
50         {{29,1},{179,1},{761185113629063949410171680543,1},{67669210599599788288233929157673,1}}
)

```

A comparison of the first 100 values of  $\text{Coeff}_{0,n}$  for prime numbers (including ‘1’) and natural numbers (the natural numbers in blue color, the prime numbers in yellow color):



It's also worth to take a look at the (linear) plot of the quotients of  $\text{Coeff}_{0,n}$  of both sequences:



### 8.9.3\_SOLUTON VECTOR: SEQUENCE OF MERSENNE PRIME EXPONENTS (WITH '1')

A concrete example:  $n = 10$ ,  $\text{intF}_k = a_k = \{1, 2, 3, 5, 7, 13, 17, 19, 31, 61\}$ . From this we get a triangular matrix, whose line values are each the coefficients of polynomials of order 0 up to 10:

{PEQS triangle based on Mersenne Prime exponents up to dimension 10}																						
1																						
-1		1																				
-1			-2				2															
-1		-3			-6		6															
-5			-4		-12		-24		24													
,		1		-25		-20		-60		-120		120										
-311			6		-150		-120		-360		-720		720									
989			-2177		42		-1050		-840		-2520		-5040		5040							
37049			7912		-17416		336		-8400		-6720		-20160		-40320		40320					
2735			333441		71208		-156744		3024		-75600		-60480		-181440		-362880		362880			
-5548769			27350		3334410		712080		-1567440		30240		-756000		-604800		-1814400		-3628800		3628800	

The first diagonal row reads:

$$\text{Coeff}_{0,k} = \{1, -1, -1, -1, -5, 1, -311, 989, 37049, 2735, -5548769\}$$

$$p_{10}(y) = -5548769 + 27350y + 3334410y^2 + 712080y^3 - 1567440y^4 + 30240y^5 - 756000y^6 - 604800y^7 - 1814400y^8 - 3628800y^9 + 3628800y^{10}$$

having the 10 complex zeros.

$$\{\text{0.985...}, \text{1.50...}, \text{-0.780... - 0.530...i}, \text{-0.780... + 0.530...i}, \text{-0.327... - 0.994...i}, \text{-0.327... + 0.994...i}, \text{0.453... - 0.960...i}, \text{0.453... + 0.960...i}, \text{0.898... - 0.374...i}, \text{0.898... + 0.374...i}\}$$

Now, the 10 solutions of the equation system read:

$$\{\text{0.985...}^n, \text{1.50...}^n, \text{-0.780... - 0.530...i}^n, \text{-0.780... + 0.530...i}^n, \text{-0.327... - 0.994...i}^n, \text{-0.327... + 0.994...i}^n, \text{0.453... - 0.960...i}^n, \text{0.453... + 0.960...i}^n, \text{0.898... - 0.374...i}^n, \text{0.898... + 0.374...i}^n\}$$

In which we use the values of 1 up to 10 for  $n$ . The first 12 values of  $\text{seqFunc}[n]$  read:

$\{1, 2, 3, 5, 7, 13, 17, 19, 31, 61, \frac{304455799}{3628800}, \frac{457468517}{3628800}, \dots\}$ . It can be seen, that the first 10 values of  $\text{seqFunc}[n]$  are identical with  $\text{intF}_n$  (respectively with  $a_n$ , using the terms from above), however can be extended beyond higher indices.

In this case ( $a_n$  ist the sequence of Mersenne prime exponents including '1') the underlying formula for  $\text{Coeff}_{0,n}$  is not (yet) known. The sequence  $\text{seqFunc}[n]$  also belongs to the group of linear recursive functions with the kernel:

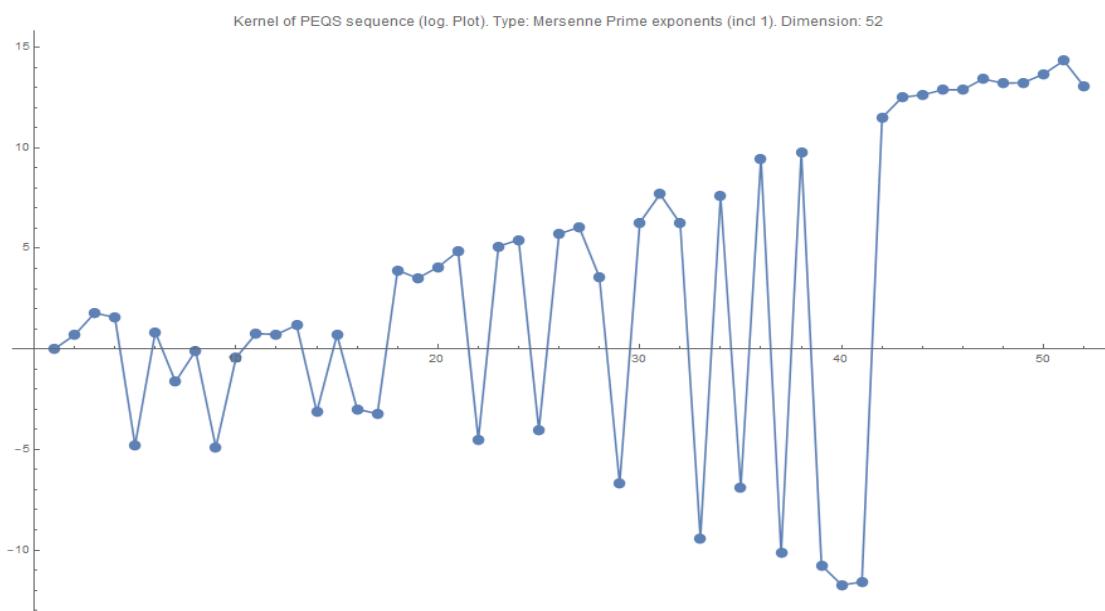
$\{1, \frac{1}{2}, \frac{1}{6}, \frac{5}{24}, -\frac{1}{120}, \frac{311}{720}, -\frac{989}{5040}, -\frac{37049}{40320}, -\frac{547}{72576}, \frac{5548769}{3628800}\}$  and it is easy to be calculated this way.

Note: If we turn this principle of construction for the triangular matrix around (construction from the first diagonal row  $\text{Coeff}_{0,n}$ ) then the triangular matrix can be extended to a square matrix (in direction to the upper left corner).

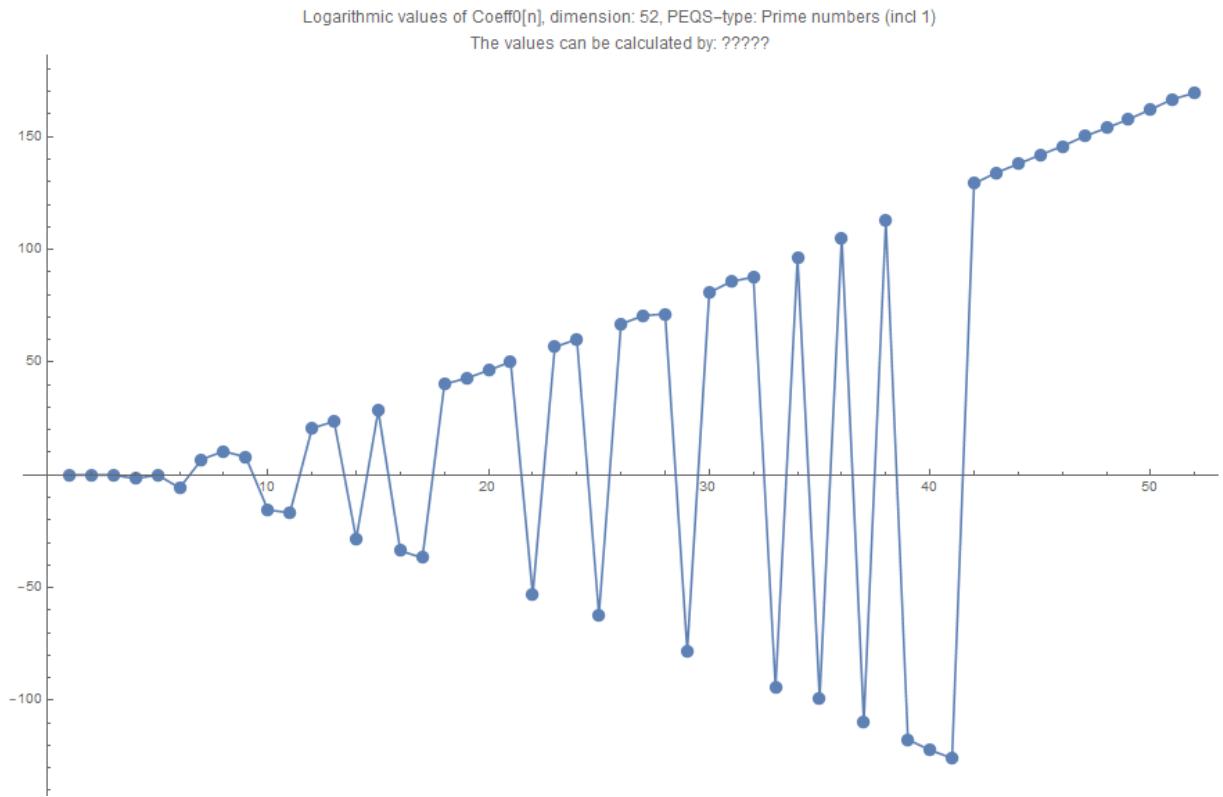
$-\frac{5548769}{362880}$	$\frac{547}{8064}$	$\frac{37049}{5040}$	$\frac{989}{720}$	$-\frac{311}{120}$	$\frac{1}{24}$	$-\frac{5}{6}$	$-\frac{1}{2}$	$-1$	$-1$
$-\frac{5548769}{362880}$	$\frac{547}{8064}$	$\frac{37049}{5040}$	$\frac{989}{720}$	$-\frac{311}{120}$	$\frac{1}{24}$	$-\frac{5}{6}$	$-\frac{1}{2}$	$-1$	$-1$
$-\frac{5548769}{362880}$	$\frac{547}{8064}$	$\frac{37049}{5040}$	$\frac{989}{720}$	$-\frac{311}{120}$	$\frac{1}{24}$	$-\frac{5}{6}$	$-\frac{1}{2}$	$-1$	$-1$
$-\frac{5548769}{362880}$	$\frac{547}{8064}$	$\frac{37049}{5040}$	$\frac{989}{720}$	$-\frac{311}{120}$	$\frac{1}{24}$	$-\frac{5}{6}$	$-\frac{1}{2}$	$-2$	$-2$
$-\frac{181440}{5548769}$	$\frac{4032}{547}$	$\frac{2520}{37049}$	$\frac{360}{989}$	$-\frac{60}{311}$	$\frac{12}{1}$	$-\frac{3}{5}$	$-1$	$-2$	$-2$
$-\frac{5548769}{60480}$	$\frac{547}{1344}$	$\frac{840}{37049}$	$\frac{120}{989}$	$-\frac{20}{311}$	$\frac{4}{1}$	$-5$	$-3$	$-6$	$-6$
$-\frac{5548769}{15120}$	$\frac{547}{336}$	$\frac{210}{37049}$	$\frac{30}{989}$	$-\frac{5}{311}$	$1$	$-20$	$-12$	$-24$	$-24$
$-\frac{5548769}{3024}$	$\frac{2735}{336}$	$\frac{42}{37049}$	$\frac{6}{989}$	$-311$	$5$	$-100$	$-60$	$-120$	$-120$
$-\frac{504}{5548769}$	$\frac{56}{2735}$	$\frac{7}{37049}$	$989$	$-1866$	$30$	$-600$	$-360$	$-720$	$-720$
$-\frac{5548769}{72}$	$\frac{2735}{8}$	$37049$	$6923$	$-13062$	$210$	$-4200$	$-2520$	$-5040$	$-5040$
$-\frac{5548769}{9}$	$2735$	$296392$	$55384$	$-104496$	$1680$	$-33600$	$-20160$	$-40320$	$-40320$
$-5548769$	$24615$	$2667528$	$498456$	$-940464$	$15120$	$-302400$	$-181440$	$-362880$	$-362880$

The rational values of the kernel can also be calculated from the quotients of the first coefficients  $c_0$  (of the n-th line) and the last coefficient  $c_n$  (of the (n+1)-th line (each marked in red and blue color). The kernel sequence can simple also be written as:  $\frac{\text{coeff}_{0,n}}{n!}$

The graphic representation of the first 52 kernel values look like this:

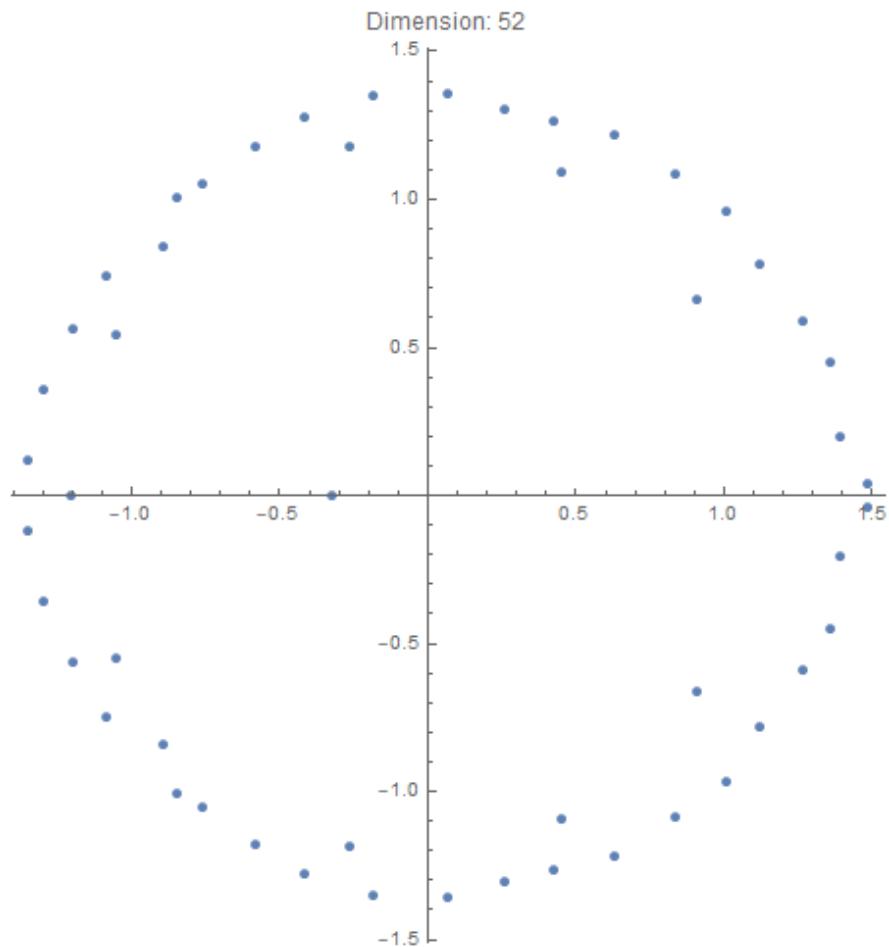


A graphical representation of the first 52 Coeff0 values:

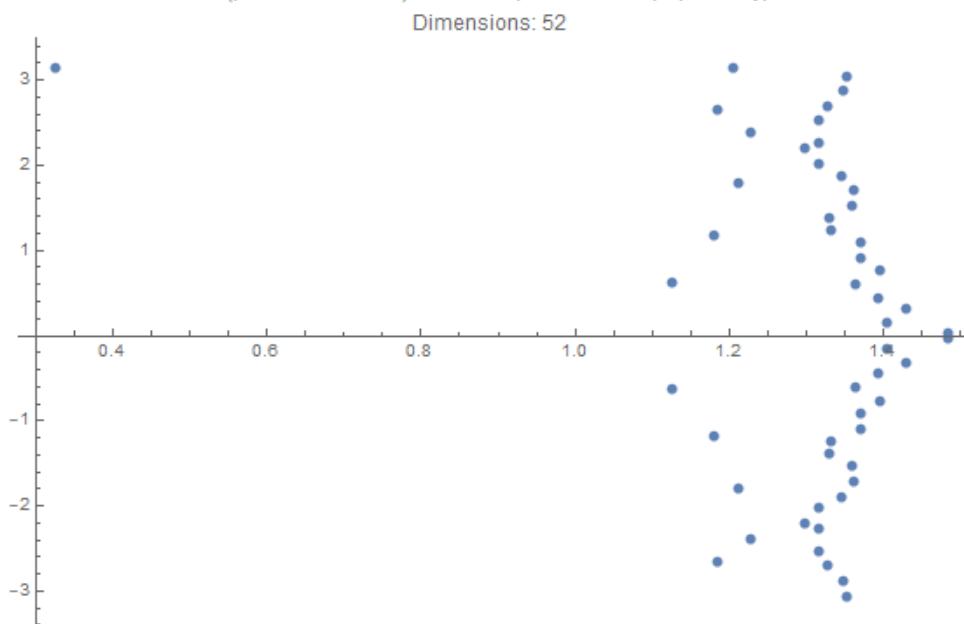


Here another example for  $n = 52$  with a graphical representation of the complex zeros based on the sequence of the Mersenne prime number exponents (including '1'), in cartesian and spherical coordinates:

location (cart. coordinates) of the complex roots for qSquence type MP1

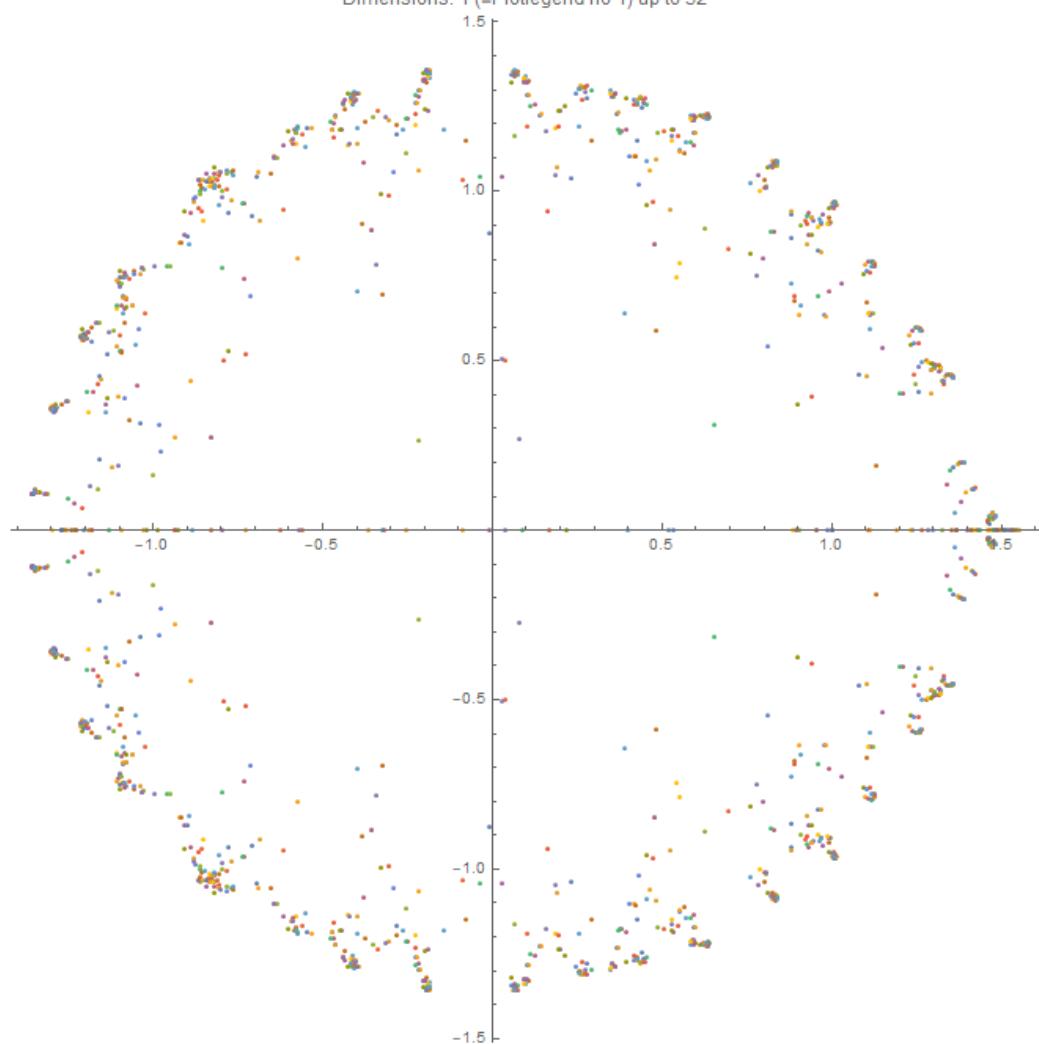


location (polar coordinates) of the complex roots for qSquence type MP1



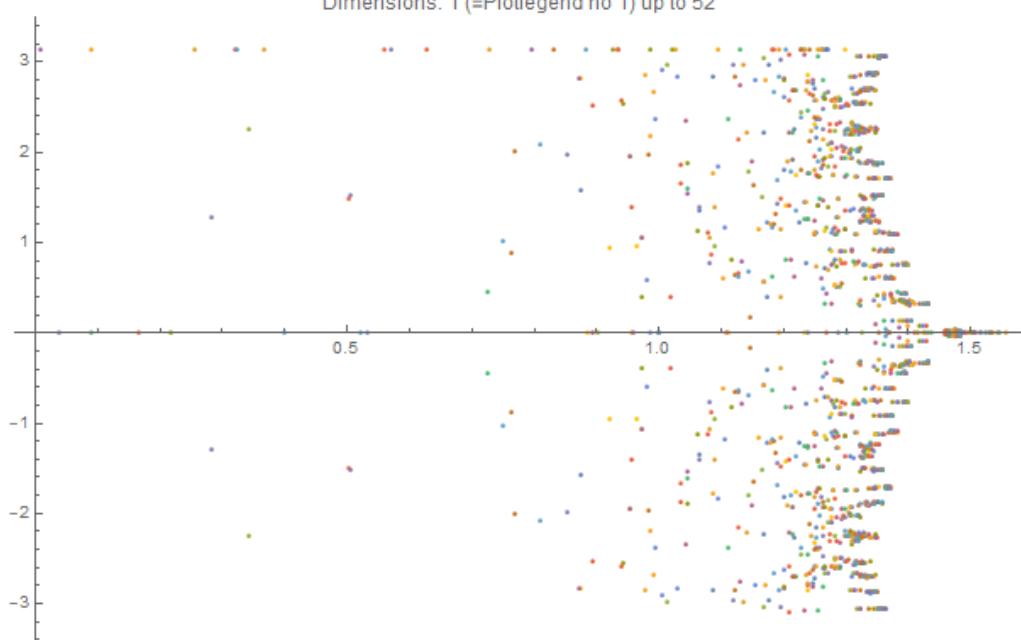
location (cart. coordinates) of the complex roots for qSequence type MP1

Dimensions: 1 (=Plotlegend no 1) up to 52



location (polar coordinates) of the complex roots for qSequence type MP1

Dimensions: 1 (=Plotlegend no 1) up to 52



And here is another look at the prime factor decompositions of the first 52 values of  $\text{Coeff}_{0,n}$ :

```

1          {{1,1}}
2          {{1,1}}
3          {{1,1}}
4          {{5,1}}
5          {{1,1}}
6          {{311,1}}
7          {{23,1},{43,1}}
8          {{37049,1}}
9          {{5,1},{547,1}}
10         {{5548769,1}}
11         {{11,1},{523,1},{3217,1}}
12         {{11,1},{89376809,1}}
13         {{11,1},{1861618579,1}}
14         {{5,2},{11,1},{66643,1},{108287,1}}
15         {{11,1},{5347,1},{45001573,1}}
16         {{11,1},{40039,1},{948934699,1}}
17         {{11,1},{67,1},{12060363522433,1}}
18         {{11,2},{6269,1},{423205940291,1}}
19         {{5,1},{11,1},{74496368234700233,1}}
20         {{11,1},{43,1},{157,1},{443353,1},{4321340407,1}}
21         {{11,1},{37363,1},{3679219,1},{4339045387,1}}
22         {{11,1},{97,1},{409,1},{457,1},{1174919,1},{431608979,1}}
23         {{11,1},{387162452943284411786279,1}}
24         {{5,1},{11,1},{602029,1},{1469856667,1},{2903134177,1}}
25         {{11,1},{31,1},{277025893213,1},{9325459350353,1}}
26         {{11,1},{41,1},{392259521,1},{696675806810700469,1}}
27         {{11,1},{71,1},{3665999,1},{917753,1},{17610852381065357,1}}
28         {{11,1},{200680884080753,1},{4889008376390953,1}}
29         {{5,1},{11,1},{89,1},{109,1},{130982653,1},{262839761,1},{389046428297,1}}
30         {{11,1},{23,1},{43,2},{73,1},{2113,1},{18260491,1},{107859559756304887,1}}
31         {{11,1},{5233,1},{15018917,1},{8563419253,1},{2451139061220503,1}}
32         {{11,1},{26648501026127,1},{461119407414864344463437,1}}
33         {{11,2},{16231,1},{5369537564926535646233398224727951,1}}
34         {{5,1},{11,2},{199,1},{433,1},{17581,1},{644702356685001291176388405761,1}}
35         {{11,2},{345679,1},{241194142549084191766614912829384391,1}}
36         {{11,2},{307,1},{1117,1},{2399933,1},{48533212210967532327610861507097,1}}
37         {{11,2},{83572067,1},{331405349937752477,1},{105112666469906764609,1}}
38         {{11,2},{89,1},{829,1},{934909,1},{963559,1},{467550868191383,1},{2382098523560753,1}}
39         {{5,2},{11,2},{11747086553,1},{2906957265248501,1},{9297370524320474620351,1}}
40         {{11,2},{838056689924826426132378086946264203294163903830879,1}}
41         {{11,2},{4481,1},{207227,1},{3632309,1},{857742281633,1},{10290855309903647200026959,1}}
42         {{11,2},{85339543,1},{13458836424230417313469439588806651411272508993,1}}
43         {{11,2},{89,1},{15427,1},{360289,1},{1515763756014093389,1},{181664976621562017499899233,1}}
44         {{5,1},{11,2},{9145825441,1},{200244966637,1},{734604976985185949004121092212457979,1}}
45         {{11,2},{71,1},{127,1},{43268260897205836543348060236496366873948076233700243233,1}}
46         {{11,2},{47,1},{5189,1},{73674210073850555446691421574871633912153208585155290003,1}}
47         {{11,2},{454969,1},{31359674865749,1},{32773811900279,1},{3100477740189438153232836937351,1}}
48         {{11,2},{16073,1},{7393777,1},{461995567073472739,1},{1020857869580026511759046442059013931,1}}
49         {{5,1},{11,2},{991,1},{3767,1},{532733,1},{284994249944021490150615533026913522129266415946753391,1}}
50         {{11,2},{43,1},{10574967671,1},{475433450508228428424421698941393092825827319072330028227,1}}
51         {{11,2},{59,1},{773,1},{148579,1},{13146241,1},{47775051807017,1},{5075219927620547247218093997004981331059,1}}
52         {{11,2},{107,1},{127,1},{354657409,1},{687691133,1},{3854014861,1},{24588314176737086300893947288566779589263,1}}
)

```

It is striking, that for the prime factor decompositions of  $\text{Coeff}_{0,n}$  (based on the Mersenne prime exponents), for indices  $\geq 11$ , the values of  $\text{Coeff}_{0,n}$  are each divisible by 11 and for indices  $\geq 33$  divisible by  $11^2$ . Neither the prime factor decompositions of  $\text{Coeff}_{0,n}$  for natural numbers nor the prime factor decompositions of  $\text{Coeff}_{0,n}$  for Prime numbers numbers show similar striking abnormalities.

Examples of integer sequences  $\text{intF}_n$  and their transformation pairs  $\text{Coeff}_{0k}$

Explicit formulas (if known) are printed in red color.

<b>no.</b>	<b><math>\text{Coeff}_{0k}</math></b>	<b><math>\text{intF}_n</math></b>
1	$\{-1, -1, -1, 1, 19, 151, 1091, 7841, 56519, 396271, 2442439, 7701409\}$ $-n! {}_1F_1(1-n; 2; 1)$	$\{1, 2, 3, 4, 5, 6, 7, \dots, n\}$ positive natural numbers $\mathbb{N}_+$ , $a_n = n$
2	$\{-1, 1, 3, 3, -21, -207, -1233, -4869, 5751, 436833, 6908571, 83211219\}$ $-n! {}_1F_1(1-n; 2; 3)$	$\{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, \dots\}$ $a_n = 3n - 2, n = 1, 2, 3, \dots$
3	$\{1, -1, -3, -7, 1, 219, 2581, 22973, 162177, 554039, -10506419, -343049631, -68464$ $(-1)^{n+1} nn! {}_2F_2(1-n, n+1; \frac{3}{2}, 2; \frac{1}{4})$	$\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \dots\}$ $a_n = n^2, n = 1, 2, 3, \dots$
4	$\{1, 2, 3, 4, 5, 6, 7, \dots, n\}$ positive natural numbers $\mathbb{N}_+$	$\{-2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, \dots\}$ $a_1 = -2, a_n = (-1)^n, n > 1$
5	$\{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31\}$ $3n - 2$	$\{-4, 9, -27, 81, -243, 729, -2187, 6561, -19683, 590$ $a_1 = -4, a_n = (-3)^n, n > 1$
6	$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots\}$ $Fibonacci(n)$	$\{-1, -1, \frac{1}{2}, \frac{1}{6}, -\frac{7}{24}, \frac{1}{24}, \frac{17}{144}, -\frac{67}{1008}, \dots\} =$ $\{-1, -1, 1, -7, 5, 85, -335, -1135, \dots\}/n!$ $a_n$ computable from the series expansion of $Ln\left(\sum_{n=1}^{\infty} Fibonacci(n) \frac{x^{n-1}}{(n-1)!}\right)$
7	$\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \dots\}$ $a_1 = 1, a_n = Prime(n-1), n > 1$	$\{-2, 1, -\frac{3}{2}, 2, -\frac{59}{24}, \frac{59}{20}, -\frac{169}{48}, \frac{2119}{504}, -\frac{67577}{13440}, \dots\} =$ $\{-2, 1, -3, 12, -59, 354, -2535, 21190, -202731, \dots\}/n!$ $a_n$ computable from the series expansion of $Ln\left(\sum_{n=1}^{\infty} Prime1(n) \frac{x^{n-1}}{(n-1)!}\right)$
8	$\{1, 0, -1, -2, -9, -52, -395, -3618, -39151, -486872, -6848361, \dots\}$	$a_n = Fibonacci(n); n = 0, 1, 2, 3, \dots$
9	$\{1, -1, 0, -2, -4, -36, -224, -2200, -22608, -281456, -3879424, \dots\}$	$a_n = Fibonacci(n); n = 1, 2, 3, 4, \dots$
10	$\{-1, -1, -1, -5, 1, -71, 2189, 17369, 272015, 2351071, 14621239, 322514659, \dots\}$	$\{1, 2, 3, 5, 7, 11, 13, \dots, n\}$ , Prime numbers (with ' $1$ '): $\mathbb{P}_1, a_1 = 1, a_n = Prime(n-1), n > 1$
11	$\{-1, -1, -1, -5, 1, -311, 989, 37049, 2735, -5548769, -18507401, 983144899, \dots\}$	$\{1, 2, 3, 5, 7, 11, 13, \dots, n\}$ , Mersenne prime exponents (with ' $1$ '): $a_1 = 1, a_n = MersennePrimeExponent(n-1), n > 1$

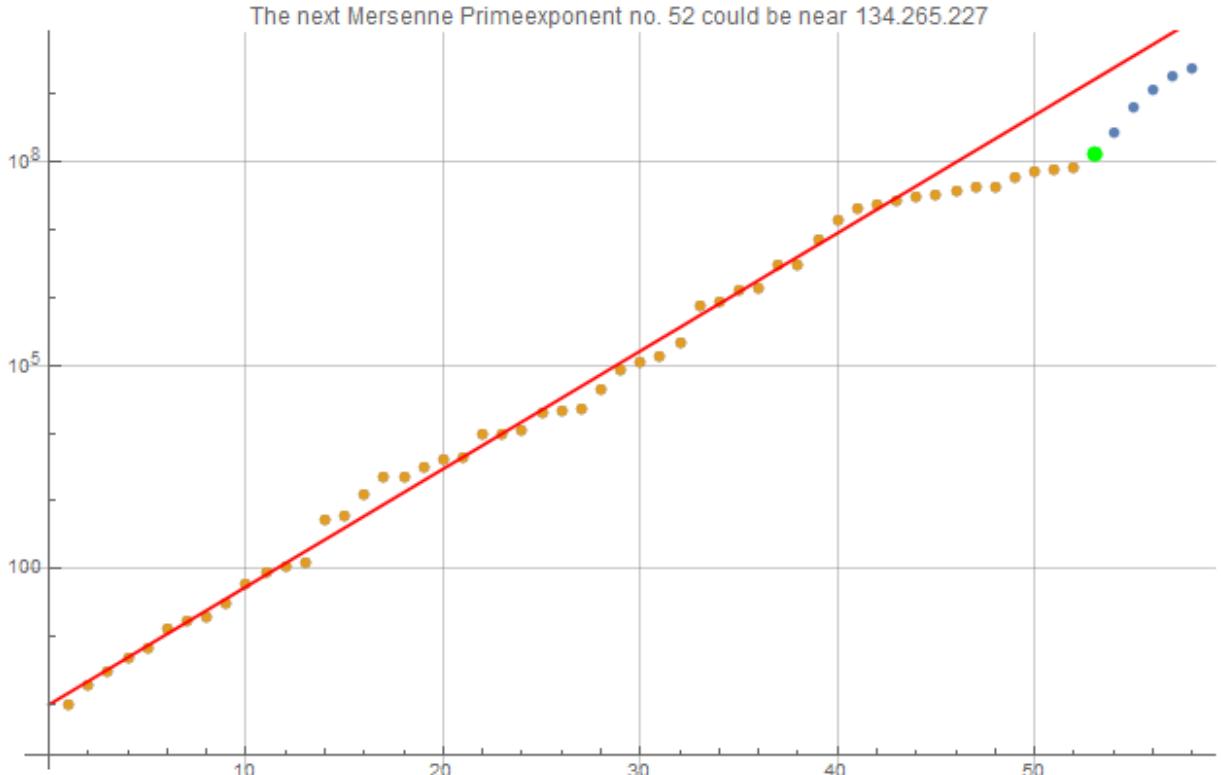
TABLE 1: EXAMPLES OF PEQS TRANSFORMATIONS PAIRS

#### 8.9.4 WHERE IS THE NEXT MERSENNE PRIME NUMBER?

---

There is a conjecture that the values  $MPExp[n]$  of the Mersenne prime exponents follow the function  $2^{\gamma n}$  on a statistical average and thus can be approximated by a straight line (in a logarithmic representation). The plot shows the straight in red color.

Here,  $\gamma$  is the Euler Mascheroni constant. If we use the method described above for determining the sequence function (which is identical with the first 52 values of the Mersenne prime number exponents, including the '1'), (marked with yellow color) and extrapolate the sequence function further on for a few indices (marked in blue color), then we get the following graph:



Of course, there is no (mathematical) reason at all to assume that the next Mersenne prime is really located in the near of  $2^{134265227} - 1$  (marked in green color). However, the extrapolated 'trend' of the sequence function was too beautiful and the author could not resist plotting the 'extrapolated' Mersenne prime exponents 52 up to 57!

If you look at the graph, you can't get rid of the feeling that the next Mersenne prime exponents are pushing towards higher values again, since the last 8 values are significantly below the predicted values.

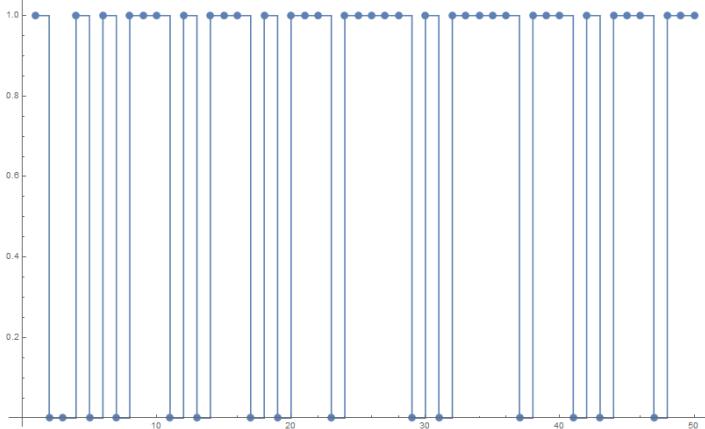
## 11.2\_ THE FUNCTIONS NOTPRIMEQ AND SIGNEDPRIMEQ

---

The function `notPrimeQ(n)` is the logical opposite of the function `PrimeQ(n)`. It returns 1 if  $n$  is not a prime number and 0 if  $n$  is a prime number.

Mathematica:

```
notPrimeQ[n_]:=1-Boole[PrimeQ[n]]
range=50;tabNotPrimeQ=Table[notPrimeQ[n],{n,1,range}]
Show[ListLinePlot[tabNotPrimeQ,InterpolationOrder\Rule]0],ListP
lot[tabNotPrimeQ]]
(* zeros at prime positions:
{1,0,0,1,0,1,0,1,1,1,0,1,0,1,1,1,0,1,0,1,1,1,0,1,1,1,1,1,0,1,0,1
,1,1,1,0,1,1,1,0,1,0,1,1,1,0,1,1,1}
*)
```

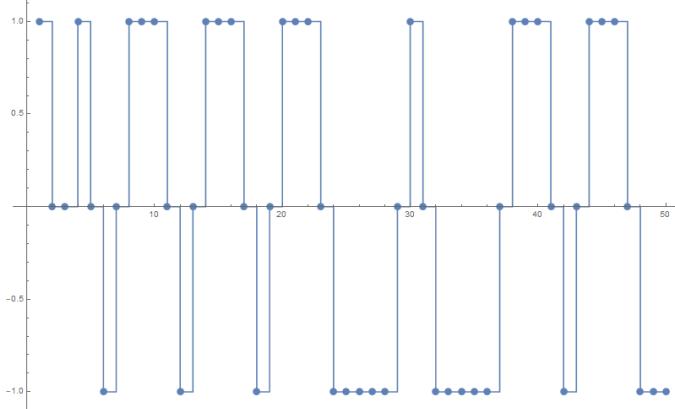


The function `signedPrimeQ(n)` returns 0 if  $n$  is a prime number and alternating values of 1 and -1 otherwise.

Mathematica:

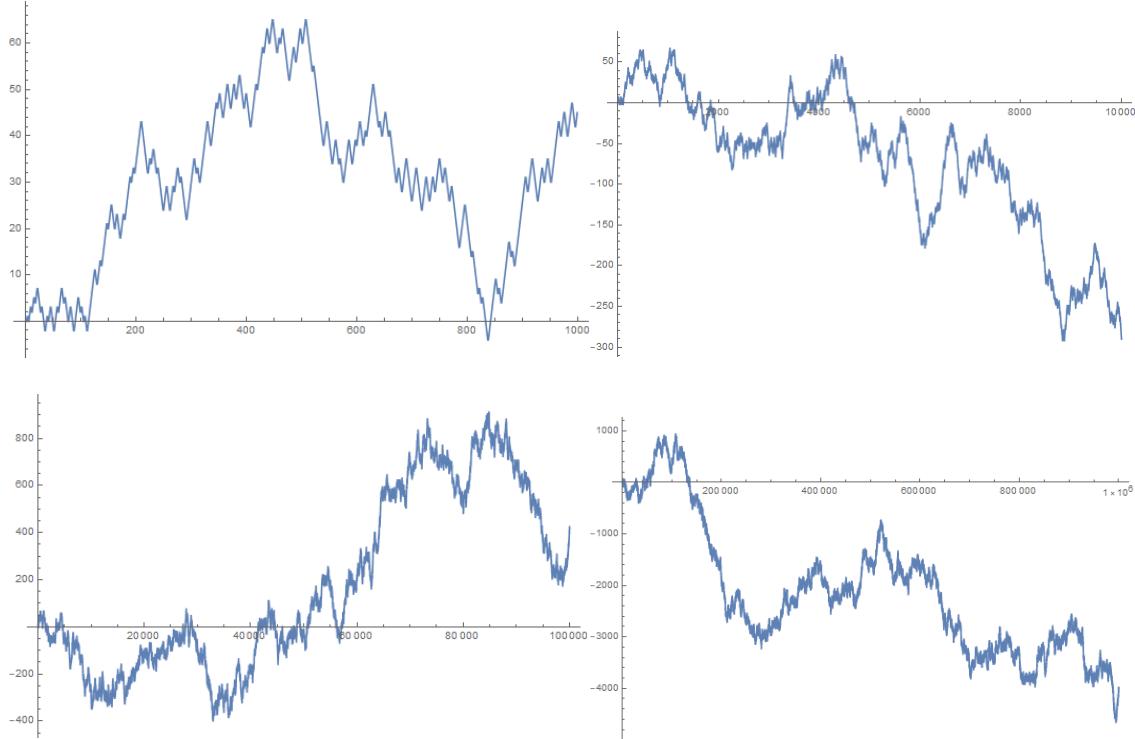
```
signedPrimeQ[n_]:=If[PrimeQ[n]==True,0,If[EvenQ[PrimePi[n]]==Tru
e,1,-1]]
(*range=50;tabSignedPrimeQ=Table[signedPrimeQ[n],{n,2,range}]*)
(*Show[ListLinePlot[tabSignedPrimeQ,InterpolationOrder\Rule]0],
ListPlot[tabSignedPrimeQ]*)
```

(\* Zeros at prime positions:\*)
{1,0,0,1,0,-1,0,1,1,1,0,-1,0,1,1,1,0,-1,0,1,1,1,0,-1,-1,-1,-1,-1,0,1,0,-1,-1,-1,-1,-1,0,1,1,1,0,-1,0,1,1,1,0,-1,-1,-1,-1}



Now, let us take a look at the summatory function of `signedPrimeQ[]`.

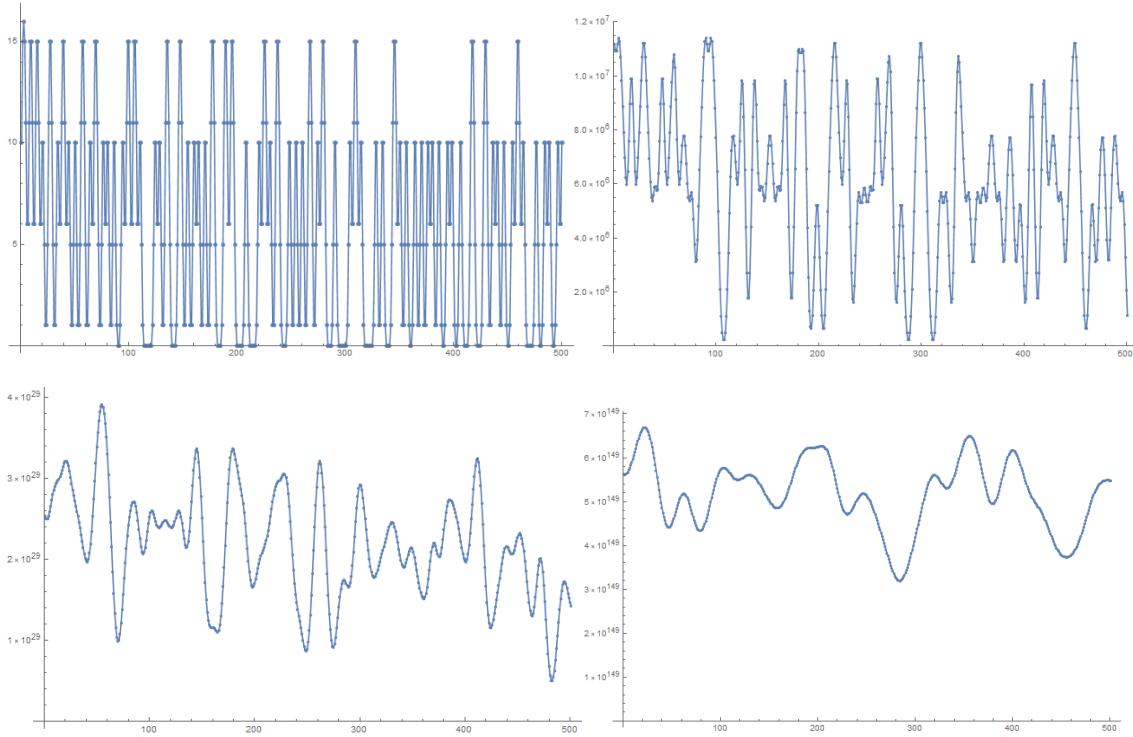
**The summatory function over `signedPrimeQ[]` is a self similar fractal on a logarithmic scale.** It has the same statistical behavior whatever number range we are considering: 1000, 10000, 100000 or 1 mio. It seems to have positive and negative values even in higher number ranges:



Mathematica:

```
range=1000;
tabSignedPrimeQ=Table[signedPrimeQ[n],{n,1,range}];
sumtabSignedPrimeQ=Accumulate[tabSignedPrimeQ];
ListLinePlot[sumtabSignedPrimeQ,ImageSize->Large]
```

The function `notPrimeQ[]` is not less interesting if we look at the differences instead of the summatory function. Building differences over `notPrimeQ[]`, however, seems to be not very interesting, if we look only at lower difference orders (up to 10<sup>th</sup> order). Higher orders of differences, however, show an unexpected, rather smooth appearance: The following plots apply to difference orders 5, 25, 100 and 500.



The author has no idea, what the reason for this effect is.

It needs to be clarified whether this behavior comes from rounding errors or insufficient precision...

Mathematica:

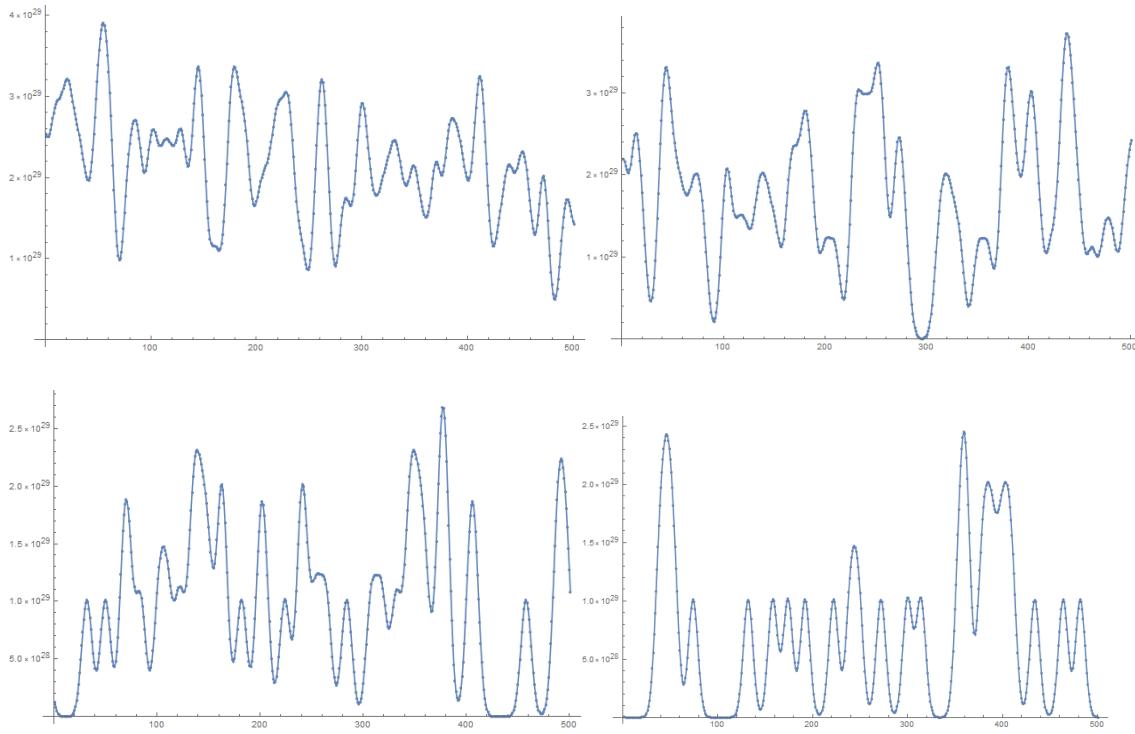
```
range=1; order=5;
tabNotPrimeQ=Table[notPrimeQ[n],{n,range,range+500+order}];
diftabNotPrimeQ=Abs[Differences[tabNotPrimeQ,order]];
Show[ListLinePlot[diftabNotPrimeQ,ImageSize->Large],ListPlot[diftabNotPrimeQ,ImageSize->Large]]
```

Going to a higher range does not change the spectral properties, except that the whole graph tends to have relatively more smaller values.

The following plots apply to order = 100 and range = 1, 1000, 1 mio., 1000 mio.

In the case of 1000 mio, the minimum values are around  $10^{21}$ , the maximum values at  $2.5 \cdot 10^{29}$ . This is a range of  $10^8$ .

This is surprising...



## 18.4\_FIELDS AND EXTENDED ARITHMETIC OPERATORS (CO-AUTHOR: ADIL BULUS)

---

### 18.4.1\_NESTING OF EXPONENTIAL- AND LOGARITHMIC FUNCTIONS

By arithmetic operators we mean the operators "+" (addition), "\*" (multiplication), " $\wedge$ " (exponentiation) and their possible generalizations or extensions.

The following method allows a very general definition of arithmetic operators. The principle is nesting and combination of exponential and logarithmic functions using a 'nesting operator', which can also be "+", "\*" or " $\wedge$ ". Formulated with the Mathematica programming language, an entire matrix of arithmetic operators can be generated by a single line of program code. The resulting operators may only become 'visible' after they have been simplified with the arithmetic laws that apply to exponential and logarithmic functions. Examples for such nested constructions are:

Without simplification:

$$\begin{pmatrix} a + b & e^{a+\log(b)} & e^{e^{a+\log(\log(b))}} & e^{e^{e^{a+\log(\log(\log(b)))}}} \\ e^{\log(a)+b} & e^{\log(a)+\log(b)} & e^{e^{\log(a)+\log(\log(b))}} & e^{e^{e^{\log(a)+\log(\log(\log(b)))}}} \\ e^{e^{\log(\log(a))+b}} & e^{e^{\log(\log(a))+\log(b)}} & e^{e^{e^{\log(\log(a))+\log(\log(b))}}} & e^{e^{e^{e^{\log(\log(a))+\log(\log(\log(b)))}}}} \\ e^{e^{e^{\log(\log(\log(a)))+b}}} & e^{e^{e^{\log(\log(\log(a))+\log(b))}}} & e^{e^{e^{e^{\log(\log(\log(a))+\log(\log(b))}}}} & e^{e^{e^{e^{e^{\log(\log(\log(a))+\log(\log(\log(b)))}}}}} \end{pmatrix}$$

After simplification:

$$\begin{pmatrix} a+b & e^a b & b^{e^a} & e^{\log^{e^a}(b)} \\ ae^b & ab & b^a & e^{\log^a(b)} \\ a^{e^b} & a^b & a^{\log(b)} & e^{a^{\log(\log(b))}} \\ e^{\log^{e^b}(a)} & e^{\log^b(a)} & e^{b^{\log(\log(a))}} & e^{\log^{\log(\log(b))}(a)} \end{pmatrix}$$

Without simplification:

$$\begin{pmatrix} ab & \log(ae^b) & \log(\log(ae^{e^b})) & \log(\log(\log(ae^{e^{e^b}}))) \\ \log(e^a b) & \log(e^{a+b}) & \log(\log(e^{a+e^b})) & \log(\log(\log(e^{a+e^{e^b}}))) \\ \log(\log(e^{e^a} b)) & \log(\log(e^{e^a+b})) & \log(\log(e^{e^a+e^b})) & \log(\log(\log(e^{e^a+e^{e^b}}))) \\ \log(\log(\log(e^{e^{e^a}} b))) & \log(\log(\log(e^{e^{e^a}+b}))) & \log(\log(\log(e^{e^{e^a}+e^b}))) & \log(\log(\log(e^{e^{e^a}+e^{e^b}}))) \end{pmatrix}$$

After simplification:

$$\begin{pmatrix} ab & \log(a) + b & \log(\log(a) + e^b) & \log(\log(\log(a) + e^{e^b})) \\ a + \log(b) & a + b & \log(a + e^b) & \log(\log(a + e^{e^b})) \\ \log(e^a + \log(b)) & \log(e^a + b) & \log(e^a + e^b) & \log(\log(e^a + e^{e^b})) \\ \log(\log(e^{e^a} + \log(b))) & \log(\log(e^{e^a} + b)) & \log(\log(e^{e^a} + e^b)) & \log(\log(e^{e^a} + e^{e^b})) \end{pmatrix}$$

Without simplification:

$$\begin{pmatrix} \log((e^a)^b) & \log((e^a)^{e^b}) & \log(\log((e^a)^{e^{e^b}})) & \log(\log(\log((e^a)^{e^{e^{e^b}}}))) \\ \log(\log((e^{e^a})^b)) & \log(\log((e^{e^a})^{e^b})) & \log(\log((e^{e^a})^{e^{e^b}})) & \log(\log(\log((e^{e^a})^{e^{e^{e^b}}}))) \\ \log(\log(\log((e^{e^{e^a}})^b))) & \log(\log(\log((e^{e^{e^a}})^{e^b}))) & \log(\log(\log((e^{e^{e^a}})^{e^{e^b}}))) & \log(\log(\log((e^{e^{e^a}})^{e^{e^{e^b}}}))) \\ \log(\log(\log(\log((e^{e^{e^{e^a}}})^b)))) & \log(\log(\log(\log((e^{e^{e^{e^a}}})^{e^b})))) & \log(\log(\log(\log((e^{e^{e^{e^a}}})^{e^{e^b}})))) & \log(\log(\log(\log((e^{e^{e^{e^a}}})^{e^{e^{e^b}}}))))) \end{pmatrix}$$

After simplification:

$$\begin{pmatrix} ab & ae^b & \log(a) + e^b & \log(\log(a) + e^{e^b}) \\ a + \log(b) & a + b & a + e^b & \log(a + e^{e^b}) \\ \log(e^a + \log(b)) & \log(e^a + b) & \log(e^a + e^b) & \log(e^a + e^{e^b}) \\ \log(\log(e^{e^a} + \log(b))) & \log(\log(e^{e^a} + b)) & \log(\log(e^{e^a} + e^b)) & \log(\log(e^{e^a} + e^{e^b})) \end{pmatrix}$$

Here are a few examples created with Mathematica with  $\text{dim} = 2$  (the value of  $\text{dim}$  can be increased as desired).

Examples of the type 'Exp-Log-Log' (with the corresponding Mathematica code):

**OpMatrix1 (commutative operators on the diagonal, nesting operator: "+"):**

```
Table[Nest[Exp,Nest[Log,a,i]+Nest[Log,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} \color{red}{a+b} & e^a b & b^{e^a} \\ ae^b & \color{red}{ab} & \color{blue}{b^a} \\ a^{e^b} & \color{blue}{a^b} & \color{red}{a^{\log(b)}} \end{pmatrix}$$

**OpMatrix2 (commutative operators on the diagonal, nesting operator: "\*"):**

```
Table[Nest[Exp,Nest[Log,a,i]*Nest[Log,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} ab & b^a & e^{\log^a(b)} \\ a^b & a^{\log(b)} & e^{a^{\log(\log(b))}} \\ e^{\log^b(a)} & e^{b^{\log(\log(a))}} & e^{\log^{\log(\log(b))}(a)} \end{pmatrix}$$

### OpMatrix3 (commutative operators on the diagonal, nesting operator: "^"):

```
Table[Nest[Exp,Nest[Log,a,i]^Nest[Log,b,j+1],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} a^{\log(b)} & e^{a^{\log(\log(b))}} & e^{e^{a^{\log(\log(\log(b)))}}} \\ e^{\log^{\log(b)}(a)} & e^{\log^{\log(\log(b))}(a)} & e^{e^{\log^{\log(\log(\log(b)))}(a)}} \\ e^{e^{\log^{\log(b)}(\log(a))}} & e^{e^{\log^{\log(\log(b))}(\log(a))}} & e^{e^{\log^{\log(\log(\log(b)))}(\log(a))}} \end{pmatrix}$$

### OpMatrix4 (no commutative operators on the diagonal, nesting operator: "^"):

```
Table[Nest[Exp,Nest[Log,a,i]^Nest[Log,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} a^b & e^{a^{\log(b)}} & e^{e^{a^{\log(\log(b))}}} \\ e^{\log^b(a)} & e^{\log^{\log(b)}(a)} & e^{e^{\log^{\log(\log(b))}(a)}} \\ e^{e^{\log^b(\log(a))}} & e^{e^{\log^{\log(b)}(\log(a))}} & e^{e^{\log^{\log(\log(b))}(\log(a))}} \end{pmatrix}$$

### Examples of type ,Log-Exp-Exp‘:

### OpMatrix5 (commutative operators on the diagonal, nesting operator: "+"):

```
Table[Nest[Log,Nest[Exp,a,i]+Nest[Exp,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} a + b & \log(a + e^b) & \log(\log(a + e^{e^b})) \\ \log(e^a + b) & \log(e^a + e^b) & \log(\log(e^a + e^{e^b})) \\ \log(\log(e^{e^a} + b)) & \log(\log(e^{e^a} + e^b)) & \log(\log(e^{e^a} + e^{e^b})) \end{pmatrix}$$

### OpMatrix6 (commutative operators on the diagonal, nesting operator: "\*"):

```
Table[Nest[Log,Nest[Exp,a,i]*Nest[Exp,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} ab & \log(a) + b & \log(\log(a) + e^b) \\ a + \log(b) & a + b & \log(a + e^b) \\ \log(e^a + \log(b)) & \log(e^a + b) & \log(e^a + e^b) \end{pmatrix}$$

### OpMatrix7 (commutative operators on the diagonal, nesting operator: "^"):

```
Table[Nest[Log,Nest[Exp,a,i+1]^Nest[Exp,b,j],Max[i+1,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} ab & ae^b & \log(a) + e^b \\ a + \log(b) & a + b & a + e^b \\ \log(e^a + \log(b)) & \log(e^a + b) & \log(e^a + e^b) \end{pmatrix}$$

## OpMatrix8 (no commutative operators on the diagonal, nesting operator: " $\wedge$ "):

```
Table[Nest[Exp,Nest[Log,a,i]^Nest[Log,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm
```

$$\begin{pmatrix} a^b & e^b \log(a) & \log(\log(a)) + e^b \\ ab & ae^b & \log(a) + e^b \\ a + \log(b) & a + b & a + e^b \end{pmatrix}$$

Some of the operators generated in this way are well known: "+", "\*" and " $\wedge$ " (exponentiation).

Less well known are the Jacobi addition ( $\log(e^a + e^b)$ ) and the logarithmic exponentiation ( $a^{\log(b)}$ ), both of which have the properties of commutativity and associativity. Neutral elements and inverse elements also exist.

**The following properties can be read directly or easily checked:**

- 1) For all operator matrices (with the exception of examples 4 and 8), commutative operators lie on the diagonal (marked in red color)
- 2) The operators outside the diagonal are not commutative.
- 3) The operators on the diagonal with the type 'Exp-Log-Log' represent the direction of 'higher arithmetic operators', those with the type 'Log-Exp-Exp' represent the direction of 'lower calculation types'
- 4) 'neighboring' operators on the diagonals form a field with regard to the set of complex numbers (exception: examples 4 and 8)  
Two operators each, who are located at the same diagonal position of two operator matrices, also form a body with regard to the set of complex numbers for the following operator-matrix pairs:  
(OpMatrix1, OpMatrix2), corresponding to the nesting operators "+" and "\*"  
(OpMatrix2, OpMatrix3), corresponding to the nesting operators "\*" and " $\wedge$ "  
(OpMatrix5, OpMatrix6), corresponding to the nesting operators "+" and "\*"

Examples of fields with the set of complex numbers are the operator pairs {  $a + b$  and  $a * b$  }, { $\log(e^a + e^b)$  and  $(a + b)$ } as well as { $(a * b)$  and  $a^{\log(b)}$  }.

We define the 'Jacobi addition operator'  $\oplus$  using Mathematica in a slightly modified form as follows (we replace  $e^x$  with  $2^x$  and the natural logarithm with binary logarithm):

```
ld[x_]:=Log[2,x];exp[x_]:=2^x;
CirclePlus[a_,b_]:=Evaluate[Nest[ld,Nest[exp,a,1]+Nest[exp,b,1],1]]//PowerExpand
//Simplify
```

This operator (explicit form:  $ld(2^a + 2^b)$ ) has, in addition to the field properties already mentioned above (such as associativity and commutativity) the following characteristics:

- 1)  $a \oplus a = a + 1$  ( $a$  'Jacobi added' with itself produces the successor of  $a$ )
- 2) Applying the operator recursively  $n$  times to  $a$  results in  $a + n$ :

$a1 = a \oplus a$  gives  $a + 1$   
 $a1 = a1 \oplus a1$  gives  $a + 2$   
 $a1 = a1 \oplus a1$  gives  $a + 3$   
 ...etc.

Consequently, the 'Jacobi addition operator' is actually the 'predecessor operator' of the normal addition in the 'hierarchy' of the arithmetic operators!

Note: In the mathematical literature, this hierarchy of arithmetic operators is described by so-called hyper operators. This method uses a completely different approach. One result of this approach are the hyper operators tetration and pentation, which cannot be constructed using the approach described here.

You can go even further and use the Jacobi addition  $\log(e^a + e^b)$  or the logarithmic exponentiation  $a^{\log(b)}$  as the nesting operator. However, this does not bring any new knowledge.

#### 18.4.2\_EXTENSION OF THE OPERATORS: INFINITE NUMBER OF ARITHMETIC OPERATIONS

---

Are there other types of arithmetic operations besides addition, multiplication and exponentiation? This question was answered at least in part in the last chapter.

An answer going further on is provided by the following ideas from the physicist **Adil Bulus** and describe some field constructions that enable a 'seamless' transition between different types of arithmetic operations.

**The original works by Adil Bulus can be found on the website**

<https://www.mathematische-notizen.de>

The methods presented here enable, among other things, a 'parameterization' between:

- Jacobi addition and addition
- addition and multiplication
- multiplication and logarithmic exponentiation

The introduction of a parameter  $n$  results in an infinite number of fields which, so to speak, merge seamlessly into one another.

**Note: Many of the following statements only apply if  $n$  is a positive odd integer, even if many statements continue to apply to rational or even real values of  $n$ . We therefore do not make a case distinction and, for the sake of simplicity, assume below that  $n$  is a positive odd integer. The values of  $a$  and  $b$  can take any complex values.**

The following limits are well known:

$$\lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n = e^x \text{ und } \lim_{n \rightarrow \infty} n(x^{1/n} - 1) = \log(x)$$

By defining  $\text{expn}(x, n) = (1 + \frac{x}{n})^n$  and  $\text{logn}(x, n) = n(x^{1/n} - 1)$  we have 'parameterized' exponential and logarithmic functions that go asymptotically against the original functions  $e^x$  and  $\log(x)$  if  $n \rightarrow \infty$ .

If we replace within the operator matrices 'OpMatrix1' and 'OpMatrix2'  $e^x$  by  $\text{expn}$  and  $\log$  by  $\text{logn}$ , we get the following operator matrices (using  $\text{dim} = 2$ ):

### OpMatrix1n (commutative operators on the diagonal, nesting operator: "+"):

`Table[Nest[expn,Nest[logn,a,i]+Nest[logn,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm`

$$\begin{pmatrix} a+b & \left(\frac{a}{n} + b^{\frac{1}{n}}\right)^n & \left(\frac{\left(\frac{a}{n} + n^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}}\right)^n + n}{n}\right)^n \\ \left(\frac{a^{\frac{1}{n}} + \frac{b}{n}}{n}\right)^n & \left(a^{\frac{1}{n}} + b^{\frac{1}{n}} - 1\right)^n & \left(\frac{\left(a^{\frac{1}{n}} + n^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1\right)^n + n}{n}\right)^n \\ \left(\frac{(n^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} + \frac{b}{n})^n + n}{n}\right)^n & \left(\frac{(n^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} + b^{\frac{1}{n}} - 1)^n + n}{n}\right)^n & \left(\frac{(n^{\frac{1}{n}}((a^{\frac{1}{n}} - 1)^{\frac{1}{n}} + (b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1)^n + n}{n}\right)^n \end{pmatrix}$$

The following applies:  $\lim_{n \rightarrow \infty} (a^{\frac{1}{n}} + b^{\frac{1}{n}} - 1)^n = ab$  (see OpMatrix1)

### OpMatrix2n (commutative operators on the diagonal, nesting operator: "\*"):

`Table[Nest[expn,Nest[logn,a,i]*Nest[logn,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm`

$$\begin{pmatrix} ab & (a(b^{\frac{1}{n}} - 1) + 1)^n & \left(\frac{(an^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - a + 1)^n + n}{n}\right)^n \\ (b(a^{\frac{1}{n}} - 1) + 1)^n & \left(n(a^{\frac{1}{n}} - 1)(b^{\frac{1}{n}} - 1) + 1\right)^n & \left(\frac{(n(a^{\frac{1}{n}} - 1)(n^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1) + 1)^n + n}{n}\right)^n \\ \left(\frac{(bn^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} - b + 1)^n + n}{n}\right)^n & \left(\frac{(n(n^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1)(b^{\frac{1}{n}} - 1) + 1)^n + n}{n}\right)^n & \left(\frac{(n(n^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1)(n^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1) + 1)^n + n}{n}\right)^n \end{pmatrix}$$

The following applies:  $\lim_{n \rightarrow \infty} (n(a^{\frac{1}{n}} - 1)(b^{\frac{1}{n}} - 1) + 1)^n = a^{\log(b)}$  (see OpMatrix2)

### OpMatrix5n (commutative operators on the diagonal, nesting operator: "+"):

`Table[Nest[logn,Nest[expn,a,i]+Nest[expn,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm`

$$\begin{pmatrix} a+b & n((a + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1) & \left(n((n((a + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1))^{\frac{1}{n}} - 1)\right)^n \\ n(((\frac{a+n}{n})^n + b)^{\frac{1}{n}} - 1) & n(((\frac{a+n}{n})^n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1) & n((n(((\frac{a+n}{n})^n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1))^{\frac{1}{n}} - 1) \\ n(((n(((\frac{a+n}{n})^n + n + b)^{\frac{1}{n}} - 1))^{\frac{1}{n}} - 1) & n((n(((\frac{a+n}{n})^n + n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1))^{\frac{1}{n}} - 1) & n((n(((\frac{a+n}{n})^n + n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1))^{\frac{1}{n}} - 1) \end{pmatrix}$$

The following applies:  $\lim_{n \rightarrow \infty} n(((\frac{a+n}{n})^n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1) = \text{Log}[e^a + e^b]$  (see OpMatrix5)

### OpMatrix6n (commutative operators on the diagonal, nesting operator: "\*"):

Table[Nest[logn,Nest[expn,a,i]\*Nest[expn,b,j],Max[i,j]],{i,0,dim},{j,0,dim}]//MatrixForm

$$\begin{pmatrix} ab & \frac{1}{a^n}(b+n) - n & n\left(\frac{\frac{1}{a^n}((\frac{b+n}{n})^n + n)}{n} - 1\right)^{\frac{1}{n}} - 1 \\ (a+n)b^{\frac{1}{n}} - n & \frac{ab}{n} + a + b & n\left(\frac{(a+n)((\frac{b+n}{n})^n + n)}{n^2} - 1\right)^{\frac{1}{n}} - 1 \\ n\left(\frac{((\frac{a+n}{n})^n + n)b^{\frac{1}{n}}}{n} - 1\right)^{\frac{1}{n}} - 1 & n\left(\frac{((\frac{a+n}{n})^n + n)(b+n)}{n^2} - 1\right)^{\frac{1}{n}} - 1 & n\left(\frac{((\frac{a+n}{n})^n + n)((\frac{b+n}{n})^n + n)}{n^2} - 1\right)^{\frac{1}{n}} - 1 \end{pmatrix}$$

The following applies:  $\lim_{n \rightarrow \infty} \frac{ab}{n} + a + b = a + b$  (see OpMatrix6)

Similar to the original operator matrices, the question of body properties naturally arises here too. The body characteristics are not so easy to examine here, but with a little hard work and perseverance the task can be solved.

From the properties of the original operator matrices, the following properties have been retained or no longer apply:

- 1) The following also applies here: For all operator matrices (with the exception of examples 4 and 8), commutative operators lie on the diagonal (marked in red color)
- 2) The following also applies here: The operators outside the diagonals are not commutative.
- 3) Also applies (is clear due to the definitions of logn and expn)
- 4) **This property no longer applies here: 'neighboring' operators on the diagonals do not form a field together with the set of complex numbers**
- 5) This property is retained: Two operators located at the same diagonal position of two operator matrices also form a field with regard to the set of complex numbers for the following operator-matrix pairs:  
(OpMatrix1n, OpMatrix2n), according to the nesting operators "+" and "\*"

Specifically, the following operator pairs located in the same diagonal positions form a field:

(OpMatrix1n(1,1) and OpMatrix2n(1,1) :  
 $a + b$  und  $a b$  (trivial, since  $n$  does not occur)

(OpMatrix1n(2,2) and OpMatrix2n(2,2):  
 $(a^{\frac{1}{n}} + b^{\frac{1}{n}} - 1)^n$  and  $(n(a^{\frac{1}{n}} - 1)(b^{\frac{1}{n}} - 1) + 1)^n$  (easy to check)

(OpMatrix1n(3,3) and OpMatrix2n(3,3):  
 $(\frac{(n^{\frac{1}{n}}((a^{\frac{1}{n}} - 1)^{\frac{1}{n}} + (b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1)^n + n}{n})^n$  and  $(\frac{(n(n^{\frac{1}{n}}(a^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1)(n^{\frac{1}{n}}(b^{\frac{1}{n}} - 1)^{\frac{1}{n}} - 1) + 1)^n + n}{n})^n$   
(difficult to verify, so far only a conjecture)

(OpMatrix5n(1,1) and OpMatrix6n(1,1)):  
 $a + b$  and  $a b$  (trivial, since  $n$  does not occur)

(OpMatrix5n(2,2) and OpMatrix6n(2,2):

$$n\left(\left(\frac{a+n}{n}\right)^n + \left(\frac{b+n}{n}\right)^n\right)^{\frac{1}{n}} - 1 \text{ and } \frac{ab}{n} + a + b \quad (\text{easy to check})$$

(OpMatrix5n(3,3) and OpMatrix6n(3,3):

$$n\left(n\left(\left(\frac{\left(\frac{a+n}{n}\right)^n+n}{n}\right)^n + \left(\frac{\left(\frac{b+n}{n}\right)^n+n}{n}\right)^n\right)^{\frac{1}{n}} - 1\right)^{\frac{1}{n}} - 1 \text{ and } n\left(n^{\frac{1}{n}}\left(\frac{\left(\frac{a+n}{n}\right)^n+n}{n^2}\left(\frac{\left(\frac{b+n}{n}\right)^n+n}{n^2}\right)\right)^{\frac{1}{n}} - 1\right)^{\frac{1}{n}} - 1$$

(difficult to verify, so far only a conjecture)

Let us now take a closer look at the interesting operator pairs (OpMatrix1n (2,2) and OpMatrix2n (2,2) as well as (OpMatrix5n (2,2) and (OpMatrix6n (2,2)) (using the operator symbols " $\oplus$ " and " $\otimes$ "). We find:

**(OpMatrix1n(2,2) and OpMatrix2n(2,2):**

,addition operator‘:	$a \oplus b := (-1 + a^{\frac{1}{n}} + b^{\frac{1}{n}})^n$
,multiplication operator‘:	$a \otimes b := (1 + (-1 + a^{\frac{1}{n}})(-1 + b^{\frac{1}{n}})n)^n$
associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	1
inverse element of addition:	$(2 - a^{\frac{1}{n}})^n$
neutral element of multiplication:	$(1 + \frac{1}{n})^n$
inverse element of multiplication:	$(1 + \frac{1}{(-1 + a^{\frac{1}{n}})n^2})^n$
law of distributivity:	is met
conditions for field properties:	is met

**(OpMatrix1(2,2) and OpMatrix2(2,2) (limit for  $n \rightarrow \infty$ ):**

,addition operator‘:	$a \oplus b := a \ b$
,multiplication operator‘:	$a \otimes b := a^{\log(b)}$
associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	1
inverse element of addition:	$\frac{1}{a}$
neutral element of multiplication:	$e$
inverse element of multiplication:	$e^{\frac{1}{\log(a)}}$
law of distributivity:	is met
conditions for field properties:	is met

**(OpMatrix1n(2,2) and OpMatrix2n(2,2) (limit for  $n \rightarrow 1$ ):**

,addition operator‘:	$a \oplus b := a + b - 1$
,multiplication operator‘:	$a \otimes b := (a - 1)(b - 1) + 1$

associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	1
inverse element of addition:	$2 - a$
neutral element of multiplication:	2
inverse element of multiplication:	$1 + \frac{1}{a-1}$
law of distributivity:	is met
conditions for field properties:	is met

**(OpMatrix5n(2,2) and OpMatrix6n(2,2):**

,addition operator‘:	$a \oplus b := n((\frac{a+n}{n})^n + (\frac{b+n}{n})^n)^{\frac{1}{n}} - 1$
,multiplication operator‘:	$a \otimes b := \frac{ab}{n} + a + b$
associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	-n
inverse element of addition:	$n(-1 + (-(\frac{a+n}{n})^n)^{\frac{1}{n}}) = -2n - a, n \text{ odd}$
neutral element of multiplication:	0
inverse element of multiplication:	$-\frac{an}{a+n}$
law of distributivity:	is met
conditions for field properties:	is met

**(OpMatrix5(2,2) and OpMatrix6(2,2) (limit for  $n \rightarrow \infty$ ):**

,addition operator‘:	$a \oplus b := \log(e^a + e^b)$
,multiplication operator‘:	$a \otimes b := a + b$
associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	$-\infty$
inverse element of addition:	$a + i\pi$ (in detail: $a + i(2k + 1)\pi, k \in \mathbb{N}$ )
neutral element of multiplication:	0
inverse element of multiplication:	-a
law of distributivity:	is met
conditions for field properties:	is met

**(OpMatrix5n(2,2) and OpMatrix6n(2,2) (limit for  $n \rightarrow 1$ ):**

,addition operator‘:	$a \oplus b := a + b + 1$
,multiplication operator‘:	$a \otimes b := a + b + a b$
associativity:	is met for both operators
commutativity:	is met for both operators
neutral element of addition :	-1
inverse element of addition:	$-2 - a$

neutral element of multiplication:	0
inverse element of multiplication:	$\frac{-a}{a+1}$
law of distributivity:	is met
conditions for field properties:	is met

Let us now consider the example **OpMatrix1n (2,2)** and the limit cases for  $n \rightarrow \infty$  and  $n = 1$ :

$n = 1$ :  $a \oplus b$  becomes  $a + b - 1$  (identical to the normal addition, except for “-1”)

$n \rightarrow \infty$ :  $a \oplus b$  becomes  $a b$  (identical to normal multiplication)

By varying the parameter  $n$  we get an infinite number of operators that represent a continuous transition from normal addition to normal multiplication.

With the example **OpMatrix5n (2,2)** and the limit cases for  $n \rightarrow \infty$  and  $n = 1$  we get:

$n = 1$ :  $a \oplus b$  becomes  $a + b + 1$  (except for 1 identical to normal addition)

$n \rightarrow \infty$ :  $a \oplus b$  becomes  $\log(e^a + e^b)$  (identical to the Jacobi addition operator)

By varying the parameter  $n$  we get an infinite number of operators that represent a continuous transition from normal addition to Jacobi addition.

#### 18.4.3 RANGE OF VALUES OF THE EXTENDED OPERATORS

---

One may ask the question, what range of values (in particular integer values) such operators have.

We assume that the original definition range for  $a$  and  $b$  is  $\mathbb{N}$ . Furthermore, we want to proceed iteratively by calculating any number of values of  $a \oplus b$  or  $a \otimes b$  and use iteratively the resulting range of values for the new definition range of the next iteration step. The interesting question now is which (positive) integer values arise in this way. The following tables could provide information:

<b>n</b>	<b>positive integer range of values for <math>a \oplus b := (-1 + a^{\frac{1}{n}} + b^{\frac{1}{n}})^n</math></b>
1	{1,2,3,4,5,6,7,8,9,10,...}: all positive integers
2	{4,8,9,12,16,18,20,24,25,27,...}: numbers, which are not square-free
3	{8,16,24,27,32,40,48,54,56,...}: numbers, which are not cube-free
4	(none)
5	(none)
6	(none)

**n positive integer range of values for  $a \otimes b := (1 + (-1 + a^{\frac{1}{n}})(-1 + b^{\frac{1}{n}})n)^n$**

- 1 {1,2,3,4,5,6,7,8,9,10,...}: all positive integers
- 2 {9,25,49,81,121,169,289,...}:  $(2n + 1)^2$  n=1,2,3,4,5,...
- 3 (???)
- 4 (???)

Note: The author gives no guarantee that this data is correct, since it was determined purely empirically.

## 20.8.2\_VALUE-PRESERVING TRANSITIONS OF POWERS OF 2 IN OCRON SPACE

We now want to consider so-called value-preserving transitions of OCRONs (or GOCRONs). Of particular interest are these transitions (one might call them transformations) for powers of two, since they offer a way of bringing the addition into play according to the formula

$$(2^a)^b = 2^{a \cdot b} \text{ as well as } (2^{2^a})^{2^b} = 2^{2^{a+b}}$$

First, let's look at an even simpler variant: the increment by the value 1.

Let  $\text{ld}(x) = \frac{\ln(x)}{\ln(2)}$  be the binary logarithm (Logarithmus Dualis).

Then obviously holds:  $x + 1 = \text{ld}(2^{x+1}) = \text{ld}(2^x \cdot 2)$ , or in terms of OCRON writing:

**O+1=ld(2<ocron>^2\*)** (whereby <ocron> stands for the string of any valid OCRON. We can formally execute the binary logarithm of an OCRON by simply omitting the prefixed '2' and the appended '^'. This requires that we consider an OCRON that has the form '2 <ocron> ^'.

Therefore, we are now faced with the problem of transforming any OCRONs (of which we know that they can be represented as a power of two, since they have the value of a power of two) in OCRONs of the form 2 <ocron> ^, which have the privilege of being quite easy to be 'de-logarithmized'. This transformation, which is generally accomplished in several, as small as possible steps, we want to call 'value preserving transitions' or the 'trace' of an OCRON.

Here is a table of the first 50 powers of two in OCRON representation, in  $(2<\text{ocron}>^2*)$  representation as well as in  $(2<\text{ocron}>^*)$  representation:

n	(G)OCRON( $2^{n-1} * 2$ )	(G)OCRON( $2^n$ )
1	2: 2	2: 2
2	22*: 40	22^*: 43
3	22^2*: 696	22P^*: 167
4	22P^2*: 2680	222^^: 687
5	222^^2*: 11000	22PP^*: 663
6	22PP^2*: 10616	222P^^: 2707
7	222P^^2*: 43320	222^P^*: 2743
8	222^P^2*: 43896	222P^^: 2719
9	222P^^2*: 43512	22P2^^: 2671
10	22P2^^2*: 42744	222PP^*: 10835
11	222PP^^2*: 173368	22PPP^*: 2647
12	22PPP^2*: 42360	222^2P^*: 43923
13	222^2P^2*: 702776	222P*P^*: 10823
14	222P*P^2*: 173176	2222^P^*: 43731

15	2222^P*^2*:	699704	22P2PP*^:	42579
16	22P2PP*^2*:	681272	2222^^^:	10943
17	2222^^^2*:	175096	222^PP^:	10967
18	222^PP^2*:	175480	222P2^*^:	43443
19	222P2^*^2*:	695096	222P^P^:	10871
20	222P^P^2*:	173944	222^2PP*^:	175699
21	222^2PP*^2*:	2811192	22P22^P*^:	170707
22	22P22^P*^2*:	2731320	222PPP*^:	43347
23	222PPP*^2*:	693560	22P2^P^:	10679
24	22P2^P^2*:	170872	222P^2P*^:	173971
25	222P^2P*^2*:	2783544	22PP2^*^:	10607
26	22PP2^*^2*:	169720	2222P*P*^:	174355
27	2222P*P*^2*:	2789688	22P2P^*^:	10655
28	22P2P^*^2*:	170488	222^22^P*^:	703187
29	222^22^P*^2*:	11251000	222PP*P^:	43335
30	222PP*P^2*:	693368	222P*2PP*^:	692819
31	222P*2PP*^2*:	11085112	22PPPP^:	10583
32	22PPPP^2*:	169336	222PP^*^:	10847
33	222PP^*^2*:	173560	22P2PPP*^:	170323
34	22P2PPP*^2*:	2725176	2222^PP*^:	174931
35	2222^PP*^2*:	2798904	22PP22^P*^:	678611
36	22PP22^P*^2*:	10857784	222^2P2^*^*^:	702899
37	222^2P2^*^*^2*:	11246392	222^2P*P^:	175687
38	222^2P*P^2*:	2811000	2222P^*^2*^:	174547
39	2222P^*^2*:	2792760	22P22P*P*^:	682259
40	22P22P*P*^2*:	10916152	222P^2PP*^:	695891
41	222P^2PP*^2*:	11134264	222P*PP^:	43287
42	222P*PP^2*:	692600	222P*22^P*^:	2771667
43	222P*22^P*^2*:	44346680	2222^P*P^:	174919
44	2222^P*P^2*:	2798712	222^2PPP*^:	702803
45	222^2PPP*^2*:	11244856	22P2^2PP*^:	683603
46	22P2^2PP*^2*:	10937656	222P2^P*^:	173779
47	222P2^P*^2*:	2780472	22P2PP*P^:	170311
48	22P2PP*P^2*:	2724984	2222^2P*^:	700307
49	2222^2P*^2*:	11204920	222^P2^*^:	43887
50	222^P2^*^2*:	702200	222PP2^*^:	173491

For the values shown in red, the  $(2<\text{ocron}>^)$  representation has a greater value than the  $(2<\text{ocron}>^2*)$  representation.

Now consider, e.g. the line 20, which describes the transition from 19 to the successor 20. Obviously are  $222P^P^2*$  (173944) and  $222^2PP*^$  (175699) equivalent (G)OCRONs. How do we find a transformation or transformation rule between these two OCRONs?

Here is an example with EOCRON data (type 'Enhanced OCRONs') describing the transition from 41 to 42: The 'trace' consists of 467 values and describes the transition between the equivalent EOCRONs  $22P*PP^2$  und  $22P*22^P*^$ .

```
Transition from 41 to 42, CodeValues: {0,1,2,3}
Output of the program,:
Computed with EOCRONs.
eocronValue: 22P*PP^2: 42078
eocronvalue1: 22P*22^P*^: 674515
myNumber=42
Number of values: 468
```

The 467 differences are:

```
spur48EOCRON={441, 768, 2940, 84, 348, 4296, 4488, 84, 60, 24, 72, 192, 1056, 24, 24, 84, 11584, 4608, 5180, 24, 24, 84, 3553, 24, 2, 6, 3, 325, 2, 8, 337, 37, 2, 8, 1142, 2, 8, 145, 1536, 4453, 2, 8, 145, 4540, 48, 5232, 748, 192, 948, 48, 4716, 24, 72, 1728, 768, 480, 24, 72, 9597, 2, 8, 32, 128, 2273, 37, 2, 8, 337, 1829, 2, 8, 809, 24, 408, 192, 144, 24, 780, 48, 2745, 2, 8, 3473, 4488, 84, 60, 24, 72, 192, 1056, 24, 24, 84, 10304, 748, 192, 948, 48, 377, 2, 8, 145, 4269, 1739, 768, 3016, 3464, 84, 348, 2869, 1764, 3072, 4, 151, 84, 348, 4296, 4488, 84, 60, 24, 72, 192, 1056, 24, 24, 84, 13337, 2, 8, 32, 128, 22, 73, 37, 2, 8, 337, 1829, 2, 8, 809, 24, 408, 192, 144, 24, 780, 48, 2745, 2, 8, 19857, 336, 5, 2, 8, 145, 1352, 192, 948, 48, 732, 84, 2988, 84, 5280, 48, 8980, 13877, 2, 8, 32, 128, 493, 19, 4989, 1536, 11417, 2, 8, 32, 5334, 2, 8, 32, 4669, 22076, 24, 24, 84, 2368, 59, 85, 24, 12, 1308, 24, 2, 6, 3, 1501, 24, 2, 6, 3, 4573, 24, 2, 6, 3, 325, 2, 8, 337, 37, 2, 8, 1142, 2, 8, 758, 2, 8, 3830, 2, 8, 182, 2, 8, 145, 37, 2, 8, 13421, 4257, 24, 2, 6, 3, 325, 2, 8, 337, 37, 2, 8, 1142, 2, 8, 145, 1536, 4453, 2, 8, 145, 1637, 2, 8, 1270, 2, 8, 145, 552, 8, 24, 24, 84, 8096, 48, 1556, 293, 2, 8, 465, 768, 7516, 4608, 5180, 24, 24, 84, 39204, 9920, 4028, 48, 8980, 9789, 24, 2, 6, 3, 85, 2, 8, 24, 5, 1, 3, 3, 338, 2, 8, 32, 96, 1, 19, 4, 1, 4, 1, 3, 3, 1, 11, 1, 5189, 2, 8, 32, 128, 3870, 24, 2, 6, 3, 1440, 37, 2, 8, 337, 192, 53, 2, 8, 32, 128, 6430, 24, 2, 6, 3, 7544, 24, 24, 84, 4068, 8957, 24, 2, 6, 3, 11541, 2, 8, 3, 2, 128, 2273, 37, 2, 8, 337, 1829, 2, 8, 809, 24, 408, 192, 144, 24, 780, 48, 2745, 2, 8, 1, 9857, 3365, 2, 8, 145, 1352, 192, 948, 48, 732, 84, 2988, 84, 5280, 48, 8980, 9820, 460, 8, 5180, 24, 24, 84, 3553, 24, 2, 6, 3, 325, 2, 8, 337, 37, 2, 8, 1142, 2, 8, 145, 1536, 445, 3, 2, 8, 145, 4540, 48, 5232, 748, 192, 948, 48, 4716, 24, 72, 1728, 768, 480, 24, 72, 95, 97, 2, 8, 32, 128, 2273, 37, 2, 8, 337, 1829, 2, 8, 809, 24, 408, 192, 144, 24, 780, 48, 27, 45, 2, 8, 3473, 4488, 84, 60, 24, 72, 192, 1056, 24, 24, 84, 10304, 748, 192};
```

**Table 1: „Trace“ of all equivalent EOCRONs for  $2^{42}$**

EGOCRON: {EOCRON} number: $2^{42}$	EGOCRON: {EOCRON} number: $2^{42}$
42078: {2, 2, P, *, P, P, ^, 2}	126599: {P, ^, 2, ^, 2, 2, *, P, ^}
42519: {2, 2, P, 2, *, P, P, ^}	126647: {P, ^, 2, ^, 2, 2, ^, P, ^}
43287: {2, 2, 2, P, *, P, P, ^}	129392: {P, ^, ^, 2, P, P, ^, *, *}
46227: {2, ^, P, *, 2, P, *, ^}	129394: {P, ^, ^, 2, P, P, ^, *, 2}
46311: {2, ^, P, *, ^, 2, P, ^}	129402: {P, ^, ^, 2, P, P, ^, 2, 2}
46659: {2, ^, P, 2, P, *, *, ^}	132875: {2, *, *, P, P, *, *, *, 2, ^}
50955: {^, *, P, ^, *, *, 2, ^}	137363: {2, *, P, 2, *, 2, P, *, ^}
55443: {^, P, 2, *, 2, P, *, ^}	137447: {2, *, P, 2, *, ^, 2, P, ^}
55527: {^, P, 2, *, ^, 2, P, ^}	137507: {2, *, P, 2, P, *, 2, *, ^}
55587: {^, P, 2, P, *, 2, *, ^}	137531: {2, *, P, 2, P, *, ^, 2, ^}
55611: {^, P, 2, P, *, ^, 2, ^}	137603: {2, *, P, 2, P, 2, *, *, ^}
55683: {^, P, 2, P, 2, *, *, ^}	137795: {2, *, P, 2, 2, P, *, *, ^}
55875: {^, P, 2, 2, P, *, *, ^}	138851: {2, *, P, ^, 2, P, 2, *, ^}
56931: {^, P, ^, 2, P, 2, *, ^}	138875: {2, *, P, ^, 2, P, ^, 2, ^}
56955: {^, P, ^, 2, P, ^, 2, ^}	138899: {2, *, P, ^, 2, 2, P, *, ^}
56979: {^, P, ^, 2, 2, P, *, ^}	138983: {2, *, P, ^, 2, ^, 2, P, ^}
57063: {^, P, ^, 2, ^, 2, P, ^}	149287: {2, P, *, P, ^, *, 2, P, ^}
68647: {P, *, *, ^, *, *, 2, P, ^}	150035: {2, P, *, 2, 2, *, P, *, ^}
73255: {P, *, P, ^, 2, *, 2, P, ^}	150227: {2, P, *, 2, 2, ^, P, *, ^}
78435: {P, *, ^, *, 2, P, 2, *, ^}	151175: {2, P, *, ^, 2, 2, *, P, ^}
78459: {P, *, ^, *, 2, P, ^, 2, ^}	151223: {2, P, *, ^, 2, 2, ^, P, ^}
78483: {P, *, ^, *, 2, 2, P, *, ^}	151600: {2, P, P, *, *, *, ^, *, *}
78567: {P, *, ^, *, 2, ^, 2, P, ^}	151602: {2, P, P, *, *, *, ^, *, 2}
82120: {P, P, *, *, *, ^, *, 2, *}	151610: {2, P, P, *, *, *, ^, 2, 2}
82144: {P, P, *, *, *, ^, 2, *, *}	151755: {2, P, P, *, *, ^, *, 2, ^}
82146: {P, P, *, *, *, ^, 2, *, 2}	156024: {2, P, 2, *, P, P, ^, 2, *}
82152: {P, P, *, *, *, ^, 2, 2, *}	157763: {2, P, 2, 2, *, P, *, *, ^}
82155: {P, P, *, *, *, ^, 2, 2, ^}	158531: {2, P, 2, 2, ^, P, *, *, ^}
82480: {P, P, *, *, 2, *, ^, *, *}	161547: {2, P, ^, P, ^, *, *, 2, ^}
82482: {P, P, *, *, 2, *, ^, *, 2}	165011: {2, 2, *, P, *, 2, P, *, ^}
82490: {P, P, *, *, 2, ^, 2, 2}	165095: {2, 2, *, P, *, ^, 2, P, ^}
82827: {P, P, *, *, ^, 2, *, 2, ^}	165443: {2, 2, *, P, 2, P, *, *, ^}
82864: {P, P, *, *, ^, 2, ^, *, *}	168312: {2, 2, P, *, P, *, 2, *}
82866: {P, P, *, *, ^, 2, ^, *, 2}	170076: {2, 2, P, 2, *, P, P, ^, *}
82874: {P, P, *, *, ^, 2, ^, 2, 2}	173148: {2, 2, 2, P, *, P, P, ^, *}
84016: {P, P, *, 2, *, *, ^, *, *}	177299: {2, 2, ^, P, *, 2, P, *, ^}
84018: {P, P, *, 2, *, *, ^, *, 2}	177383: {2, 2, ^, P, *, ^, 2, P, ^}
84026: {P, P, *, 2, *, *, ^, 2, 2}	177731: {2, 2, ^, P, 2, P, *, *, ^}
84171: {P, P, *, 2, *, *, ^, 2, ^}	182027: {2, ^, *, P, *, *, *, 2, ^}
85707: {P, P, *, ^, 2, *, *, 2, ^}	186515: {2, ^, P, 2, *, 2, P, *, ^}
90160: {P, P, 2, *, *, *, ^, *, *}	186599: {2, ^, P, 2, *, ^, 2, P, ^}
90162: {P, P, 2, *, *, *, ^, *, 2}	186659: {2, ^, P, 2, P, *, 2, *, ^}
90170: {P, P, 2, *, *, *, ^, 2, 2}	186683: {2, ^, P, 2, P, *, ^, 2, ^}
90315: {P, P, 2, *, *, *, ^, *, 2, ^}	186755: {2, ^, P, 2, P, 2, *, *, ^}
94855: {P, P, ^, *, 2, 2, *, P, ^}	186947: {2, ^, P, 2, 2, P, *, *, ^}
94903: {P, P, ^, *, 2, 2, ^, P, ^}	188003: {2, ^, P, ^, 2, P, 2, *, ^}
100135: {P, 2, *, P, ^, *, 2, P, ^}	188027: {2, ^, P, ^, 2, P, ^, 2, ^}
100883: {P, 2, *, 2, 2, *, P, *, ^}	188051: {2, ^, P, ^, 2, 2, P, *, ^}
101075: {P, 2, *, 2, 2, *, P, *, ^}	188135: {2, ^, P, ^, 2, ^, 2, P, ^}
102023: {P, 2, *, ^, 2, 2, *, P, ^}	201472: {^, *, P, *, ^, *, *, *, *, *}
102071: {P, 2, *, ^, 2, 2, ^, P, ^}	201474: {^, *, P, *, ^, *, *, *, 2, }
106787: {P, 2, 2, *, P, *, 2, *, ^}	201482: {^, *, P, *, ^, *, *, 2, 2}
106811: {P, 2, 2, *, P, *, ^, 2, ^}	201514: {^, *, P, *, ^, *, *, 2, 2, 2}
106883: {P, 2, 2, *, P, 2, *, *, ^}	201642: {^, *, P, *, ^, 2, 2, 2, 2}
108611: {P, 2, 2, 2, *, P, *, *, ^}	203915: {^, *, P, ^, *, 2, *, 2, ^}
109379: {P, 2, 2, 2, ^, P, *, *, ^}	203952: {^, *, P, ^, *, 2, ^, *, *}
109859: {P, 2, 2, ^, P, *, 2, *, ^}	203954: {^, *, P, ^, *, 2, ^, *, 2}
109883: {P, 2, 2, ^, P, *, ^, 2, ^}	203962: {^, *, P, ^, *, 2, ^, 2, 2}
109955: {P, 2, 2, ^, P, 2, *, *, ^}	204299: {^, *, P, ^, 2, *, *, 2, ^}
119552: {P, ^, P, *, ^, *, *, *, *}	206128: {^, *, 2, P, P, *, ^, *, *}
119554: {P, ^, P, *, ^, *, *, *, 2}	206130: {^, *, 2, P, P, *, ^, *, 2}
119562: {P, ^, P, *, ^, *, *, 2, 2}	206138: {^, *, 2, P, P, *, ^, 2, 2}
119594: {P, ^, P, *, ^, *, 2, 2, 2}	206947: {^, *, 2, 2, *, P, 2, *, ^}
119722: {P, ^, P, *, ^, 2, 2, 2, 2}	206971: {^, *, 2, 2, *, P, ^, 2, ^}
121995: {P, ^, P, ^, *, 2, *, 2, ^}	207379: {^, *, 2, 2, 2, 2, *, P, *, ^}
122032: {P, ^, P, ^, *, 2, ^, *, *}	207571: {^, *, 2, 2, 2, 2, ^, P, *, ^}
122034: {P, ^, P, ^, *, 2, ^, *, 2}	207715: {^, *, 2, 2, 2, ^, P, 2, *, ^}
122042: {P, ^, P, ^, *, 2, ^, 2, 2}	207739: {^, *, 2, 2, 2, ^, P, ^, 2, ^}
122379: {P, ^, P, ^, 2, *, *, 2, ^}	208519: {^, *, 2, 2, ^, 2, 2, *, P, ^}
124208: {P, ^, 2, P, P, *, *, *, *}	208567: {^, *, 2, ^, 2, 2, ^, P, ^}
124210: {P, ^, 2, P, P, *, ^, *, 2}	211312: {^, *, ^, 2, P, P, ^, *, *}
124218: {P, ^, 2, P, P, *, ^, 2, 2}	211314: {^, *, ^, 2, P, P, ^, *, 2}
125027: {P, ^, 2, 2, *, P, 2, *, ^}	211322: {^, *, ^, 2, P, P, ^, 2, 2}
125051: {P, ^, 2, 2, *, P, ^, 2, ^}	231179: {^, 2, *, P, ^, *, *, *, 2, ^}
125459: {P, ^, 2, 2, 2, *, P, *, ^}	234544: {^, 2, P, P, *, *, *, *, 2}
125651: {P, ^, 2, 2, 2, ^, P, *, ^}	234546: {^, 2, P, P, *, *, ^, *, 2}
125795: {P, ^, 2, 2, ^, P, 2, *, ^}	234554: {^, 2, P, P, *, *, ^, 2, 2}
125819: {P, ^, 2, 2, ^, P, ^, 2, ^}	234699: {^, 2, P, P, *, *, *, 2, ^}

EGOCRON: {EOCRON} number: 2^42	EGOCRON: {EOCRON} number: 2^42
236051: {^,2,P,2,2,*^,P,*^,^}	360674: {P,P,2,*^,*^,^,2,*^,2}
236243: {^,2,P,2,2,^,P,*^,^}	360680: {P,P,2,*^,*^,^,2,2,*}
237191: {^,2,P,^,2,2,*^,P,^}	360683: {P,P,2,*^,*^,^,2,2,^}
237239: {^,2,P,^,2,2,^,P,^}	361008: {P,P,2,*^,2,*^,^,*^,^}
237971: {^,2,2,*^,P,2,P,*^,^}	361010: {P,P,2,*^,*^,2,*^,^,*^,2}
238055: {^,2,2,*^,P,^,2,P,^}	361018: {P,P,2,*^,*^,2,*^,^,2,2}
241043: {^,2,2,^,P,2,P,*^,^}	361355: {P,P,2,*^,*^,^,2,*^,2,^}
241127: {^,2,2,^,P,^,2,P,^}	361392: {P,P,2,*^,*^,^,2,^,*^,^}
246407: {^,^,*^,*^,2,2,*^,P,^}	361394: {P,P,2,*^,*^,^,2,^,*^,2}
246455: {^,^,*^,*^,2,2,^,P,^}	361402: {P,P,2,*^,*^,^,2,^,2,2}
255435: {^,^,2,P,^,*^,2,^}	362544: {P,P,2,*^,2,*^,*^,^,*^,^}
269312: {P,*^,*^,P,^,*^,*^,*^,*^}	362546: {P,P,2,*^,2,*^,*^,^,*^,2}
269314: {P,*^,*^,P,^,*^,*^,*^,*^}	362554: {P,P,2,*^,2,*^,*^,^,2,2}
269322: {P,*^,*^,P,^,*^,*^,*^,2,2}	362699: {P,P,2,*^,2,*^,*^,*^,2,^}
269354: {P,*^,*^,P,^,*^,*^,2,2,2}	364235: {P,P,2,*^,^,2,^,*^,*^,2}
269482: {P,*^,*^,P,^,*^,2,2,2,2}	368688: {P,P,2,2,*^,*^,*^,^,*^,^}
269975: {P,*^,*^,P,^,2,2,P,^}	368690: {P,P,2,2,*^,*^,*^,^,*^,2}
269994: {P,*^,*^,P,^,2,2,2,2,2}	368698: {P,P,2,2,*^,*^,*^,^,2,2}
274983: {P,*^,*^,^,*^,2,*^,2,P,^}	368843: {P,P,2,2,*^,*^,*^,^,*^,2,^}
276519: {P,*^,*^,^,2,*^,*^,2,P,^}	370480: {P,P,2,2,P,^,*^,*^,*^,^}
287936: {P,*^,P,2,P,*^,*^,*^,*^}	370482: {P,P,2,2,P,^,*^,*^,*^,2}
287938: {P,*^,P,2,P,*^,*^,*^,*^,2}	370490: {P,P,2,2,P,^,*^,*^,*^,2,2}
287946: {P,*^,P,2,P,*^,*^,2,2}	371760: {P,P,2,2,^,*^,*^,*^,*^}
287978: {P,*^,P,2,P,*^,*^,2,2,2}	371762: {P,P,2,2,^,*^,*^,*^,*^,2}
293312: {P,*^,P,^,2,P,^,*^,*^}	371770: {P,P,2,2,^,*^,*^,*^,2,2}
293314: {P,*^,P,^,2,P,^,*^,*^,2}	371915: {P,P,2,2,^,*^,*^,*^,2,^}
293322: {P,*^,P,^,2,P,^,*^,2,2}	377443: {P,P,^,*^,*^,2,P,2,*^,^}
293354: {P,*^,P,^,2,P,^,*^,2,2}	377467: {P,P,^,*^,*^,2,P,^,*^,2}
298023: {P,*^,2,*^,*^,*^,2,P,^}	377491: {P,P,^,*^,*^,2,2,P,*,^}
320099: {P,*^,*^,2,*^,2,P,2,*^,*^}	377575: {P,P,^,*^,*^,2,^,*^,2,P,^}
320123: {P,*^,*^,2,*^,2,P,^,*^,2}	385671: {P,P,^,*^,2,2,*^,P,^}
320147: {P,*^,*^,2,*^,2,2,P,*^,*^}	385719: {P,P,^,*^,2,2,^,*^,2,P,^}
320231: {P,*^,*^,2,*^,2,^,*^,2,P,^}	387275: {P,P,^,*^,2,2,*^,*^,*^,2}
322599: {P,*^,*^,2,^,*^,*^,2,P,^}	387568: {P,P,^,*^,2,2,P,^,*^,*^,^}
328584: {P,P,*^,*^,*^,*^,2,*^,*^}	387570: {P,P,^,*^,2,2,P,^,*^,*^,2}
328608: {P,P,*^,*^,*^,*^,2,2,*^,*^}	387578: {P,P,^,*^,2,2,P,^,*^,2,2}
328620: {P,P,*^,*^,*^,*^,2,2,^,*^}	388043: {P,P,^,*^,2,2,^,*^,*^,2,^}
329928: {P,P,*^,*^,2,*^,*^,*^,2,*^}	388811: {P,P,^,*^,2,2,^,*^,*^,2}
329952: {P,P,*^,*^,2,*^,*^,*^,2,*^}	396327: {P,P,^,*^,*^,*^,*^,2,P,^}
329954: {P,P,*^,*^,2,*^,*^,*^,2,*^}	400935: {P,P,^,*^,*^,*^,2,*^,2,P,^}
329960: {P,P,*^,*^,2,*^,*^,2,2,*^}	406115: {P,P,^,*^,*^,2,P,2,*^,*^}
329963: {P,P,*^,*^,2,*^,*^,2,2,^}	406139: {P,P,^,*^,*^,2,P,^,*^,2}
331464: {P,P,*^,*^,*^,2,^,*^,2,*^}	406163: {P,P,^,*^,*^,2,2,P,*,^}
331488: {P,P,*^,*^,*^,2,^,*^,2,*^}	406247: {P,P,^,*^,*^,*^,2,2,P,^}
331490: {P,P,*^,*^,*^,2,^,*^,2,*^}	445451: {P,P,^,*^,*^,*^,*^,*^,2,*^}
331496: {P,P,*^,*^,*^,2,^,*^,2,2,*^}	455371: {P,P,^,*^,*^,2,^,*^,*^,2,*^}
331499: {P,P,*^,*^,*^,2,^,*^,2,2,^}	459399: {P,P,^,*^,*^,*^,2,2,P,^}
336072: {P,P,*^,2,*^,*^,*^,2,*^}	459447: {P,P,^,*^,*^,*^,2,2,^,*^,P,^}
336096: {P,P,*^,2,*^,*^,*^,2,*^,*^}	468427: {P,P,^,*^,2,P,^,*^,2,*^}
336098: {P,P,*^,2,*^,*^,*^,2,*^,2}	478216: {P,P,^,*^,*^,*^,*^,2,*^}
336104: {P,P,*^,2,*^,*^,*^,2,2,*^}	478240: {P,P,^,*^,*^,*^,*^,2,*^}
336107: {P,P,*^,2,*^,*^,*^,2,2,^}	478242: {P,P,^,*^,*^,*^,*^,2,*^}
336432: {P,P,*^,2,*^,*^,*^,*^,*^}	478248: {P,P,^,*^,*^,*^,*^,2,2,*^}
336434: {P,P,*^,2,*^,*^,*^,*^,2,*^}	478251: {P,P,^,*^,*^,*^,*^,2,2,*^}
336442: {P,P,*^,2,*^,2,*^,*^,2,*^}	478336: {P,P,^,*^,*^,*^,*^,2,*^,*^}
336779: {P,P,*^,2,*^,*^,*^,2,*^,2,^}	478338: {P,P,^,*^,*^,*^,*^,2,*^,*^,2}
336816: {P,P,*^,2,*^,*^,*^,2,^,*^,*^}	478346: {P,P,^,*^,*^,*^,2,2,*^,2,2}
336818: {P,P,*^,2,*^,*^,*^,2,^,*^,2}	478370: {P,P,^,*^,*^,*^,2,2,*^,2}
336826: {P,P,*^,2,*^,*^,*^,2,2,^}	478375: {P,P,^,*^,*^,*^,2,2,P,^}
337968: {P,P,*^,2,2,*^,*^,*^,*^}	478376: {P,P,^,*^,*^,*^,2,2,2,*^}
337970: {P,P,*^,2,2,*^,*^,*^,*^,2}	478379: {P,P,^,*^,*^,*^,2,2,2,^}
337978: {P,P,*^,2,2,*^,*^,*^,2,2}	478382: {P,P,^,*^,*^,*^,2,2,^,*^}
338736: {P,P,*^,2,2,^,*^,*^,*^,*^}	478720: {P,P,^,*^,*^,*^,2,*^,*^,*^,*^}
338738: {P,P,*^,2,2,^,*^,*^,*^,2}	478722: {P,P,^,*^,*^,*^,2,*^,*^,*^,2}
338746: {P,P,*^,2,2,^,*^,*^,2,2}	478730: {P,P,^,*^,*^,*^,2,*^,*^,2,2}
342576: {P,P,*^,*^,2,2,^,*^,*^,*^}	478762: {P,P,^,*^,*^,*^,2,*^,2,2,2}
342578: {P,P,*^,*^,2,2,^,*^,*^,2}	478858: {P,P,^,*^,*^,2,2,*^,2,2}
342586: {P,P,*^,*^,2,2,^,*^,*^,2,2}	478859: {P,P,^,*^,*^,2,2,2,*^,2,^}
342768: {P,P,*^,*^,2,2,^,*^,*^,*^}	478878: {P,P,^,*^,*^,2,2,P,*,2}
342770: {P,P,*^,*^,2,2,^,*^,*^,2}	478882: {P,P,^,*^,*^,2,2,2,2,*^,2}
342778: {P,P,*^,*^,2,2,^,*^,2,2}	478883: {P,P,^,*^,*^,2,2,2,2,*^,*^}
342923: {P,P,*^,*^,2,2,^,*^,2,^}	478887: {P,P,^,*^,*^,2,2,2,P,^}
342960: {P,P,*^,*^,2,2,^,*^,*^,*^}	478888: {P,P,^,*^,*^,2,2,2,2,2,*^}
342962: {P,P,*^,*^,2,2,^,*^,*^,2}	478891: {P,P,^,*^,*^,2,2,2,2,2,^}
342970: {P,P,*^,*^,2,2,^,*^,2,2}	478894: {P,P,^,*^,*^,2,2,2,2,^,*^}
356391: {P,P,P,^,*^,*^,*^,2,P,^}	478895: {P,P,^,*^,*^,2,2,2,2,^,*^}
360648: {P,P,2,*^,*^,*^,*^,2,*^}	478906: {P,P,^,*^,*^,2,2,2,2,^,*^}
360672: {P,P,2,*^,*^,*^,*^,2,*^}	478907: {P,P,^,*^,*^,2,2,2,2,^,*^}

EGOCRON: {EOCRON} number: 2^42	EGOCRON: {EOCRON} number: 2^42
484096: {P,^,P,2,*,^,*,*,*,*,*}	602723: {2,P,*,^,*,2,P,2,*,^}
484098: {P,^,P,2,*,^,*,*,*,2}	602747: {2,P,*,^,*,2,P,^,2,^}
484106: {P,^,P,2,*,^,*,*,2,2}	602771: {2,P,*,^,*,2,2,P,*,^}
484138: {P,^,P,2,*,^,*,2,2,2}	602855: {2,P,*,^,*,2,^,2,P,^}
484266: {P,^,P,2,*,^,2,2,2,2}	606408: {2,P,P,*,*,*,^,*,2,*}
488136: {P,^,P,^,*,2,^,*,2,*}	606432: {2,P,P,*,^,*,*,^,2,*,*}
488160: {P,^,P,^,*,2,^,2,*,*}	606434: {2,P,P,*,^,*,*,^,2,*,2}
488162: {P,^,P,^,*,2,^,2,*,2}	606440: {2,P,P,*,^,*,*,^,2,2,*,*}
488168: {P,^,P,^,*,2,^,2,2,*,*}	606443: {2,P,P,*,^,*,*,^,2,2,^}
488171: {P,^,P,^,*,2,^,2,2,2,^}	606768: {2,P,P,*,^,*,2,*,^,*,*}
489611: {P,^,P,^,*,2,*,2,*,2,^}	606770: {2,P,P,*,^,*,2,*,^,*,2}
489648: {P,^,P,^,*,2,*,2,^,*,*}	606778: {2,P,P,*,^,*,2,*,^,2,2}
489650: {P,^,P,^,*,2,*,2,^,*,2}	607115: {2,P,P,*,^,*,^,2,*,2,^}
489658: {P,^,P,^,*,2,*,2,2,2,2}	607152: {2,P,P,*,^,*,^,2,*,*,*}
489995: {P,^,P,^,2,2,*,*,2,2,^}	607154: {2,P,P,*,^,*,^,2,*,^,2,2}
490187: {P,^,P,^,2,2,^,*,2,2,^}	607162: {2,P,P,*,^,*,^,2,^,2,2}
490240: {P,^,P,^,2,^,*,*,*,*,*}	608304: {2,P,P,*,^,2,*,*,^,*,*}
490242: {P,^,P,^,2,^,*,*,*,2}	608306: {2,P,P,*,^,2,*,*,^,*,2}
490250: {P,^,P,^,2,^,*,*,*,2,2}	608314: {2,P,P,*,^,2,*,*,^,2,2}
490282: {P,^,P,^,2,^,*,*,2,2,2}	608459: {2,P,P,*,^,2,*,^,*,2,^}
490410: {P,^,P,^,2,^,2,2,2,2,2}	609995: {2,P,P,*,^,2,^,*,*,2,2,^}
496840: {P,^,2,P,P,*,^,*,2,2,*}	614448: {2,P,P,2,*,^,*,*,*,*}
496864: {P,^,2,P,P,*,^,2,*,*}	614450: {2,P,P,2,*,*,*,^,*,2}
496866: {P,^,2,P,P,*,^,2,*,2}	614458: {2,P,P,2,*,^,*,*,^,2,2}
496872: {P,^,2,P,P,*,^,2,2,*,*}	614603: {2,P,P,2,*,^,*,^,*,2,^}
496875: {P,^,2,P,P,*,^,2,2,2,^}	619143: {2,P,P,^,*,2,2,*,P,^}
504419: {P,^,2,^,*,2,P,2,*,^}	619191: {2,P,P,^,*,2,2,*,P,^}
504443: {P,^,2,^,*,2,P,^,2,^}	624423: {2,P,2,*,P,^,*,2,P,^}
504467: {P,^,2,^,*,2,2,P,*,^}	625171: {2,P,2,*,2,2,*,P,*,^}
504551: {P,^,2,^,*,2,^,2,P,^}	625363: {2,P,2,*,2,2,^,P,*,^}
508619: {P,^,^,*,*,2,^,2,*,2,^}	626311: {2,P,2,*,^,2,2,*,P,^}
517576: {P,^,^,2,P,P,^,*,2,^}	626359: {2,P,2,*,^,2,2,^,P,^}
517600: {P,^,^,2,P,P,^,2,*,*}	631075: {2,P,2,2,*,P,*,2,*,^}
517602: {P,^,^,2,P,P,^,2,*,2}	631099: {2,P,2,2,*,P,*,^,2,^}
517608: {P,^,^,2,P,P,^,2,2,*,*}	631171: {2,P,2,2,*,P,2,*,^,*,*}
517611: {P,^,^,2,P,P,^,2,2,2,^}	632899: {2,P,2,2,2,*,P,*,^,*,*}
529152: {2,*,*,P,*,^,*,*,*,*,*}	633667: {2,P,2,2,2,^,P,*,^,*,*}
529154: {2,*,*,P,*,^,*,*,*,2}	634147: {2,P,2,2,^,P,*,2,*,^}
529162: {2,*,*,P,*,^,*,*,*,2,2}	634171: {2,P,2,2,^,P,*,^,2,^}
529194: {2,*,*,P,*,^,*,2,2,2}	634243: {2,P,2,2,2,^,P,2,*,^,*,*}
529322: {2,*,*,P,*,^,2,2,2,2,2}	643840: {2,P,^,P,*,^,*,*,*,*,*}
531595: {2,*,*,P,^,*,2,*,2,*,2,^}	643842: {2,P,^,P,*,^,*,*,*,*,2}
531632: {2,*,*,P,^,*,2,*,*,*,*}	643850: {2,P,^,P,*,^,*,*,^,2,2}
531634: {2,*,*,P,^,*,2,^,*,2,2}	643882: {2,P,^,P,*,^,*,2,2,2}
531642: {2,*,*,P,^,*,2,^,2,2,2}	644010: {2,P,^,P,*,^,2,2,2,2}
531979: {2,*,*,P,^,2,*,2,*,2,^}	646283: {2,P,^,P,^,*,2,*,2,*,2,^}
533808: {2,*,*,2,P,P,*,^,*,*}	646320: {2,P,^,P,^,*,2,^,*,*,*}
533810: {2,*,*,2,P,P,*,^,*,*}	646322: {2,P,^,P,*,^,2,*,*,*,*}
533818: {2,*,*,2,P,P,*,^,2,2}	646330: {2,P,^,P,^,*,2,^,2,2}
534627: {2,*,*,2,2,*,P,2,*,^}	646667: {2,P,^,P,^,2,*,*,2,^}
534651: {2,*,*,2,2,*,P,*,2,^}	648496: {2,P,^,2,P,P,*,^,*,*,*}
535059: {2,*,*,2,2,2,*,P,*,^}	648498: {2,P,^,2,P,P,*,^,*,2}
535251: {2,*,*,2,2,2,2,^,P,*,^}	648506: {2,P,^,2,P,P,*,^,2,2}
535395: {2,*,*,2,2,2,2,^,P,2,*,^}	649315: {2,P,^,2,2,*,P,2,*,^}
535419: {2,*,*,2,2,2,2,^,P,^,2,^}	649339: {2,P,^,2,2,*,P,^,2,^}
536199: {2,*,*,2,2,2,2,2,^,P,^}	649747: {2,P,^,2,2,2,2,^,P,*,^}
536247: {2,*,*,2,2,2,2,2,2,^,P,^}	649939: {2,P,^,2,2,2,2,2,^,P,*,^}
538992: {2,*,*,2,2,2,2,2,2,^,P,*,^}	650083: {2,P,^,2,2,2,2,2,2,^,P,*,^}
538994: {2,*,*,2,2,2,2,2,2,2,^,P,2,*,^}	650107: {2,P,^,2,2,2,2,2,2,2,^,P,2,*,^}
539002: {2,*,*,2,2,2,2,2,2,2,2,^,P,2,*,^}	650887: {2,P,^,2,2,2,2,2,2,2,2,^,P,2,*,^}
558859: {2,*,2,*,P,^,*,2,2,^}	650935: {2,P,^,2,2,2,2,2,2,2,2,^,P,2,*,^}
562224: {2,*,2,P,P,*,*,*,*,*}	653680: {2,P,^,2,2,2,2,2,2,2,2,2,^,P,2,*,^}
562226: {2,*,2,P,P,*,^,*,2,2}	653682: {2,P,^,2,2,2,2,2,2,2,2,2,2,2,^,P,2,*,^}
562234: {2,*,2,P,P,*,^,*,2,2,2}	653690: {2,P,^,2,2,2,2,2,2,2,2,2,2,2,2,^,P,2,*,^}
562379: {2,*,2,P,P,*,^,*,2,2,2}	657163: {2,2,*,*,P,^,*,*,2,2,^}
563731: {2,*,2,P,2,2,*,P,*,^}	661651: {2,2,*,P,2,*,2,P,*,^}
563923: {2,*,2,P,2,2,2,^,P,*,^}	661735: {2,2,*,P,2,*,2,2,P,*,^}
564871: {2,*,2,P,^,2,2,2,*,P,^}	661795: {2,2,*,P,2,2,P,*,2,*,2,^}
564919: {2,*,2,P,^,2,2,2,2,*,P,^}	661819: {2,2,*,P,2,P,*,^,2,2,^}
565651: {2,*,2,2,*,P,2,P,*,^}	661891: {2,2,*,P,2,P,2,*,*,2,2,^}
565735: {2,*,2,2,2,*,P,^,2,P,^}	662083: {2,2,*,P,2,2,P,*,*,2,2,^}
568723: {2,*,2,2,2,2,*,P,2,P,*,^}	663139: {2,2,*,P,2,P,*,2,2,P,*,^}
568807: {2,*,2,2,2,2,2,*,P,2,P,^}	663163: {2,2,*,P,^,2,P,2,*,2,2,^}
574087: {2,*,^,*,*,2,2,2,*,P,^}	663187: {2,2,*,P,^,2,2,2,P,*,^}
574135: {2,*,^,*,*,2,2,2,2,*,P,^}	663271: {2,2,*,P,^,2,2,2,2,*,P,^}
583115: {2,*,^,2,P,P,^,*,2,2,^}	673575: {2,2,P,*,P,^,*,2,P,^}
592935: {2,P,*,*,*,*,*,2,P,^}	674323: {2,2,P,*,2,2,2,*,P,*,^}
597543: {2,P,*,P,^,2,*,2,P,^}	674515: {2,2,P,*,2,2,2,2,*,P,*,^}

---

### 20.8.3\_VIRTUAL OCRONS AND MATULA-GOEBEL NUMBERS

---

As we saw in Chapter 20.8.1, there are virtual OCRONS of order 0, 1, or 2,. We observed that every natural number can be represented as a virtual OCRON of order 0, 1, or 2. We summarize again:

#### Virtual OCRONS of order 0:

These are not really virtual OCRONS but normal OCRONS that have a "\*" -free OCRON representation. In the number range from 1 to 10000 there are 95 virtual OCRONS of order 0:

```
Mathematica (needs OCRON-Library):
numbersOrder0=Select[Range[10000], kVirt0Q]
{2,3,4,5,7,8,9,11,16,17,19,23,25,27,31,32,49,53,59,64,67,81,83,97,103,
121,125,127,128,131,227,241,243,256,277,289,311,331,343,361,419,431,50
9,512,529,563,625,661,691,709,719,729,739,961,1024,1331,1433,1523,1543
,1619,1787,1879,2048,2063,2187,2221,2309,2401,2437,2809,2897,3001,3125
,3481,3637,3671,3803,4091,4096,4489,4637,4913,4943,5189,5381,5441,5519
,5623,6561,6859,6889,7573,8161,8192,9409}
```

These are all prime numbers and their powers, which can be represented as "\*" -free OCRONS. Here are the representations of the first 54 virtual OCRONS of order 0:

```
Mathematica (needs OCRON-Library):
ocronsOrder0={};

For[i=1,i<= 54,i++,
AppendTo[ocronsOrder0,nToVirtualOCRONLevel0[numbersOrder0[[i]]];
Print["i= ",i," ,number= ",numbersOrder0[[i]],": ",ocronsOrder0[[i]]];

(*virtual OCRONS order 0: Matula-Goebel numbers of generalized Bethe
trees *)
i= 1, number= 2: {2}
i= 2, number= 3: {2,P}
i= 3, number= 4: {2,2,^}
i= 4, number= 5: {2,P,P}
i= 5, number= 7: {2,2,^,P}
i= 6, number= 8: {2,2,P,^}
i= 7, number= 9: {2,P,2,^}
i= 8, number= 11: {2,P,P,P}
i= 9, number= 16: {2,2,2,^,^}
i= 10, number= 17: {2,2,^,P,P}
i= 11, number= 19: {2,2,P,^,P}
i= 12, number= 23: {2,P,2,^,P}
i= 13, number= 25: {2,P,P,2,^}
i= 14, number= 27: {2,P,2,P,^}
i= 15, number= 31: {2,P,P,P,P}
i= 16, number= 32: {2,2,P,P,^}
i= 17, number= 49: {2,2,^,P,2,^}
i= 18, number= 53: {2,2,2,^,^,P}
i= 19, number= 59: {2,2,^,P,P,P}
i= 20, number= 64: {2,2,^,2,P,^}
i= 21, number= 67: {2,2,P,^,P,P}
i= 22, number= 81: {2,P,2,2,^,^}
i= 23, number= 83: {2,P,2,^,P,P}
i= 24, number= 97: {2,P,P,2,^,P}
```

```

i= 25, number= 103: {2,P,2,P,^,P}
i= 26, number= 121: {2,P,P,P,2,^}
i= 27, number= 125: {2,P,P,2,P,^}
i= 28, number= 127: {2,P,P,P,P,P}
i= 29, number= 128: {2,2,2,^,P,^}
i= 30, number= 131: {2,2,P,P,^,P}
i= 31, number= 227: {2,2,^,P,2,^,P}
i= 32, number= 241: {2,2,2,^,^,P,P}
i= 33, number= 243: {2,P,2,P,P,^}
i= 34, number= 256: {2,2,2,P,^,^}
i= 35, number= 277: {2,2,^,P,P,P,P}
i= 36, number= 289: {2,2,^,P,P,2,^}
i= 37, number= 311: {2,2,^,2,P,^,P}
i= 38, number= 331: {2,2,P,^,P,P,P}
i= 39, number= 343: {2,2,^,P,2,P,^}
i= 40, number= 361: {2,2,P,^,P,2,^}
i= 41, number= 419: {2,P,2,2,^,^,P}
i= 42, number= 431: {2,P,2,^,P,P,P}
i= 43, number= 509: {2,P,P,2,^,P,P}
i= 44, number= 512: {2,2,P,2,^,^}
i= 45, number= 529: {2,P,2,^,P,2,^}
i= 46, number= 563: {2,P,2,P,^,P,P}
i= 47, number= 625: {2,P,P,2,2,^,^}
i= 48, number= 661: {2,P,P,P,2,^,P}
i= 49, number= 691: {2,P,P,2,P,^,P}
i= 50, number= 709: {2,P,P,P,P,P,P}
i= 51, number= 719: {2,2,2,^,P,^,P}
i= 52, number= 729: {2,P,2,^,2,P,^}
i= 53, number= 739: {2,2,P,P,^,P,P}
i= 54, number= 961: {2,P,P,P,2,^}

```

These numbers are not unknown in mathematical literature and they come from graph theory:

They are the so-called Matula-Goebel numbers for generalized "Bethe trees". Once again, relationships arise from completely different mathematical disciplines, in our case from graph theory and the theory of prime numbers! Thus, we identified the sequence of virtual OCRONs of type 0 as the so-called Matula-Goebel numbers, which appear in the study of generalized Bethe trees.

In [oeis.org](https://oeis.org) they are listed as [A214577](#).

### **Virtual OCRONs of order 1:**

This applies to all OCRONs that can be written as the product of different "\*" -free factors, hence all composed numbers that can be written as a product of Matula-Goebel numbers (numbers that have a virtual OCRON order 0). Many composed numbers can be represented as virtual OCRONs of order 1 (in the range of 1 to 100 these are 62 of them):

```

Mathematica (needs OCRON-Library):
numbersOrder1=Select[Range[100],kVirt1Q]
{4,6,8,9,10,12,14,15,16,18,20,21,22,24,25,27,28,30,32,33,34,35,36,38,4
0,42,44,45,46,48,49,50,51,54,55,56,57,60,62,63,64,66,68,69,70,72,75,76
,77,80,81,84,85,88,90,92,93,95,96,98,99,100}
ocronsOrder1={};
For[i=1,i<= 62,i++,
AppendTo[ocronsOrder1,nToVirtualOCRONLevel1[numbersOrder1[[i]]]];

```

```

Print["i= ",i," ,number= ",numbersOrder1[[i]],": ",ocronsOrder1[[i]]];

i= 1, number= 4: {2,^,2}
i= 2, number= 6: {2,^,2,P}
i= 3, number= 8: {2,^,2,2,^}
i= 4, number= 9: {2,P,^,2,P}
i= 5, number= 10: {2,^,2,P,P}
i= 6, number= 12: {2,2,^,^,2,P}
i= 7, number= 14: {2,^,2,2,^,P}
i= 8, number= 15: {2,P,^,2,P,P}
i= 9, number= 16: {2,2,^,^,2,2,^}
i= 10, number= 18: {2,^,2,P,2,^}
i= 11, number= 20: {2,2,^,^,2,P,P}
i= 12, number= 21: {2,P,^,2,2,^,P}
i= 13, number= 22: {2,^,2,P,P,P}
i= 14, number= 24: {2,2,P,^,^,2,P}
i= 15, number= 25: {2,P,P,^,2,P,P}
i= 16, number= 27: {2,P,^,2,P,2,^}
i= 17, number= 28: {2,2,^,^,2,2,^,P}
i= 18, number= 30: {2,^,2,P,^,2,P,P}
i= 19, number= 32: {2,2,^,^,2,2,P,^}
i= 20, number= 33: {2,P,^,2,P,P,P}
i= 21, number= 34: {2,^,2,2,^,P,P}
i= 22, number= 35: {2,P,P,^,2,2,^,P}
i= 23, number= 36: {2,2,^,^,2,P,2,^}
i= 24, number= 38: {2,^,2,2,P,^,P}
i= 25, number= 40: {2,2,P,^,^,2,P,P}
i= 26, number= 42: {2,^,2,P,^,2,2,^,P}
i= 27, number= 44: {2,2,^,^,2,P,P,P}
i= 28, number= 45: {2,P,2,^,^,2,P,P}
i= 29, number= 46: {2,^,2,P,2,^,P}
i= 30, number= 48: {2,2,2,^,^,^,2,P}
i= 31, number= 49: {2,2,^,P,^,2,2,^,P}
i= 32, number= 50: {2,^,2,P,P,2,^}
i= 33, number= 51: {2,P,^,2,2,^,P,P}
i= 34, number= 54: {2,^,2,P,2,P,^}
i= 35, number= 55: {2,P,P,^,2,P,P,P}
i= 36, number= 56: {2,2,P,^,^,2,2,^,P}
i= 37, number= 57: {2,P,^,2,2,P,^,P}
i= 38, number= 60: {2,2,^,^,2,P,^,2,P,P}
i= 39, number= 62: {2,^,2,P,P,P,P}
i= 40, number= 63: {2,P,2,^,^,2,2,^,P}
i= 41, number= 64: {2,2,P,^,^,2,2,P,^}
i= 42, number= 66: {2,^,2,P,^,2,P,P,P}
i= 43, number= 68: {2,2,^,^,2,2,^,P,P}
i= 44, number= 69: {2,P,^,2,P,2,^,P}
i= 45, number= 70: {2,^,2,P,P,^,2,2,^,P}
i= 46, number= 72: {2,2,P,^,^,2,P,2,^}
i= 47, number= 75: {2,P,^,2,P,P,2,^}
i= 48, number= 76: {2,2,^,^,2,2,P,^,P}
i= 49, number= 77: {2,2,^,P,^,2,P,P,P}
i= 50, number= 80: {2,2,2,^,^,^,2,P,P}
i= 51, number= 81: {2,P,2,^,^,2,P,2,^}
i= 52, number= 84: {2,2,^,^,2,P,^,2,2,^,P}
i= 53, number= 85: {2,P,P,^,2,2,^,P,P}
i= 54, number= 88: {2,2,P,^,^,2,P,P,P}
i= 55, number= 90: {2,^,2,P,2,^,^,2,P,P}
i= 56, number= 92: {2,2,^,^,2,P,2,^,P}
i= 57, number= 93: {2,P,^,2,P,P,P,P}
i= 58, number= 95: {2,P,P,^,2,2,P,^,P}

```

```
i= 59, number= 96: {2,2,P,P,^,^,2,P}
i= 60, number= 98: {2,^,2,2,^,P,2,^}
i= 61, number= 99: {2,P,2,^,^,2,P,P,P}
i= 62, number= 100: {2,2,^,^,2,P,P,2,^}
```

Order 1 virtual OCRONs can be easily constructed by defining the product operator " $\wedge$ " (which looks exactly like the OCRON power operator " $\wedge$ ") and simply writing the product using factors that consist of Matula-Goebel numbers. Here an example:

$6 = 2 * 3$ :

2 (product operator)  $2P \rightarrow ,2\wedge 2P$ .

Another example:  $90 = 2 * 3^2 * 5$ :

$90 = 2\wedge 2P 2\wedge\wedge 2PP$

A few numbers can not be represented as virtual OCRONs of order 0 nor in the order 1. In the range of 1 to 100, these are 23 such numbers:

```
Mathematica (needs OCRON-Library):
strangeNumbers=Select[Range[100],kvirt0Q[#]==False && kvirt1Q[#]
==False&]
{13,26,29,37,39,41,43,47,52,58,61,65,71,73,74,78,79,82,86,87,89,91,94}
```

These are all numbers which contain in their prime factorization one of the "-containing" prime numbers (13, 29, 37, 41, 43, etc.).

The OCRON library contains the function `findMatulaProduct[]`, with which numbers can be decomposed into matula factors.

Some numbers can be represented both as virtual OCRONs of order 0 and of order 1 (these are powers of Matula-Goebel numbers).

## Virtual OCRONs of order 2:

This applies to all OCRONs that can be written as the sum of different "\*" -free summands (and thus all natural numbers, since all natural numbers (strictly speaking  $\geq 4$ ) can be written as the sum of Matula-Goebel numbers). Every natural number can be represented as a virtual OCRON of order 2. Many numbers can even be written as the sum of only two matula summands. However, three matula summands are sufficient in any case (the reader will have noticed by now that this book contains almost no proofs, so the proof is still pending ...). Here are a few examples of numbers that need to be broken down into three matula summands to be represented as a virtual OCRON of order 2 (there are exactly 4 such numbers in the range up to 100):

```
Mathematica (needs OCRON-Library):

For[i=1,i<=100,i++,mgSum=findMatulaGoebelSum[i];
If[Length[mgSum]> 2,Print["i= ",i,": ",mgSum]]];
i= 45: {32,11,2}
i= 77: {67,8,2}
i= 79: {67,9,3}
i= 93: {83,8,2}
```

Order 2 virtual OCRONs can be easily constructed by defining the sum operator " $\wedge\wedge 2$ " (which looks exactly like the OCRON string " $\wedge\wedge 2$ ") and simply write down the sum using summands consisting of Matula-Goebel numbers.

Here an example:  $13 = 5 + 8$ . Since  $5 = 2PP$  and  $8 = 22P^{\wedge}$  it holds:

**2PP** (sum operator) **22P<sup>^</sup>** -> „**2PP<sup>^</sup>222P<sup>^</sup>**“.

Thus, we have found a virtual OCRON of order 2 for the number 13, by breaking the number 13 into the two matula summands 5 and 8: **2PP<sup>^</sup>222P<sup>^</sup>**.

As a reminder: virtual OCRONs of order 2 are calculated by first exposing them twice and taking the binary logarithm twice at the end of the calculation:

`13=ld(ld(Value(222PP^222P^)))`. For the case that one wants to use the number 1 in a matula sum, there is the ‘SumPlus1Operator’ „ $\wedge\wedge 2^2$ “, which can be applied the same way as the sum operator. An Example::  $5 = 2 + 2 + 1$  gives: „ $2^{\wedge\wedge} 2^2$ “

As another example, the virtual OCRONs of the first 32 Mersenne prime number exponents (including their order) are listed here:

```
Mathematica: (needs OCRON-Library)
For[i=1,i<=50,i++,Print["i= ",i,"",
nToVirtualOCRON[myMersennePrimeExponent[i]]]]
i= 1,{{2},0}
i= 2,{{2,P},0}
i= 3,{{2,P,P},0}
i= 4,{{2,2,^,P},0}
i= 5,{{2,P,P,^,^,2,2,2,P,^},2}
i= 6,{{2,2,^,P,P},0}
i= 7,{{2,2,P,^,P},0}
i= 8,{{2,P,P,P,P},0}
i= 9,{{2,2,P,^,^,^,2,2,2,2,^,^,P},2}
i= 10,{{2,P,P,2,^,^,^,2,2,2,^,2,P,^},2}
i= 11,{{2,2,^,^,^,2,2,P,2,P,^,P},2}
i= 12,{{2,P,P,P,P,P},0}
i= 13,{{2,P,2,^,^,^,2,2,2,P,2,^,^},2}
i= 14,{{2,P,2,P,^,P,P,^,^,2,2,P,2,P,^,^,2,2,2,^,P,P},2}
i= 15,{{2,2,^,2,P,P,^,^,^,2,2,2,2,^,P,^,^,^,2,2,P,P,P,P},2}
i= 16,{{2,2,2,^,^,^,^,2,2,P,2,2,^,P,^},2}
i= 17,{{2,2,P,^,P,P,P,P,^,^,2,2,2,2,^,P,P},2}
i= 18,{{2,P,P,2,P,P,^,^,2,2,P,2,^,P,P,^,^,2,2,P,2,^},2}
i= 19,{{2,2,^,2,^,2,P,^,^,^,2,2,P,P,2,P,^},2}
i= 20,{{2,2,^,2,^,2,P,^,^,^,2,2,2,^,2,P,^,^,^,2,2,2,2,^,^},2}
i= 21,{{2,P,P,2,^,P,2,^,^,^,2,2,2,^,P,P,P,P,^,^,2,2,P},2}
i= 22,{{2,P,P,2,^,P,2,^,^,^,2,2,P,2,^,^,^,2,2,P},2}
i= 23,{{2,2,2,P,^,^,^,^,2,2,P,P,2,P,^,P},2}
i= 24,{{2,P,2,P,2,^,^,^,^,2,2,P,2,P,^,^,^,2,2,P,P,P},2}
i= 25,{{2,2,^,P,2,2,^,^,P,^,^,2,2,P,2,P,^,^,^,2,2,2,P,^,P,P},2}
i= 26,{{2,2,P,^,P,2,^,P,P,^,^,2,2,2,2,^,P,2,^,P,P,^,^,2,2,2,^,P,2,^},2}
i= 27,{{2,2,P,^,P,P,2,^,P,^,^,2,2,P,2,P,P,^,^,2,2,P,P,P},2}
i= 28,{{2,2,^,2,P,^,P,^,^,2,2,2,P,^,P,P,P,P,^,^,2,2,P,P},2}
i= 29,{{2,P,2,2,P,^,^,P,^,^,2,2,P,P,2,P,P,^,P,^,^,2,2,P,P,P,P,P,2,^},2}
i= 30,{{2,2,2,^,P,P,^,^,2,2,P,P,P,2,^,^,^,2,2,2,2,^,^},2}
i= 31,{{2,P,2,2,^,P,P,^,^,2,2,2,^,P,2,^,^,^,2,2,P,P},2}
i= 32,{{2,P,P,2,P,^,P,P,P,^,^,2,2,2,2,^,^,P,P,P,P,P,^,^,2,2,P,2,^,P},2}
```

**Interestingly, the conspicuously short virtual OCRON for the 12th Mersenne prime number exponent (127) contains no <sup>^</sup> operators: 2PPPPP.**

At the same time, it seems to be the last Mersenne prime number exponent that can be represented as a virtual OCRON of order 0. All subsequent values are only representations of order 2!

Note:

The additive principle in the construction of virtual OCRONs of order 2 also makes it possible to disassemble a number into matula summands, which consist only of powers of two, whereby the powers should consist recursively only of sums of powers of two. This would result in a representation of virtual OCRONs of order 2 where the P-symbol no longer occurs.

### A pure number representation using the symbols 2 and ^!

The author did not pursue this idea any further, since the whole thing probably only amounts to a kind of 'mislabeled' or at least complicated binary representation.

## Virtual OCRONs of order 3:

Actually, virtual OCRONs of the order 0,1 and 2 could be 'enough', since we can represent all natural numbers with them. However, the whole thing can be pushed even further and virtual OCRONs of third order can also be examined. Such virtual OCRONs are practically no longer numerically evaluable, since utopically high numbers are generated during evaluation. Nevertheless, we still want to take a look at it. Analogous to the orders 1 and 2 we define a virtual operator: „ $\wedge\wedge\wedge 2^2$ “, which we will call 'Jacobi addition operator', because it turns out that it corresponds to the so-called 'Jacobi addition' in the equivalent number range. Let's take a quick trip to the world of Jacobi arithmetic:

Mathematica:

```
CirclePlus[a_,b_]:=  
Quiet[FullSimplify[Distribute[PowerExpand[Log[2^a+2^b]/Log[2]]]]]  
CircleMinus[a_,b_]:=CirclePlus[a,(b+(I \[Pi])/Log[2])]
```

Some examples:

Mathematica:

$3 \oplus 3$  gives 4,  $4 \oplus 4$  gives 5,  $x \oplus x$  gives  $x + 1$

A number, taken ,Jacobi-added' by itself, increments this number by 1.

$x \oplus x == x + 1$  gives True

$((x \oplus x) \oplus x) \oplus x == x + 2$  gives True

$((((x \oplus x) \oplus x) \oplus x) \oplus x) \oplus x == x + 3$  gives True

A number  $x$ ,  $2^n - 1$  times ,Jacobi-added' by itself results to  $x + n$ .

One could classify the Jacobi addition in the hierarchy of arithmetic one level lower than the addition, since the addition arises from the Jacobi addition by successive applying the Jacobi addition. If we consider  $x \oplus x$  recursively, applying  $x = x \oplus x$   $n$  times results in the value  $x + n$ .

Thus, the Jacobi addition is actually the 'predecessing' kind of arithmetic for the addition, much like the normal addition is the 'predecessor' of the multiplication, since the multiplication arises from the addition by repeated application of the addition.

The Jacobi addition is associative, commutative, and has a neutral element. There is also an inverse to each element so that the Jacobi addition yields the neutral element. Also the distributive law is valid. The Jacobi addition thus forms, together with the addition, a field with respect to the set of integers.

Here is a proof, as well as a few derived laws of calculation, with the help of Mathematica:

```
(* Properties of the Jacobi-Addition *)
ClearSystemCache[];
ClearAll[a];ClearAll[b];ClearAll[c];Clear[CirclePlus];
Clear[CirclePlusN];
ld[x_]:=Log[2,x]
f0[a_,b_]:=FullSimplify[PowerExpand[ld[ld[ld[ld[(2^2^2^a)^2^2^b]]]]]] (*Jacobi-Addition*)
CirclePlus[a_,b_]:=Quiet[FullSimplify[Distribute[PowerExpand[Log[2^a+2^b]/Log[2]]]]] (*Jacobi-Addition*)
CircleMinus[a_,b_]:=CirclePlus[a,(b+(I*Pi)/Log[2])]
simplify[term_]:=Distribute[PowerExpand[term]]

(*associativity: (a⊕b)⊕c == a⊕(b⊕c) *)
(a⊕b)⊕c == a⊕(b⊕c)
(*commutativity: (a⊕b) == (b⊕a) *)
(a⊕b) == (b⊕a)
(*distributivity : (a⊕b)+c == (a+c)⊕(b+c)*)
(a⊕b)+c == simplify[(a+c)⊕(b+c)]
(*neutral element *)
Print["neutral element: ",-Infinity];
(*Applying Jacobi addition with the neutral element a⊕(-∞) *)
a⊕(-∞)
(*inverse element of a: *)
Print["inverse element of a: "];
simplify[Log[2,-2^a]]
(*applying Jacobi addition with the inverse element of a gives the neutral element: *)
Print["applying Jacobi addition with the inverse element: "];

$$\left(a + \frac{i\pi}{\log 2}\right) \oplus a$$


$$a \oplus \left(a + \frac{i\pi}{\log 2}\right)$$

(*applying n times the operator ⊕ a gives a+ld[n+1] *)
simplify[a⊕a]
simplify(a⊕a)⊕a
simplify((a⊕a)⊕a)⊕a
simplify(((a⊕a)⊕a)⊕a)⊕a
simplify((((a⊕a)⊕a)⊕a)⊕a)⊕a
(*Laws of power: 2^(a⊕b) gives:*)
2a⊕b
(*Laws of power: 2^(a⊖b) gives in:*)
2(a⊖b)
(*more formulas:*)
(*a⊖a results in:*)
a⊖a
(*New logarithmic law of addition: ld[a]⊕ld[b]: *)
(*gives ld[a]⊕ld[b] *)
Log[2,a]⊕Log[2,b]
```

Output of the Mathematica program:

associativity:  $(a \oplus b) \oplus c == a \oplus (b \oplus c)$

```

True
commutativity: (a⊕b) == (b⊕a)
True
distributivity: (a⊕b)+c == (a+c)⊕(b+c)
True
neutral element: -∞
applying Jacobi addition with the neutral element  $a \oplus (-\infty)$ :
 $a$ 
inverse element of  $a$ :
 $a + \frac{i\pi}{\text{Log}[2]}$ 
applying Jacobi addition with the inverse element of  $a$ :
 $-\infty$ 
applying  $n$  times the operator  $\oplus a$  gives  $a + \text{ld}[n+1]$ 
 $1 + a, a + \frac{\text{Log}[3]}{\text{Log}[2]}, 2 + a, a + \frac{\text{Log}[5]}{\text{Log}[2]}, 3 + a$ 

laws of power:  $2^a(a \oplus b)$  gives:
 $2^a + 2^b$ 
laws of power:  $2^a(a \ominus b)$  gives:
 $2^a - 2^b$ 

 $a \ominus a$  gives:
 $-\infty$ 

```

Back to our third-order virtual OCRONs.

Order 3 virtual OCRONs can be easily constructed by defining the Jacobi sum operator "**^^^22**" (which looks exactly like the OCRON string "**^^^22**") and simply write down the OCRON using Jacobi sums, which consist of Matula-Goebel numbers.

Here an example:

$3 = 2 \oplus 2$ . Now, we have: **2** (Jacobi sum operator) **2**: „**2\*\*\*222**“.

Thus we have found a virtual OCRON of order 3 for the number 3 by breaking the number 3 into the two Jacobi-Matula-summands 2 and 2: „**2<sup>3</sup><sup>3</sup><sup>2</sup>2<sup>2</sup>**“.

## Another example:

$$7 = ((5 \oplus 5) \oplus 5) \oplus 5$$

Our third order virtual OCRON for the number 7 then reads:

2PP<sup>^^^</sup>222PP<sup>^^^</sup>222PP<sup>^^^</sup>222PP

The traditional form of the number 7 in virtual order-2-OCRON representation is:

**Log[2, Log[2, (2^(2^2))^2^Prime[Prime[2]]]]** (Mathematica input-form)

$$\log_2(\log_2((2^{2^2})^{2^{p_{p_2}}})) \quad (\text{Traditional form})$$

The reader may want to find out the corresponding representation for the third order OCRON above!

### **Viewed from the other side:**

**Is there a corresponding number for each string consisting of the symbols "2", "^" and "P" (which at first glance seems to be a virtual OCRON)?**

We find that not every such string represents a 'well-formed' virtual OCRON.

The set of 'well-formed', interpretable virtual OCRONS can be easily found out with our OCRON library.

Here is an example of the first 50 well-formed interpretable virtual OCRONS:

```
Mathematica: (needs OCRON-Library)
list=createAscendingVirtualOcron4List[50]
Output:
{{2}, {P, 2}, {2, P}, {P, P, 2}, {P, 2, P}, {2, ^, 2}, {2, P, P}, {2, 2, ^}, {P, ^, 2, 2}, {P, P, P, 2}, {P, P, 2, P}, {P, 2, ^, 2}, {P, 2, P, P}, {P, 2, 2, ^}, {2, ^, P, 2}, {2, ^, 2, P}, {2, P, ^, 2}, {2, P, P, P}, {2, P, 2, ^}, {2, 2, ^, P}, {2, 2, P, ^}, {P, ^, P, 2, 2}, {P, ^, 2, P, 2}, {P, ^, 2, 2, P}, {P, P, ^, 2, 2}, {P, P, P, P, 2}, {P, P, P, 2, P}, {P, P, 2, ^, 2}, {P, P, 2, P, P}, {P, P, 2, 2, ^}, {P, 2, ^, P, 2}, {P, 2, ^, 2, P}, {P, 2, P, ^, 2}, {P, 2, P, P, P}, {P, 2, P, 2, ^}, {P, 2, 2, ^, P}, {P, 2, 2, P, ^}, {2, ^, ^, 2, 2}, {2, ^, P, P, 2}, {2, ^, P, 2, P}, {2, ^, 2, P}, {2, ^, 2, 2}, {2, ^, 2, P, P}, {2, ^, 2, 2, ^}, {2, P, ^, P, 2}, {2, P, ^, 2, P}, {2, P, P, ^, 2}, {2, P, P, P, P}, {2, P, P, 2, ^}, {2, P, 2, ^, P}, {2, P, 2, P, ^}}
```

This list contains (with mixed ocron-orders), the first 50 interpretable virtual OCRONS of the order 0,1 and 2.

If we look at the resulting numerical values of these virtual OCRONS, we find that the resulting numbers are not always natural numbers, but can also contain terms with binary logarithms.

Note: The first column contains the (in the 3-based number system) 'Gödelized' numeric value of the virtual OCRON; the second column contains the virtual OCRON, the third column contains the interpreted value together with its virtual order (0,1 or 2):

```
(* the first 50 virtual OCRONS, (needs OCRON-Library) *)
For[i=1,i<=50,i++,
val=virtualOCRON4ToN[list[[i]]];
Print[goedelSymbolListToNForVirtualOCRONS[list[[i]]],": ",list[[i]],": ",
",val];]

2: {2}: {2,0}
5: {P,2}: {Log[9]/Log[2],1}
7: {2,P}: {3,0}
14: {P,P,2}: {Log[25]/Log[2],1}
16: {P,2,P}: {Log[27]/Log[2],1}
20: {2,^,2}: {4,1}
22: {2,P,P}: {5,0}
24: {2,2,^}: {4,0}
35: {P,^,2,2}: {Log[12]/Log[2],2}
41: {P,P,P,2}: {Log[121]/Log[2],1}
43: {P,P,2,P}: {Log[125]/Log[2],1}
47: {P,2,^,2}: {Log[81]/Log[2],1}
49: {P,2,P,P}: {Log[243]/Log[2],1}
51: {P,2,2,^}: {Log[81]/Log[2],1}
59: {2,^,P,2}: {Log[49]/Log[2],1}
61: {2,^,2,P}: {6,1}
65: {2,P,^,2}: {6,1}
```

```

67: {2,P,P,P}: {11,0}
69: {2,P,2,^}: {9,0}
73: {2,2,^,P}: {7,0}
75: {2,2,P,^}: {8,0}
98: {P,^,P,2,2}: {Log[Log[130321]/Log[2]]/Log[2],2}
104: {P,^,2,P,2}: {Log[27]/Log[2],2}
106: {P,^,2,2,P}: {Log[24]/Log[2],2}
116: {P,P,^,2,2}: {Log[20]/Log[2],2}
122: {P,P,P,P,2}: {Log[961]/Log[2],1}
124: {P,P,P,2,P}: {Log[1331]/Log[2],1}
128: {P,P,2,^,2}: {Log[625]/Log[2],1}
130: {P,P,2,P,P}: {Log[3125]/Log[2],1}
132: {P,P,2,2,^}: {Log[625]/Log[2],1}
140: {P,2,^,P,2}: {Log[529]/Log[2],1}
142: {P,2,^,2,P}: {Log[729]/Log[2],1}
146: {P,2,P,^,2}: {Log[729]/Log[2],1}
148: {P,2,P,P,P}: {Log[177147]/Log[2],1}
150: {P,2,P,2,^}: {Log[19683]/Log[2],1}
154: {P,2,2,^,P}: {Log[2187]/Log[2],1}
156: {P,2,2,P,^}: {Log[6561]/Log[2],1}
170: {2,^,^,2,2}: {4,2}
176: {2,^,P,P,2}: {Log[289]/Log[2],1}
178: {2,^,P,2,P}: {Log[343]/Log[2],1}
182: {2,^,2,^,2}: {8,1}
184: {2,^,2,P,P}: {10,1}
186: {2,^,2,2,^}: {8,1}
194: {2,P,^,P,2}: {Log[361]/Log[2],1}
196: {2,P,^,2,P}: {9,1}
200: {2,P,P,^,2}: {10,1}
202: {2,P,P,P,P}: {31,0}
204: {2,P,P,2,^}: {25,0}
208: {2,P,2,^,P}: {23,0}
210: {2,P,2,P,^}: {27,0}

```

The numerical values of the first row (in our terms: virtual GOCRONs) we can also get more easier:

```

(* the first 50 virtual 'wellformed' GOCRONs, (needs OCRON-Library) *)
Select[Range[2,210],checkVirtualOCRON4[nToGoedelSymbolListForVirtualOCRONS[#]==True&]

{2,5,7,14,16,20,22,24,35,41,43,47,49,51,59,61,65,67,69,73,75,98,104,106,116,122,124,128,130,132,140,142,146,148,150,154,156,170,176,178,182,184,186,194,196,200,202,204,208,210}

```

This sequence is known (\* oeis.org: A031457 \*) and can also be calculated as follows:  
`Select[Range[210],DigitCount[#,3,0]==DigitCount[#,3,2]-1&]`

This is not surprising since the sequence simply contains all the numbers whose number of '0's (represented in the 3-based number system) is exactly 1 less than the number of '2's. Exactly the same applies to our virtual OCRONs, as we have chosen the following 'Gödelization':

, 2 = 2, , P = 1 , , ^ = 0. Since '^' is a binary operator (that is, requires 2 operands), the total number of '^' symbols in a 'well-formed' virtual OCORN is always 1 less than the number of '2's.

If we restrict ourselves to such virtual OCRONs, which only produce natural numbers when interpreted, we have exactly 11 values in the range between 1 and 100:

```
(* the first virtual 'wellformed' (G)OCRONS in the Range up to 100,
which yield natural numbers (needs OCRON-Library) *)
createIntValuesListFromAscendingVirtualOcron4s[100,2,100]
```

Note: The OCRON after ,> is the (to normal OCRON format) expanded version of the corresponding virtual OCRON.

```
0 i= 2: {2}->{2}
1 i= 7: {2,P}->{2,P}
2 i= 20: {2,^,2}->{2,2,^,2,^}
3 i= 22: {2,P,P}->{2,P,P}
4 i= 24: {2,2,^}->{2,2,^}
5 i= 61: {2,^,2,P}->{2,2,^,2,P,^}
6 i= 65: {2,P,^,2}->{2,2,P,^,2,^}
7 i= 67: {2,P,P,P}->{2,P,P,P}
8 i= 69: {2,P,2,^}->{2,P,2,^}
9 i= 73: {2,2,^,P}->{2,2,^,P}
10 i= 75:{2,2,P,^}->{2,2,P,^}
```

If we only want to look at the Gödel values of the virtual OCRONS, things also work faster (e.g for the range up to 210, see above):

```
Select[Range[2,210],IntegerQ[(virtualGOCRONToN[#])[[1]]]==True&&(virtualGOCRONToN[#])[[1]]>0&]
{2,7,20,22,24,61,65,67,69,73,75,170,182,184,186,196,200,202,204,208,210}
```

### 20.10.5 ZEROS OF THE RAMANUJAN TAU-L-FUNCTION $L(s)$

Table: The first 128 zeros of the Ramanujan Tau-L-function along the critical line  $\text{Re}(s) = 6$  with a precision of 40 digits

n	n-th Zero (imaginary part)
1	9.222379399921084797142611932940781116486
2	13.907549861392134005200205137953162193298
3	17.44277697823447326186396821867674589157
4	19.65651314195496013326192041859030723572
5	22.33610363720986669022749993018805980682
6	25.27463654811236537511831556912511587143
7	26.80439115835040198021488322410732507706
8	28.83168262418687532999683753587305545807
9	31.17820949836026045431935926899313926697
10	32.77487538223120822067357948981225490570
11	35.19699584121007518433543737046420574188
12	36.74146297671030936271563405171036720276
13	37.75391597562427392631434486247599124908
14	40.21903437422132299161603441461920738220
15	41.73049228930784693147870711982250213623
16	43.59174123557517077642842195928096771240
17	45.04007921377559853226557606831192970276
18	46.19731875314330693527153925970196723938
19	48.35905247802367057374794967472553253174
20	49.27605353655818021252343896776437759399
21	51.15656028143634870275491266511380672455
22	53.06671423542580612320307409390807151794
23	54.09995263156227451872837264090776443481
24	55.21778745348462535957878571934998035431
25	56.71529404472536839421081822365522384644
26	58.58016100791407154702028492465615272522
27	59.78593800331714191997889429330825805664
28	61.13672295792679989290263620205223560333
29	62.66499232630715710001823026686906814575
30	64.08664571892624906013224972411990165710
31	64.84864127982825721119297668337821960449
32	66.49476926718958225137612316757440567017
33	67.93860977475046070139796938747167587280
34	69.04339787488993351871613413095474243164
35	71.11465341424647590429231058806180953979
36	71.74750419616562169267126591876149177551
37	72.81406066758940198724303627386689186096
38	74.09582544001794701671315124258399009705
39	75.77216168976411836410989053547382354736
40	77.10183189348964560849708504974842071533
41	77.68461125026033187168650329113006591797
42	79.79293909123566663765814155340194702148
43	80.56019206809750698994321282953023910522
44	82.00757620451852858423080760985612869263
45	82.84252583957207605180883547291159629822
46	83.97564035576498042701132362708449363708
47	85.46221814858006382564781233668327331543
48	86.7559721882552850474320026114583015442
49	88.07513099425673885889409575611352920532

50	89.02289034074360074555443134158849716187
51	90.45103289616260155980853596702218055725
52	91.11271853147249544235819485038518905640
53	92.44292549472127973331225803121924400330
54	93.76912394743676770758611382916569709778
55	95.13807853977348827356763649731874465942
56	95.62492107704515831301250727847218513489
57	97.34104088984686597996187629178166389465
58	98.70980408818076057286816649138927459717
59	99.74664890030413744170800782740116119385
60	100.22461499968198950227815657854080200195
61	101.34359353371037570923363091424107551575
62	103.16663591563629154279624344781041145325
63	103.81733899744642712903441861271858215332
64	105.22181333799052538324758643284440040588
65	106.29382213420061020769935566931962966919
66	107.42670755392653347826126264408230781555
67	108.47543790163686594496539328247308731079
68	109.39169607602677558588766260072588920593
69	110.70966268400202636712492676451802253723
70	111.53473540163911081890546483919024467468
71	112.75715359897023404300853144377470016479
72	113.84343404772059216156776528805494308472
73	115.06276556053481385788472834974527359009
74	116.46348398369597987311863107606768608093
75	117.11654084727238966934237396344542503357
76	118.14687073684822848917974624782800674438
77	119.08216779664660123216890497133135795593
78	119.99454209523629799605259904637932777405
79	121.78633067852094029603904346004128456116
80	122.55731782502655846656125504523515701294
81	123.21241716312161429414118174463510513306
82	124.60624049116798062186717288568615913391
83	125.94289344930038510028680320829153060913
84	126.75939204586923381157248513773083686829
85	127.55580316015350206271250499412417411804
86	128.62383894451065202702011447399854660034
87	129.60342208412549780405242927372455596924
88	130.94859240739617689541773870587348937988
89	131.70819904811898481966636609286069869995
90	132.96854278614409849978983402252197265625
91	134.34729668877156427697627805173397064209
92	135.07869588873938937467755749821662902832
93	135.55289998752846258867066353559494018555
94	137.09033471100445922274957410991191864014
95	137.70022292031720212435175199061632156372
96	139.28400855168445104936836287379264831543
97	139.93658439005704963165044318884611129761
98	140.89653322681010649830568581819534301758
99	142.1411519890185388703685021027922630310
100	143.0835552634784448855498339980840682983
101	144.3547263694031244085635989904403686523
102	145.1653120064068502870213706046342849731
103	146.1487705718024301404511788859963417053

104	146.4097883646259958823065971955657005310
105	148.1177541226128084872470935806632041931
106	149.0412678815713718449842417612671852112
107	150.2750742969780901603371603414416313171
108	150.9064237539794532949599670246243476868
109	152.1344343784803641028702259063720703125
110	153.1151471940314081621181685477495193481
111	154.0518290966241181649820646271109580994
112	154.7953122295758987547742435708642005920
113	155.7320793911374607887410093098878860474
114	157.0957831922944762936822371557354927063
115	157.9127528865146530279162107035517692566
116	158.6608139225808713490550871938467025757
117	159.6686139103367452207749010995030403137
118	161.3063702811864743580372305586934089661
119	161.8503586051299976134032476693391799927
120	162.8714549225416021727141924202442169189
121	163.5474941087671822970150969922542572021
122	164.3389052284337310538830934092402458191
123	165.6101228957916760009538847953081130981
124	166.5807970056847295836632838472723960876
125	167.6436347091075731441378593444824218750
126	168.659124784726088192977290600538253784
127	169.2457741065447009987110504880547523499
128	170.5979320487521135873976163566112518311

Mathematica program: Please contact the author.

```
startValues={9,14,17.5,20,22.5,25,27,29,31,33,35,37,38,40,42,44,45,46,48,49,5
1,53,54,55,57,58.5,60,61,62.5,64,65,66.5,68,69,
71,72,73,74,76,77,78,80,80.5,82,83,84,85.5,87,88,89,90.5,91,92.5,94,95,95.5,9
7.5,98.5,99.5,100.5,
101,103,104,105,106,107.5,108.5,109.5,110.6,111.5,112.8,114,115,116.5,117,118
,119,120,122,122.5,123.2,124.6,126,126.8,127.6,128.6,129.6,131,131.7,133,134.
4,135,135.6,137,137.6,139.2,140,141,142.1,143.1,144.4,145.1,146.1,146.5,148,1
49,150.2,151,152,153,154,154.9,155.8,157.1,158,158.7,159.6,161.3,161.9,162.9,
163.5,164.3,165.6,166.6,167.6,168.7,169.3,170.5};
```

```
tabZeros=SetPrecision[Table[{n,FindRoot[RamanujanTauL[6+I*t],{t,startValues[[n]]}],AccuracyGoal->40,PrecisionGoal->40},{n,1,Length[startValues]}],40];
ramanujanTauLZerosTab128=Table[Re[(tabRZeros[[n,2]])[[1,2]]],{n,1,Length[tabR
Zeros]}]
ramanujanTauLZerosTab128=Table[Re[(tabZeros[[n,2]])[[1,2]]],{n,1,Length[tabZe
ros]}]
```

The application of Euler's product representation for Ramanujan's tau-L function  $L(s)$  along the critical line with real part 6 yields clear minima at the positions of the first 59 zeros. The similarity of the situation to the calculations for the Riemann zeta function in chapter 5.3 is obvious!

The product representation for Ramanujans Tau-L-Funktion  $L(s)$  along the critical line reads:

$$L(6 + t \cdot i) = \prod_{p \in \mathbb{P}}^{\infty} \frac{1}{1 - \tau(p)p^{-6-t \cdot i} + p^{11-2(6+t \cdot i)}}, \quad t > 0$$

Returns the following representation (using the first 128 primes):

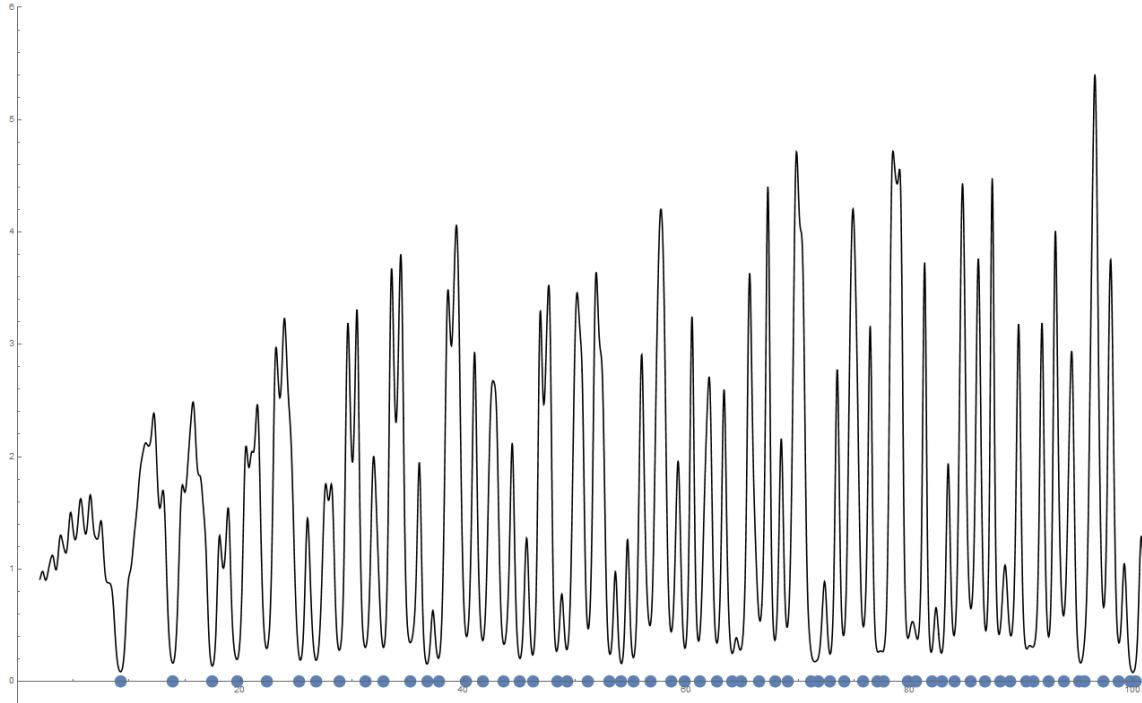


Figure: Ramanujan-Tau-L-function along the critical line, computed by the product formula in the range 0 up to 100, showing clear minima at the zero positions

Mathematica-program (please contact the author):

```
crterm[n_,x_]:=1/(1-RamanujanTau[Prime[n]]*(Prime[n])^(-
(6+x*I))+Prime[n]^(11.0-2*(6+x*I)));
myRFunc[x_]:=Product[crterm[n,x],{n,1,128}]
xmax=101;
Show[ListPlot[Table[{ramanujanTauLZerosTab128[[i]],0},{i,1,60}],
PlotRange->{{0,xmax},{-0.2,6}}],
Plot[Abs[myRFunc[x]],[x,2,xmax],PlotStyle->Black],PlotRange->{{0,xmax},{-
0.2,6}}]
```

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## 20.10.6 REPUNIT NUMBERS AND REPUNIT PRIME NUMBERS

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(Coming soon!)

A repunit number is a number that contains only the digit 1, e. g. ‘111’ oder ‘1111’ or ‘1111111111111111’. Its value depends on the base of the number system we are using. The term comes from “repeated unit”. A repunit prime is a repunit which is also a prime number. Of special interest are the repunit primes in the base-2 number system: These are the Mersenne prime numbers.

Here are some self-explaining tables which show the prime number decompositions of repunits in different number systems. The first columns on the left side show which repunits are also prime numbers in different bases (the first line above shows the base, prime numbers are framed).